t-Delta Sets of Numeric Semigroups

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Abstract

A Numerical Semigroup is a cofinite submonoid of $(\mathbb{N}_0, +)$ containing all linear combinations of a finite number of coprime integer generators. These semigroups allow for non-unique factorizations, meaning that elements can often be expressed as sums of the generators in multiple ways. Traditionally, the length of these factorizations has been measured using the 1-norm, with Delta sets consisting of gaps, which are the differences between consecutive lengths of an element when in ascending order. We introduce a method of computing lengths using t-norms for various t, and identify properties of the associated Delta sets for different families of numeric semigroups. In particular, for $t = 0$, the Δ_0-sets of all semigroups up to three generators, as well as maximal embedding dimension semigroups, semigroups with generators in generalized arithmetic progression, and semigroups with generators in a compound sequence, are explicitly given. For $t = \infty$, the Δ_∞-sets of semigroups with two generators are explicitly given, and the contents of other generalized families, including semigroups with generators in generalized arithmetic progression, are analyzed. The periodicity of the Δ_0 and Δ_∞ sets of individual semigroup elements is also proven, along with general results for t-lengths between 1 and ∞. We also relate semigroup trade structure, t-catenary degree, and Δ_t sets.

Contents

1 Introduction

Numerical semigroups are a common object of study in factorization theory, with applications to number theory, algebraic geometry, discrete optimization, and many other fields [2]. In particular, they are very accessible objects of study and application due to their straightforward definition and construction. We begin our investigation with a

Definition. A numerical semigroup S is a subset of $\mathbb{Z}_{\geq 0}$ such that

1. For all $b_1, b_2 \in S, b_1 + b_2 \in S$;
2. $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$;
3. $0 \in S$.

More simply put, a numerical semigroup is a cofinite set of positive integers, together with 0, which is closed under addition. A simple and common way of constructing numerical semigroups is using a finite set of coprime *generating elements*, or generators, for example:

Example 1.0.1.

$$\text{🐔} = \langle 6, 9, 20 \rangle = \{c_1 \cdot 6 + c_2 \cdot 9 + c_3 \cdot 20 : c_1, c_2, c_3 \in \mathbb{Z}_{\geq 0}\} \tag{1}$$

is a numerical semigroup containing all linear combinations of 6, 9, and 20. So $0 = 0 \cdot 6 \in \text{🐔}$, $35 = 6 + 9 + 20 \in \text{🐔}$, $60 = 4 \cdot 6 + 4 \cdot 9 \in \text{🐔}$. This semigroup is commonly referred to as the McNugget semigroup, since it was once possible to order Chicken McNuggets from McDonalds in quantities of 6, 9, or 20. Thus, elements of the McNugget semigroup represent all the possible quantities of Chicken McNuggets that could be ordered using these sizes.

The requirement that the generators of a numerical semigroup be coprime ensures that the semigroup itself is cofinite. Were the generators to have a collective gcd exceeding one, the semigroup would be isomorphic (through divison by this common factor) to a numerical semigroup with coprime generators, so it is conventional include the convenient stipulation that a numerical semigroup be cofinite.

Another important consequence of the cofinitude of these semigroups is that the elements which are *not* in a semigroup are greatly outnumbered by those that are. In fact, past a certain point, all integers are contained within a numerical semigroup. More formally, for any numerical semigroup S, there exists $N \in \mathbb{Z}_{\geq 0}$ such that for any integer $n > N$, $n \in S$. It is helpful to define the point at which this integer-density begins:

Definition. Let S be a numerical semigroup. The *Frobenius number* of S, denoted $F(S)$, is the largest integer not contained within S.

While the Frobenius number can be explicitly calculated for any numerical semigroup, calculating this invariant without significant computation is nontrivial [30][8].

Example 1.0.2. $F($🐔$) = 43$, since 43 cannot be written as a linear combination of 6, 9, and 20, but 44, 45, 46, 47, 48, and 49 all can. Thus, any integer greater than 43 is in 🐔, since all such integers are either one of $\{44, 45, 46, 47, 48, 49\}$ or can be expressed as one of these numbers plus a multiple of 6.

Notice that in 🐔, 6, 9, and 20, are the only elements of the semigroup (apart from 0) which cannot be "broken up" and expressed as a combination of multiple copies of generators. Letting S be a numerical semigroup, if $a \in S$ is an element which cannot be expressed as the sum of two nonzero elements of S, we say that a is *irreducible*, or an *atom* of S. Note that although the atoms of a numerical semigroup are necessarily generators, the converse does not hold. For instance, in Example 1.0.1 we could write

$$\text{🐔}^* = \langle 6, 9, 15, 20 \rangle = \{c_1 \cdot 6 + c_2 \cdot 9 + c_3 \cdot 15 + c_4 \cdot 20 : c_1, c_2, c_3, c_4 \in \mathbb{Z} \geq 0\}$$

to represent the same semigroup; since $15 = 6 + 9$, any multiple of 15 can also be expressed as a multiple of $6 + 9$, and so is contained in the set defined in Equation 1. This means that since 15 is reducible, our generating set is not minimal. Going forward, we will adopt the convention of using only minimal generating sets for our numerical semigroups, that is, generating sets containing only irreducible elements.

The reader may also notice that in Example 1.0.1, the element $60 \in$ 🐔 is written as the sum $4 \cdot 6 + 4 \cdot 9$, but could just as easily be written using some different combination of atoms, such as $3 \cdot 20$. This exhibits an important property of numerical semigroups, namely that they include many elements with *non-unique factorizations.* In the context of numerical semigroups, a factorization is a particular additive combinations of generators which sum to an element. There are many ways of writing out the various factorizations of a semigroup element, but unless otherwise specified we will employ the following notation:

Definition. Let $S = \langle a_1, a_2, ..., a_k \rangle$ be a numerical semigroup. We say that $f = (f_1, f_2, ..., f_k)$ is a factorization of $x \in S$ if x can be written as $x = f_1 a_1 + f_2 a_2 + \cdots f_k a_k$, $f_1, f_2, \ldots, f_k \in \mathbb{Z}_{\geq 0}$. The set of all factorizations of x is denoted $Z(x)$.

Example 1.0.3. Once again letting 🐔 $= \langle 6, 9, 20 \rangle$, the factorizations for 60 are:

$$Z(60) = \{(10, 0, 0),\ (7, 2, 0),\ (4, 4, 0),\ (1, 6, 0),\ (0, 0, 3)\} \tag{2}$$

Factorizations are commonly characterized by their *length*, defined as follows:

Definition. Let $S = \langle a_1, a_2, \ldots, a_n \rangle$ be a numerical semigroup, and let $x \in S$ have a factorization $f = (f_1, f_2, \ldots, f_k) \in Z(x)$.

The *length* of f is $|f|_1 = f_1 + f_2 + \cdots f_k$.

The *length set* of x is $\mathscr{L}_1(x) = \{|z|_1 : z \in Z(x)\}$.

This definition of length and length sets is used to study factorization lengths in numerical semigroups ([20][21][19][22][9][23][15][28][28][27]). Length sets are of particular use in studying the trade structure of semigroups ([32][34][29][17]), local and global elasticity of semigroups ([3][5]), and delta sets ([7][25][24][10][11][14]), which we will discuss shortly. We specify in the above definition that the 1-norm is traditionally used for these length computations. Our research introduces a modification to this concept of length, namely the usage of t-norms to compute t-lengths:

Definition. Let $S - \langle a_1, a_2, \ldots, a_n \rangle$ be a numerical semigroup, and let $x \in S$ have a factorization $f = (f_1, f_2, \ldots, f_k) \in Z(x)$.

The *t-length* of f is $|f|_t = \begin{cases} f_1^t + f_2^t + \cdots + f_k^t & 0 \leq t \leq 1 \\ (f_1^t + f_2^t + \cdots + f_k^t)^{1/t} & t > 1 \\ \max\{f_1, f_2, \ldots, f_t\} & t = \infty \end{cases}$

The *t-length set* of x is $\mathscr{L}_t(x) = \{|z|_t : z \in Z(x)\}$

Note that in the case for $t = \infty$, $|f|_t = \max\{f_1, f_2, \ldots, f_k\} = \lim_{t\to\infty}(f_1^t + f_2^t + \cdots + f_k^t)^{1/t}$, making this definition a natural extension of the norms defined for finite t. It may also be helpful to note that for $t = 0$, the length of a factorization is simply the count of non-zero elements in that factorization.

As previously mentioned, one important topic of study relating to length sets is the Delta Set, which has previously been studied using the 1-norm definition of length but which we define for arbitrary t-length:

Definition. Let $S = \langle a_1, a_2, ..., a_k \rangle$ be a numerical semigroup. The $t-$*delta set* of $x \in S$, denoted as $\Delta_t(x)$, is the set of differences between consecutive elements in $\mathscr{L}_t(x)$ when the lengths are in ascending order. The $t-$delta set of S is $\Delta_t(S) = \bigcup\limits_{x \in S} \Delta_t(x)$.

Example 1.0.4. Consider $60 \in$. Recall from Example 1.0.3 that

$$Z(60) = \{(10,0,0),\ (7,2,0),\ (4,4,0),\ (1,6,0),\ (0,0,3)\}.$$

It is simple to compute

- $\mathscr{L}_1(60) = \{3, 7, 8, 9, 10\}$; $\Delta_1(60) = \{4, 1\}$
- $\mathscr{L}_0(60) = \{1, 2\}$; $\Delta_0(60) = \{1\}$
- $\mathscr{L}_\infty 60 = \{3, 4, 6, 7, 10\}$; $\Delta_\infty(60) = \{1, 2, 3\}$

The delta set of numerical semigroups has been studied in great depth, yielding many important results and many unanswered questions. The following are some notable properties of 1-delta set:

1. For a numerical semigroup S, $\min \Delta_1(S) = \gcd \Delta_1(S)$ ([26], Proposition 1.4.4)
2. 1-delta sets of elements in a semigroup are eventually periodic ([14])

Furthermore, the 1-delta set for certain classes of numerical semigroups is explicitly known, and the 1-delta set for many other numerical semigroups is easily computed ([16][24]). However, it is still unknown whether any set T satisfying $\min(T) = \gcd(T)$ is realized as the delta set of some numerical semigroup.

In the sections that follow we investigate the t-delta sets of numerical semigroups, specifically with $t = 0$ and $t = \infty$. We prove that the above properties of the 1-delta set hold for both $t = 0$ and $t = \infty$. We also prove explicit results regarding the content of t-delta sets for certain classes of numerical semigroups using $t = 0$ and $t = \infty$. We introduce a new invariant for numerical semigroups, Minimal Trade Support, defining both it and catenary degree for various t and relating these both to t-delta sets. Finally, we briefly discuss the structure of t-delta sets for $t \neq 0, \infty$ in connection to the preceding results.

2 Δ_0

2.1 Initial results

Since the 0-length of a factorization counts the number of distinct atoms contained in that factorization, there is a natural limit on the 0-length for factorizations of elements of a finitely generated semigroup. The following lemma formalizes this:

Lemma 2.1.1. Let $S = \langle a_1, \ldots, a_k \rangle$ be a finitely-generated semigroup. For all $x \in S$, $\mathscr{L}_0(x) \subseteq \{1, ..., k\}$; $\Delta_0(x) \subseteq \{1, ..., k-1\}$, and thus $\Delta_0(S) \subseteq \{1, ..., k-1\}$.

The other extremum is not only bounded, but known:

Theorem 2.1.2. For any numerical semigroup S, $\min(\Delta_0(S)) = \gcd(\Delta_0(S)) = 1$.

Proof. Let $x \in S$ have nonunique, non-0-half-factorial factorization, and let A_1, A_2 be the sets of atoms present in two factorizations of x of different 0-lengths. WLOG let $|A_1| > |A_2|$. Then $|A_1/A_2| \geq |A_1| - |A_2| - 1$, so we may pick $|A_1| - |A_2| - 1$ atoms in $|A_1/A_2|$. Adding them to x does not change the atom set of the first factorization, but adds $|A_1| - |A_2| - 1$ new atoms to the second factorization, resulting in two factorizations of 0-length $|A_1|$, $|A_1| - 1$. QEΔ

Remark 1. An atomic semigroup S is called *half-factorial* if, for all $x \in S$, $|\mathscr{L}_1(x)| = 1$. This concept is easily extended to the t-length of an element, with S being *t-half-factorial* if, for all $x \in S$, $|\mathscr{L}_t(x)| = 1$. Theorem 2.1.2 in fact holds for any atomic commutative semigroup which is *not* 0-half-factorial; that is, any semigroup for which elements exist whose factorizations have different 0-lengths. In subsequent remarks, we will refer to such semigroups as non-0-half-factorial.

In the two-generated case, these bounds meet.

Corollary 2.1.3. If $S = \langle a_1, a_2 \rangle$ is a 2-generated semigroup, then $\Delta_0(S) = \{1\}$.

In addition, the limiting behavior of $\Delta_0(x)$ within any particular numerical semigroup is known.

Theorem 2.1.4. For all numerical semigroups S, there exists N such that if $n > N$, $\Delta_0(n) = \{1\}$.

Proof. Let $S := \langle a_1, ..., a_k \rangle$ and $l := \max(\{\mathrm{lcm}(a_i, a_j) : i, j \in [1, k]\})$. Suppose $n > kl$. We will prove by induction that $\mathscr{L}_0(n) = [l_0(n), k]$.

n, by definition, has a factorization of length $l_0(n)$.

Suppose $f = (f_1, ..., f_k)$ has $|f|_0 < k$, i.e., that $f_j = 0$ for some $j \in \{1, ..., k\}$. If $f_i a_i \leq l$ for all $i \in \{1, ..., k\}$, $n = \sum_{i=1}^{n} f_i a_i \leq kl$, a contradiction. So $l' := \mathrm{lcm}(a_i, a_j) \leq l < f_i a_i$ for some i, meaning $f' = (f_1, ...f_i - \frac{l'}{a_i}, ..., f_j + \frac{l'}{a_j} ..., a_n)$ is a factorization with $|f'|_0 = |f|_0 + 1$.

QEΔ

These results together prove that the same notable properties of the 1-delta set which were previously mentioned hold in the 0-delta case, as well.

2.2 Classification of Δ_0 of 3-generated Numerical Semigroups

The result on the Δ_0 set of 2-generated semigroups naturally inspires inquiry into the 3-generated case. Lemmas 2.1.1 and 2.1.2 imply that such a semigroup has a Δ_0 set of either $\{1\}$ or $\{1, 2\}$. The following section classifies numerical semigroups into these categories.

Before presenting the classification, some preliminaries must be established.

Definition. A numerical semigroup S is said to be a *gluing* if there exist semigroups $S_1 = \langle c_1, ..., c_k\rangle, S_2 = \langle b_1, ..., b_m\rangle$ such that for some $a_1 \in S_1$, $a_2 \in S_2$ we have that $S = a_1S_2 + a_2S_1 = \langle a_1b_1, ..., a_1b_m, a_2c_1, ..., a_2c_k\rangle$.

In particular, S is a monoscopic gluing if there exists S_1 such that $S = a_1S_1 + a_2\langle 1\rangle$ for some $a_1 \in \mathbb{Z}^+$, $a_2 \in S_1$.

Example 2.2.1. The semigroup 🐔 $= \langle 6, 9, 20\rangle$ is a gluing; in particular, it can be constructed using multiple different monoscopic gluings. For instance,

$$20 \in \langle 2, 3\rangle, \text{ so } 🐔 = 3\langle 2, 3\rangle + 20\langle 1\rangle$$

$$9 \in \langle 3, 10\rangle, \text{ so } 🐔 = 2\langle 3, 10\rangle + 9\langle 1\rangle$$

Whether or not a semigroup is a gluing reveals a great deal about its structural properties; in particular, the *minimal presentation* of a gluings is conveniently inherited from its component semigroups. Before beginning work with minimal presentations, it is necessary to first define *trades* between elements of factorizations.

Definition. Let $S = \langle a_1, a_2, \ldots, a_k\rangle$ be a numerical semigroup, and let $x \in S$ be an element with at least two distinct factorizations, $z, w \in Z(x)$. We say that

$$(z_1, z_2, \ldots, z_k) \sim (w_1, w_2, \ldots, w_k)$$

is a *trade* in S.

A trade is said to be *minimal* if $\gcd\{z, w\} = 0$; that is, z and w share no common support. Accordingly, we may define, for each element a factorization, or ∇-*graph* whose vertices are the factorizations of the element and whose edges exist between and only between the factorizations with nondisjoint support (nonzero gcd). A *Betti element* is an element with multiple components in its ∇-graph.

The *minimal presentation* of S is a set of minimal trades

$$T = \{t_1 \sim t_2,\ t_3 \sim t_4, \ldots\}$$

that is sufficient to span all factorizations of all elements of S and that contains no redundant trades. Minimal presentations may be found by bridging all components of the ∇-graphs of all Betti Elements.

Example 2.2.2. Consider again the semigroup 🐔 $= \langle 6, 9, 20\rangle$, with $60 \in$ 🐔. Recall from Example 1.0.3 that $Z(60) = \{(10, 0, 0),\ (7, 2, 0),\ (4, 4, 0),\ (1, 6, 0),\ (0, 0, 3)\}$. So, $(10, 0, 0) \sim (7, 2, 0)$ is a trade in 🐔, exchanging $10\cdot 6$ to be represented as $7\cdot 6+2\cdot 9$. However, note that $\gcd\{(10, 0, 0),\ (7, 2, 0)\} = (7, 0, 0)$, so this trade is not minimal. Removing this common divisor, we can instead apply the minimal trade $(3, 0, 0) \sim (0, 2, 0)$ to connect the factorizations $(10, 0, 0)$ and $(7, 2, 0)$.

From $Z(60)$ we also see $(0, 0, 3)$ shares support with no other factorizations of 60. This means that 60 is a Betti element of 🐔. Therefore, any trade between $(0, 0, 3)$ and some other factorization, such as $(10, 0, 0)$, is a minimal trade. Together with $(3, 0, 0) \sim (0, 2, 0)$, this trade is sufficient to span all factorizations of any $x \in$ 🐔. So $T = \{(3, 0, 0) \sim (0, 2, 0),\ (10, 0, 0) \sim (0, 0, 3)\}$ is a minimal presentation for 🐔.

Note that instead of $(10, 0, 0) \sim (0, 0, 3)$, we could have instead selected the trade $(7, 2, 0) \sim (0, 0, 3)$, which is also minimal. This still produces a minimal presentation of 🐔, $T = \{(3, 0, 0) \sim (0, 2, 0),\ (7, 2, 0) \sim (0, 0, 3)\}$.

This example demonstrates that minimal presentations of semigroups are not unique; however, by definition, the same factorizations of elements are obtainable using any minimal presentation. The relationship between factorizations, Betti elements, and minimal presentations is explored further in the section on Catenary Degree and Minimal Trade Support. For now, these descriptions provide the necessary background to understand the following results for gluings and t-delta sets of three generated semigroups.

Theorem 2.2.3. If $S = \langle a_1, a_2, a_3\rangle$ is a 3-generated numerical semigroup with $\Delta_0(S) = \{1, 2\}$, then S has at most 1 gluing.

Proof. Because $2 \in \Delta_0(S)$, there exists $s \in S$ such that, up to relabeling, S has a factorization $f_1 = (a_1, 0, 0)$ with $|f_1|_0 = 1$ and a factorization $f_2 = (b_1, b_2, b_3)$ with $|f_2|_0 = 3$, but no factorization with exactly two nonzero entries. Equivalently, it must be possible, with every minimal presentation of S, to move from $(a_1, 0, 0)$ to (b_1, b_2, b_3) without landing on a factorization with exactly 2 nonzero entries.

Suppose for contradiction that S has two gluings. Thus, there is, up to relabeling n_2 and n_3, a minimal presentation of S consisting of $T_1 : (x_1, 0, 0) \sim (0, x_2, 0)$ and $T_2 : (y_1, y_2, 0) \sim (0, 0, y_3)$. Thus, the trade $(a_1, 0, 0) \sim (b_1, b_2, b_3)$ must be able to be constructed from T_1 and T_2 without landing on a factorization with exactly 2 nonzero entries. More specifically, either T_1 or T_2 must be able to be applied to $(a_1, 0, 0)$ without resulting in a factorization of length 2. Clearly, T_2 is not able to be applied to $(a_1, 0, 0)$. If $x_1 < a_1$, T_1 results in a factorizaton of length 2, so is unable to be applied. If $x_1 = a_1$, we have $(a_1, 0, 0) \sim (0, x_2, 0)$, which cannot progress further. If $x_1 > a_1$, T_1 cannot be applied. Therefore, it is impossible to construct $(a_1, 0, 0) \sim (b_1, b_2, b_3)$ from T_1 and T_2, contradicting the fact that T_1 and T_2 form a minimal presentation of S. We conclude S has at most 1 gluing.

QEΔ

This implies that if S has more than 1 gluing, $\Delta_0(S) \neq \{1, 2\}$; accordingly, $\Delta_0(S)$ must equal $\{1\}$. The converse of this theorem is true as well.

Theorem 2.2.4. If $S = \langle n_1, n_2, n_3 \rangle$ is a 3-generated numerical semigroup with $\Delta_0(S) = \{1\}$, then S has (at least) two gluings.

Proof. S has a minimal presentation P containing a "norm 3" trade of the form $(x_1, x_2, 0) \sim (0, 0, x_3)$, up to reordering of generators. Let P be such a presentation with the fewest possible norm-3-trades. If S has fewer than two gluings, P still contains at least one norm-3-trade. Let $T_1 := (x_1, x_2, 0) \sim (0, 0, x_3)$ be such a trade among those norm-3-trades with the smallest possible associated Betti element. Now consider the element with factorizations $v_2 := (x_1, x_2, 1) \sim (0, 0, x_3 + 1) := v_1$. By the delta set assumption, $v_1 \sim v_2 \sim f$ for some factorization f with $|f|_0 = 2$. Since T_1 only changes v_1 into v_2, we must reach f by first applying some other trade.

If S has no gluings, P contains two other trades of the forms $(0, y_2, 0) \sim (y_1, 0, y_3)$ and $(z_1, 0, 0) \sim (0, z_2, z_3)$. Neither of these may apply to v_1, so they must be applied to v_2. WLOG we may suppose a trade of the first form applies. So we may subtract either the left or right side of that trade from v_2 to get a legal factorization of an element of S. If $(x_1, x_2, 1) - (0, y_2, 0)$ is a legal factorization, then $(x_1, x_2, 0) - (0, y_2, 0)$ is as well, violating minimality of T. If $(x_1, x_2, 1) - (y_1, 0, y_3) = (x_1 - y_1, x_2, 1 - y_3)$ is a legal factorization, then $x_1 \geq y_1$ and $y_3 = 1$. Furthermore, to preserve minimality, we must have $(y_1, 0, 1) \cdot (n_1, n_2, n_3) = (0, y_2, 0) \cdot (n_1, n_2, n_3) \geq (x_1, x_2, 0) \cdot (n_1, n_2, n_3) = (0, 0, x_3) \cdot (n_1, n_2, n_3)$. This gives us $n_3 + y_1 n_1 \geq x_1 n_1 + x_2 n_2$, or $n_3 \geq (x_1 - y_1) n_1 + x_2 n_2 \geq x_2 n_2$, and $x_2 n_2 \geq y_2 n_2 \geq n_3 x_3$. Putting these together gives $n_3 \geq x_2 n_2 n_3 x_3$, which implies that $x_3 = 1$ and that $n_3 \geq x_2 n_2 \geq n_3$, or $n_3 = x_2 n_2$. Then n_3 is a multiple of n_2, a contradiction.

If S has exactly one gluing, P contains one other trade of the form $T_2 := (y_1, 0, 0) \sim (0, y_2, 0)$. Once again, we must reach f by first applying this trade to v_2. Doing so any number of times will either lower the first or second coordinate below x_1 or x_2, making application of T_1 impossible. So f must be obtained purely through application of T_2 to v_2, meaning, because T_2 cannot modify the third coordinate, that $f = (0, k, 1)$ or $(k, 0, 1)$ for some k. This implies that $(x_1, x_2, 1) \sim (0, k, 1)$ or $(k, 0, 1)$, or that $(x_1, x_2, 0) \sim (0, k, 0)$ or $(k, 0, 0)$. This allows us to replace T_1 with a trade not of norm 3, violating minimality of P.

QEΔ

With this classification completed, we can explicitly describe the 0-delta set of 🐔. Recall from Example 2.2.1 that 🐔 has two gluings. Therefore, by the contrapositive of Theorem 2.2.3, $\Delta_0(🐔) = \{1\}$.

The next section introduces the catenary degree and minimal trade support invariants for semigroups their elements, and applies these invariants to prove more results involving gluings and other families of semigroups.

2.3 Catenary Degree, Minimal Trade Support, and Gluings

The preceding results indicate that properties of trade structure deserve more scrutiny. In particular, the gap between norms of factorizations relies heavily on the norms of the two vectors on either side of the minimal trade bridging the two factorizations. This implies that minimal presentations may simplify the computation of 0-delta sets of semigroups by providing semigroup-wide information about all elements in a semigroup, greatly simplifying the task of examining the 0-delta set of each one.

The following theorem follows through on those expectations by defining a semigroup invariant based on minimal presentations and relating it to the Δ set.

Theorem 2.3.1. Let $p \in [0, \infty]$ be a norm parameter, S be a semigroup, and $\mathcal{T}$ (also denoted $\mathcal{T}(S)$) be the set of minimal presentations of S. For $T = \{t_1 \sim t_2;\ t_3 \sim t_4;\ \cdots;\ t_{2x} \sim t_{2x+1}\} \in \mathcal{T}$, let the p-weight of a trade $t_i \sim t_{i+1}$ be $\max(|t_i|_t, |t_{i+1}|_0)$ and let $M_p(T) := \max\{|t_i|_0 : i \in [1, 2x+1]\}$ (also referred to as *minimal trade support*). Then $\max(\Delta_p(S)) \leq M_p(S) := \min\{M_p(T) : T \in \mathcal{T}\}$.

Proof. Let $T \in \mathcal{T}$, $x \in S$, and f, f' be two factorization vectors of x related by the trade $t_{2i} \sim t_{2i+1}$ such that $f - t_{2i} + t_{2i+1} = f'$. WLOG let $|f|_p \leq |f'|_p$. Then because t_{2i}, t_{2i+1} have nonnegative coordinates, $|f'|_p = |f - t_{2i} + t_{2i+1}|_p \leq |f + t_{2i+1}|_p$, meaning by the triangle inequality property of norms, $|f'|_p \leq |f|_p + |t_{2i+1}|_p$, or $|f'|_p - |f|_p \leq |t_{2i+1}|_p \leq M_p(T)$. Thus, two factorization vectors of x related by a trade in T differ in p-norm by at most $M_p(T)$.

Now suppose $d > M_p(T)$ and $d \in \Delta_p(x)$. There then exists x with factorizations f, f' of x such that $|f'|_0 - |f|_0 = d$ and with no factorizations h with $|h|_0 \in (|f|_0, |f'|_0)$. Since T is spanning, there is a sequence of factorizations of x, $f, f_1, ..., f_k, f'$, such that each successive pair of factorizations is related by a trade in T. Let x be the first factorization with 0-norm at least $|f'|_p$ and y be the immediate predecessor of x in the sequence. Since $|x|_p \geq |f'|_p = |f| + d \geq |y|_p$, $|x|_p - |y|_p \geq |f'|_p - |f|_p = d$. But since x is related to y by a trade in T, $||x|_p - |y|_p| \leq M(T) < d$, a contradiction.

So $\max(\Delta_p(x)) \leq M_p(T)$ for all $x \in S$, $T \in \mathcal{T}$, meaning $\max(\Delta_0(S)) \leq M_p(S)$.

QEΔ

Example 2.3.2. The has $M_0() = 1$ and $M_\infty() = 4$, as witnessed by the minimal presentations

$$(4, 4, 0) \sim (0, 0, 3);\ (3, 0, 0) \sim (0, 2, 0)$$

and

$$(10, 0, 2) \sim (0, 0, 3);\ (3, 0, 0) \sim (0, 2, 0)$$

.

This definition allows us to revisit the classification of 3-generated numerical semigroups from a new angle. Inherent to the classification of Δ_0 of 3-generated numerical semigroups is the relationship between trade structure and gluings. The following result, presented as a separate theorem, illustrates that the core mechanic behind the Δ_0 classification is trade structure, for which gluings are simply a proxy.

Theorem 2.3.3. All 3-generated numerical semigroups S satisfy $\max(\Delta_0(S)) = M(S)$.

Proof. By 2.3.14, $M(S) \leq 2$.

If $M(S) = 1$, then by 2.3.1, $\max(\Delta_0(S)) \leq 1$, so $\max(\Delta_0(S)) = 1$ as well.

If $M(S) = 2$, then all minimal presentations of S have a norm-3-trade of the form described in the proof of 2.2.4. Said proof showed that this condition and $\Delta_0(S) = \{1\}$ are contradictory, so $\Delta_0(S) = \{1, 2\}$, and thus $\max(\Delta_0(S)) = 2 = M(S)$.

QEΔ

In fact, all numerical semigroups built from gluings have finely determined trade structure, not just 3-generated ones. However, 3-generated numerical semigroups with more than two gluings can be expressed as monoscopic gluings, which have especially tractable properties and trades. This series of theorems illustrates some of the general properties of such gluings:

Theorem 2.3.4. Let $S' = xS + y\langle 1\rangle$ be a monoscopic gluing. Let $m \in S$ be the unique element such that, for all $n \in S$ with $n \leq m$, $\cup\Delta_0(n) = \Delta_0(S)$ and $(\cup\Delta_0(n)) - \{\Delta_0(m)\} \neq \Delta_0(S)$. If $y > m$, then $\Delta_0(S) \subseteq \Delta_0(S')$.

Proof. There exists a minimal presentation of S' of the form $P \cup \{t\}$ where P is a minimal presentation of S and t is the new trade occuring at xy. Because $xn < xy$, t is not a trade that can be used at xn. Thus, since each factorization of n in S is a factorization of xn in S', n and xn have the same factorizations in S and S' respectively. Therefore, $\cup\Delta_0(n) = \cup\Delta_0(xn)$, where $\cup\Delta_0(xn)$ considers factorizations with respect to S'. Because $\cup\Delta_0(n) = \Delta_0(S)$ and $\cup\Delta_0(xn) \subseteq \Delta_0(S')$, $\Delta_0(S) \subseteq \Delta_0(S')$ QEΔ

There is no characterization of the Δ_0 set of the gluing of two semigroups based on their respective Δ_0 sets. However, it is understood how monoscopic gluings change the Δ_0 set of semigroups based on various properties of the semigroups to which the gluings are applied.

Theorem 2.3.5. For a semigroup $S' = \langle n_1, n_2, ..., n_k\rangle$ with $\Delta_0(S') = \{1\}$, a monoscopic gluing $S = dS' + a\langle 1\rangle$ has $\Delta_0(S) = \{1\}$ if and only if a is the multiple of any single generator.

Proof. Suppose $\Delta_0(S) = \{1\}$. Now suppose that a is not the multiple of any single generator. Let the smallest number of nonzero generators that a can be written as a linear combination of be q generators. Then, we have the following new trade as a result of this gluing, $de_{k+1} \sim (0, ..., b_1, ..., b_2, ..., b_q, ..., 0)$. Consider $(d+1)e_{k+1} \sim (0, ..., b_1, ..., b_2, ..., b_k, ..., 1)$, no more trades can be applied that involve the $k+1$th position. So, if we can apply a trade to decrease the length of this factorization, then we can write a as a linear combination of strictly fewer than q generators which is a contradiction.

Suppose a is the multiple of some generator. Since $\Delta_0(S') = \{1\}$, it follows that for all $x \in S'$ we have that $\Delta_0(x) \subseteq \{1\}$. So, we can get between all factorizations of x by increasing or decreasing the length by 1 or 0. Gluing an element to S' that is the multiple of a single generator n_i, say $a = m \cdot n_i$, gives us a new trade $de_{k+1} \sim me_i$. Applying this new trade can change the length of a factorization by 1 or 0. And so, since we can get between all factorizations of any give element in S' by changing the length by 1 or 0 as well, it follows that we have a presentation of trades of S such that we can get between all the factorizations of any element in S by changing the length by 1 or 0. Thus, we have that $\Delta_0(S) = \{1\}$. QEΔ

Example 2.3.6. Consider 🐔 $= \langle 6, 9, 20\rangle = 3\langle 2, 3\rangle + 2\langle 10\rangle$, we have a minimal presentation of trades $(3, 0, 0) \sim (0, 2, 0)$ and $(0, 0, 3) \sim (10, 0, 0)$ and so applying each of these trades changes the factorization length by 1 or 0. If we pick an arbitrary element in 🐔, say 45, we can get between the factorizations of 45, which are $(6, 1, 0), (3, 3, 0)$, and $(0, 5, 0)$, using the trades in this presentation. If we apply the trade $(3, 0, 0) \sim (0, 2, 0)$ to $(6, 1, 0)$ we get $(3, 3, 0)$, and if we apply $(3, 0, 0) \sim (0, 2, 0)$ to $(3, 3, 0)$ we get $(0, 5, 0)$.

The following corollary utilizes the fact that each gluing requires exactly one new trade to be introduced between the generators belonging to each component semigroup. In particular, for monoscopic gluings this means we have one new trade between the newly introduced generator and all other generators. So, for a semigroup constructed solely from monoscopic gluings where the newest element is a multiple of a single generator, there is a presentation of trades where every trade occurs between exactly two elements (in particular, we are trading x copies of generator a_i for y copies of generator a_j).

Corollary 2.3.7. For a semigroup, S, constructed by successive monoscopic gluings where the element being glued on is the multiple of any single generator has $M_0(S) = 1$.

Proof. Since S has a minimal presentation of trades, T, where each trade occurs between exactly two elements, we have $M_0(T) = 1$. We know there does not exist T' such that $M_0(T') = 0$ since we must have trades in S. Thus, $M_0(S) = 1$. QEΔ

The following theorem illustrates that once we have a more complicated Δ_0 set, in that something other than 1 is in it, we cannot monoscopically glue on an element in order to get the simplest Δ_0 set of $\{1\}$.

Theorem 2.3.8. If $q \in \Delta_0(S')$,where $q \in \mathbb{Z}^+ \setminus \{1\}$, then $\Delta_0(S) \neq \{1\}$ for any S constructed by a monoscopic gluing to S'.

Proof. Let S' be a semigroup such that $\Delta_0(S') \neq \{1\}$. Suppose we can glue an element to S' to get a semigroup with Δ_0 set $\{1\}$. Then it must have an element with something, say q, other than 1 in its Δ_0 set. Suppose we can glue an element onto S' to get a semigroup, $S = dS' + \langle a\rangle$, with $\Delta_0(S) = \{1\}$.

Suppose that $q = 2$, then we have an element, say x, in S with $2 \in \Delta_0(x)$. So, we have at least one gap of 2 in the length set of x, so x has factorizations of length $\ell, \ell + 2$, but not $\ell + 1$. To get a factorization of length $\ell + 1$ from the factorization of length $\ell + 2$ we must add a new trade that takes two nonzero positions, say b_i, b_j to the new position, those two positions would need to become zero simultaneously. So, the trade we would need to apply is $(0, ..., c_i, ..., c_j, ...0) \sim (0, ..., 0, d)$, since b_i, b_j must become 0 simultaneously, $tc_i = b_i, tc_j = b_j$ for some $t \in \mathbb{Z}^+$. Consider the factorization $(0, ..., 0, d + 1) \sim (0, ..., c_i, ..., c_j, ...0, 1)$, there are clearly no more trades involving the last position so if we have a trade taking this factorization to length 2, then it must take c_i, c_j to the same position using a trade that was available in the preceding semigroup. Since $c_i = tb_i, c_j = tb_j$ for some $t \in \mathbb{Z}^+$, this would mean that there was a trade taking b_i and b_j to the same place if applied t times, which would contradict our assumption unless $t = 1$ and we have a trade simultaneously taking b_i and b_j to a nonzero position, in which case we will get that the element corresponding to the factorization $(0, 0, ..., 0, d+1) + e_k$ has 2 in its Δ_0 set where k is the position that b_i and b_j are taken by some trade in the preceding semigroup.

If we have that $q > 2$, then the new trade must take multiple positions to 0 during different applications of the trade in order to get all the factorization lengths needed to ensure nothing other than 1 is in the Δ_0 set of x, which is impossible. QEΔ

Example 2.3.9. Consider $S = 3\langle 2, 3\rangle + 11\langle 1\rangle = \langle 6, 9, 11\rangle$. We have a minimal presentation of trades $(3, 0, 0) \sim (0, 2, 0)$ and $(0, 0, 3) \sim (1, 3, 0)$. Since this semigroup has 3 generators, to get a gap of 2 an element must have factorization(s) of length 1 and 3, and no factorizations of length 2. Consider the factorizations of the element corresponding to $(0, 0, 4)$, which is 44; its factorizations are $(0, 0, 4) \sim (1, 3, 1) \sim (4, 1, 1)$ so $2 \in \Delta_0(44)$. In $S' = 2\langle 6, 9, 11\rangle + \langle 33\rangle = \langle 12, 18, 22, 33\rangle$ we have a new trade in the minimal presentation, $(0, 0, 0, 2) \sim (4, 1, 0, 0)$. Now we have that $(0, 0, 4, 0) \sim (0, 0, 1, 2)$ and so the element corresponding to $(0, 0, 4, 0)$ has Δ_0 set $\{1\}$. But, the element corresponding to $(0, 0, 1, 3)$, which is 121, has factorizations $(0, 0, 1, 3) \sim (4, 1, 1, 1) \sim (1, 3, 1, 1) \sim (0, 0, 4, 1)$, and so $2 \in \Delta_0(121)$.

Theorem 2.3.10. Let S be a semigroup constructed by successive monoscopic gluings, then $\Delta_0(S) = \{1\}$ if and only if $M_0(S) = 1$.

Proof. Suppose $M_0(S) = 1$. It is immediate that $\Delta_0(S) = \{1\}$ since $M_0(S)$ is an upper bound for $\max \Delta_0(S)$.
Suppose $\Delta_0(S) = \{1\}$. Suppose at least one of the elements glued on to construct S is not a multiple of any single generator. Then, from Theorem 2.3.5, it follows that if a semigroup, S, constructed by successive monoscopic gluings has $m \in \Delta_0(S)$ where $m \neq 1$, then at some point an element that was not a multiple of any single generator was glued on. Consider the first such gluing where the element being glued on is not the multiple of any single generator, $dS'' + \langle a\rangle$. Let the smallest number of nonzero generators that a can be written as a linear combination of be k generators. Then, we have the following new trade as a result of this gluing, $de_{k+1} \sim (0, ..., b_1, ..., b_2, ..., b_k, ..., 0)$. Consider $(d+1)e_{k+1} \sim (0, ..., b_1, ..., b_2, ..., b_k, ..., 1)$, no more trades can be applied that involve the $k+1$th position. Since all preceding monoscopic gluings involved gluing on an element that was a multiple of any single generator, any trades we can apply to this factorization will occur between exactly two generators, so each trade will lead to the factorization length increasing by 1, decreasing by 1, or not changing. Suppose we can apply trades until we get to a factorization of length 2, then since the $k+1$th position is untouched, this means that $de_{k+1} \sim ce_j$ for some $j < k+1$, which means that a is the multiple of a single generator, a contradiction. Thus, it follows that $q \in \Delta_0(S)$ for some q such that $1 < q \leq k$. From Theorem 2.3.8, we know that we cannot glue on any element to get a semigroup with Δ_0 set $\{1\}$. And so, it follows from Theorem 2.3.5 that all the elements glued on to construct S must be a multiple of any single generator, and so we can construct a minimal presentation for the trades of S' such that all trades occur between exactly two elements, which means that $M_0(S) = 1$. QEΔ

This result confirms, for semigroups constructed by monoscopic gluings, the intuition that we cannot construct a semigroup with a simple Δ_0 set yet large catenary degree.

The following corollaries also illustrate the flexibility and power of the $M_p(S)$ result.

Corollary 2.3.11. All compound sequence numerical semigroups S have $\Delta_0(S) = \{1\}$.

Proof. $1 = \min \Delta_0(S)$, by 2.1.1. Because $M(S) = 1$ for all compound sequence numerical semigroups, $\max(\Delta_0(S)) = 1$. So $\Delta_0(S) = \{1\}$.

QEΔ

Incidentally, this shows that the bound given by Lemma 2.1.1 may be arbitrarily loose.

Corollorollary 2.3.12. For all $n \in \mathbb{N}$, there exists a numerical semigroup S with n generators and $\Delta_0(S) = \{1\}$.

Proof. Since we can have compound sequences with an arbitrary number of generators, let S be an n-generated numerical semigroup generated by a compound sequence. Then by by Corollary 2.3.11, $\Delta_0(S) = \{1\}$.

QEΔ

Remark 2. (Relating to Corollorollary 2.3.12). Since numerical semigroups generated by a compound sequence can be expressed as a sequence of gluings, this result demonstrates that it is possible to apply an arbitrary number of gluings to a semigroup while retaining the delta set of the original semigroup. As an example, consider the following (arbitrary) compound sequence and numerical semigroup:

$$\begin{aligned}
(a) &= a_1 a_2 \cdots a_k \quad \text{such that } a_i > 2 \\
(b) &= b_1 b_2 \cdots b_k \quad \text{such that } b_i > a_i,\ \gcd(b_i, a_i a_{i+1} \cdots a_k) = 1
\end{aligned}$$

$$\begin{aligned}
n_0 &= a_1 a_2 \cdots a_k \\
n_1 &= b_1 a_2 \cdots a_k \\
&\vdots \\
n_k &= b_1 b_2 \cdots b_k
\end{aligned}$$

$$\begin{aligned}
S_1 &= \langle a_1, b_1 \rangle \\
S_2 &= a_2 S_1 + b_1 b_2 = \langle a_1 a_2, b_1 a_2, b_1 b_2 \rangle \\
&\vdots \\
S_k &= a_k S_{k-1} + b_1 b_2 \cdots b_k = \langle a_1 a_2 \cdots a_k, b_1 a_2 \cdots a_k, \ldots, b_1 b_2 \cdots b_k \rangle = \langle n_0, n_1, \ldots, n_k \rangle
\end{aligned}$$

Since the final semigroup S_k is a compound sequence numerical semigroup, $\Delta_0(S_k) = \{1\}$, even after numerous applications of monoscopic gluings.

Maximal embedding dimension numerical semigroups are, in the technical sense, "almost all" numerical semigroups. They are numerical semigroups whose embedding dimension (number of generators) equals their smallest generator.

Corollary 2.3.13. All numerical semigroups S with maximal embedding dimension satisfy $\Delta_0(S) = \{1, 2\}$.

Proof. All numerical semigroups S with maximal embedding dimension d have a minimal presentation of the form $T = \{(f_1, 0, ..., f_i = 1, ..., f_j = 1, ..., 0) : i, j \in [2, d]\} \cup \{(f_1, 0..., f_i = 2, 0, ...) : i \in [2, d]\}$, with two types of trades. Because $M_0(S) \geq M_0(T) = 2$, $2 \geq \max \Delta_0(S)$, so $\Delta_0(S) = \{1\}$ or $\{1, 2\}$.

Now consider the element x with the factorization $f = (1, 1, ..., 1)$, which has length d. Applying a trade of the first type gives a factorization of length $d-2$. Suppose there were a factorization $f' = (f'_1, ..., 0, f'_{i+1}, ..., f'_d)$ of length $d-1$. Subtracting $gcd(f, f') = (1, 1, ..., 0, 1, ..., 1)$ gives $(f'_1 - 1, ..., 0, f'_{i+1} - 1, ..., f_d - 1) \sim (0, 0, ..., 1, ..., 0)$ as a valid trade. This trade implies that the ith generator is generated by the others, a contradiction. So $d, d-2 \in \mathscr{L}_0(x)$, while $d - 1 \notin \mathscr{L}_0(x)$. So $2 \in \Delta_0(x)$.

QEΔ

As in Lemma 2.1.1, a natural bound exists on the minimal trade support of a finitely generated semigroup.

Lemma 2.3.14. All semigroups $S = \langle a_1, ..., a_k \rangle$ satisfy $M_0(S) \leq k - 1$.

Proof. Let $S =< a_1, ..., a_k >= \hat{a}$ be a semigroup with minimal presentation T. By 2.1.1, $M_0(T) \leq k$. Consider the set W (possibly empty) of trades in T witnessing $M_0(T) = k$. These trades are of the form $\hat{x} = (x_1, ..., x_k) \sim (y_1, ..., y_k) = \hat{y}$; WLOG let $|y|_0 = k$. If $x_i < y_i$ for all $i \in [1, k]$, $\hat{x} \cdot \hat{a} < \hat{y} \cdot \hat{a}$, a contradiction. So some i must satisfy $x_i \geq y_i$, meaning we may replace t by $(x_1, ..., x_i - y_i, ..., x_k) \sim (y_1, ..., 0, ..., y_k)$ to get a new minimal presentation. Doing this for all trades in W produces a minimal presentation with $M_0(T) < k$.

Therefore, $M_0(S) \leq k - 1$.

QEΔ

This leads to a simple result that holds when bounds are tight.

Corollary 2.3.15. Given a semigroup $S = \langle a_1, ..., a_k \rangle$, if $\max(\Delta_0(S)) = k - 1$, then $M_0(S) = k - 1$.

Proof. By 2.3.1, $M_0(S) \geq k - 1$, and by 2.3.14, $M_0(S) \leq k - 1$, so $M(S) = k - 1$.

QEΔ

Another angle from which to analyze trade structure is the *catenary degree.*

Definition. Let x be an element of the semigroup S. The p-catenary degree $c_p(x)$ and p'-catenary degree $c'_p(x)$ are defined as follows:

1. Create a complete graph with vertex set $Z(x)$.
2. For each edge zz', set $z'' := gcd(z, z')$ and assign zz' the weight $max(|z|_p - |z''|_p, |z'|_p - |z''|_p)$ for $c_p(x)$ and $max(|z - z''|_p, |z' - z''|_p)$ for $c'_p(x)$.
3. Remove edges in decreasing order of weight until the graph is disconnected.

4. The weight of the last edge removed is $c_p(x)$ (or $c'_p(x)$).

The semigroup invariants $c_p(S)$, $c'_p(S)$, $C_p(S)$, $C'_p(S)$ are then defined as $sup(c_p(x) : x \in S)$, $sup(c'_p(x) : x \in S)$, $\{c_p(x) : x \in S\}$, $\{c'_p(x) : x \in S\}$, respectively.

Example 2.3.16. The factorization graph of $54 \in$ 🐦 is shown below, with greater and lesser ∞-catenary weights labeled. The element has greater and lesser catenary degree of 3, as shown by the highlighted edges.

$$\begin{tikzcd}
(9,0,0) \arrow[r, "1/1"] \arrow[d, "1/1"] \arrow[dr, "1/1"] & (0,6,0) \arrow[d, "1/1"] \\
(6,2,0) \arrow[r, "1/0"] \arrow[ur, "1/1"] & (3,4,0)
\end{tikzcd}$$

Figure 1: Factorization graph for $54 \in$ 🐦; vertices labeled with greater/lesser 0-catenary weights.

As both minimal trade support and the catenary degrees relate intimately to trade structure, one would expect them to behave similarly. The following result shows that they, in fact, are exactly equal across a semigroup. This is due largely to the existence of Betti elements, where minimal presentation trades occur between factorizations with no gcd.

Theorem 2.3.17. All numerical semigroups S satisfy $c_p(S) = M_p(S) = c'_p(S)$.

Proof. Let T be a minimal presentation of witnessing $M(S)$. For all $x \in S$, $z, z' \in S$, note that by the triangle inequality property of norms, the c_p edge weight of zz' is strictly less than the c'_p edge weight of zz'. Furthermore, the c'_p edge weight of zz' corresponds to the p-weight of the trade $z - gcd(z, z') \sim z' - gcd(z, z')$ connecting them. Since all factorization graphs in S may be connected solely using trades in T, all factorization graphs in S may be connected solely by edges of c'_p (and c_p) weight at most $M(T)$. So $c_p(x) \leq c'_p(x) \leq M(T) = M(S)$ for all $x \in S$, so $c_p(x) \leq c'_p(S) \leq M(S)$.

On the other hand, let b be a Betti element that witnesses $M_p(T)$. Since the Nabla graph of b has the same vertices as the c'_p graph of b, the edge removal of the c_p and c'_p graphs of b cannot remove enough edges to disconnect the Nabla graph. This means that the c_p and c'_p graphs of b must, throughout the removal process, retain an edge corresponding to a trade of weight at least $M(S)$. Since that edge $t_1 \sim t_2$ is also component-bridging edge in the Nabla graph, it also satisfies $gcd(t_1, t_2) = \hat{0}$. So c'_p, c_p, and trade weight all agree at that edge. So $c'_p(S), c_p(S) \geq c'_p(b), c_p(b) \geq M_p(T) = M_p(S)$.

QEΔ

Note that this does not show that $c_p(x) = c'_p(x)$ for all *elements* $x \in S$; it does not trivialize the distinction between catenary degrees.

The asymptotic behavior of the lesser 0-catenary degree mirrors that of the Δ_0 set.

Proposition 1. For all numerical semigroups $S = \langle a_1, ..., a_k \rangle$, there exists N such that if $n > N$, $c_0(n) = 1$.

Proof. Set $l := \max\{\text{lcm}(a_i, a_j) : i, j \in [1, k]\}$ and $N := kl$. Suppose $n > N$ and that $f = (f_1, ..., f_k)$ be a factorization of n.

If $|f|_0 < k$, then there exists i such that $f_i = 0$. Furthermore, there exists j such that $f_j a_j > l$. Then $l' := \text{lcm}(a_i, a_j) \leq l < f_j a_j$, meaning $f' := f - \frac{l'}{a_j} e_j + \frac{l'}{a_i} e_i$ is a valid factorization of n. Note that since $|\gcd(f, f')|_0 = |f|_0$ and $|f'|_0 = |f|_0 + 1$, the c_0 weight of ff' is 1.

Else, $|f|_0 = k$. Suppose f' is some other factorization of n with maximal 0-norm. Then $|\gcd(f, f')|_0 = k$ as well, so the c weight of ff' is 0.

Together, these links imply that any factorization has a 1-chain to a factorization of maximal length, all of which have 0-chains between them. So any two factorizations have 1-chains between them, so $c(n) = 1$.

QEΔ

Furthermore, various properties of norms, (notably Holder's inequality and the Triangle Inequality), allow for much structure in the relationship between greater and lesser catenary degrees derived from different norm parameters.

Theorem 2.3.18. For all elements x of numerical semigroups S and valid norm parameters p, q with $p < q$:

1. If $p, q \in [0, 1]$, $c_p(x) \leq c'_p(x)$. If $p, q \in [1, \infty]$, $c_q(x) \leq c'_p(x)$.
2. $c_p(x) \leq c_q(x)$; $c'_p(x) \leq c'_q(x)$.

Proof. Let f, f' be two factorizations of x with gcd f''. If $p, q \in [0, 1]$, the p-norm of a vector is at most the q-norm of a vector [prove this in a lemma?], $|f - f''|_p \leq |f - f''|_q$; $|f' - f''|_p \leq |f' - f''|_q$, and thus $\max(|f - f''|_p, |f' - f''|_p) \leq \max(|f - f''|_q, |f' - f''|_q)$. Similarly, $\max(|f|_p - |f''|_p, |f'|_p - |f''|_p) \leq \max(|f|_q - |f''|_q, |f'|_q - |f''|_q)$. If $p, q \in [1, \infty]$, the q-norm is at most the p-norm. So all inequalities in the preceding sentences are reversed.

Furthermore, by the triangle inequality property of norms, $|f|_p - |f''|_p \leq |f - f''|_p$ and $|f'|_p - |f''_p| \leq |f' - f''|_p$, so $\max(|f|_p - |f''|_p, |f'|_p - |f''|_p) \leq \max(|f - f''|_p, |f' - f''|_p)$.

This means that if $p, q \in [0, 1]$, any c_q-weighted N-chain is also a c_p-weighted N-chain, any c'_q-weighted N-chain is also a c'_p-weighted N-chain, and if $p, q \in [1, \infty]$, any c_p-weighted N-chain is also a c_q-weighted N-chain, and any c'_p-weighted N-chain is also a c'_q-weighted N-chain. Finally, for all p, any c'_p-weighted N-chain is also a c_p-weighted N-chain. QEΔ

The semigroup invariants, as expected, also obey these inequalities.

Corollary 2.3.19. For all numerical semigroups S and norm parameters p, q with $p < q$, if $p, q \in [0, 1]$, $c_p(S) = c'_p(S) \leq c_q(S) = c'_q(S)$, and if $p, q \in [1, \infty]$, $c_p(S) = c'_p(S) \geq c_q(S) = c'_q(S)$.

Proof. This follows from Theorem 2.3.18 and Theorem 2.3.17.

QEΔ

Finally, the p-delta set is also bounded by the catenary degree.

Theorem 2.3.20. For all semigroups S and valid norm parameters p, $\max(\Delta_p(x)) \le c_p(x) \le c'_p(x)$.

Proof. For any two factorizations f, f' with gcd f'' (WLOG let $|f|_p \ge |f'|_p$), the difference in p-length, $|f|_p - |f'|_p$, equals $|f|_p - |f''|_p - |f'|_p + |f''|_p \le |f|_p - |f''|_p$, the c_p weight of ff'. Therefore, since there exist $c_p(x)$-chains between any two factorizations of x, all factorizations of x are connected by sequences of factorizations whose consecutive entries differ in p-norm by at most $c_p(x)$.

QEΔ

Remark 3. *Contrasting* $(1-)$ *and* $(0-), (0'-)$ *Catenary Degree*

Existing literature contains many results on 1-catenary degree. The following remarks contrast them with their 0-catenary analogues.

Both inequalities in Theorem 2.3.18 may be tight. For example, in the the Alabio monoid $Ab_6 = \langle 32, 48, 56, 76, 94, 153\rangle$ (see Section 2.6), $c_0(459) = c'_0(459) = c_1(459) = 5$.

The inequality in Theorem 2.3.19 may be tight. Again, the Ab_6 is an example.

Also, the proof of Theorem 2.3.17 shows that, as with $(1-)$ catenary degree, the maximal values of $(0-)$ and $(0'-)$ catenary degree occur at a Betti element. However, unlike the $(1-)$ catenary degree, the minimal nonzero value of the $(0-)$ catenary degree does not always happen at a Betti element. In particular, the numerical semigroup $\langle 23, 28, 33, 38\rangle$ has $c_0(b) = c'_0(b) = 2$ at all of its Betti elements b.

Thirdly, while $c_1(x) \le 2$ implies $|\mathscr{L}_1(x)| = 1$, for every embedding dimension d, there are numerical semigroups with infinite elements satisfying $c_0(x) = 1$ and $\mathscr{L}_0(x) = [1, d]$. The proven bounds for Lemmas 1 and 2.1.4 are the same; all elements above those bounds witness this.

Fourthly, while every element x with multiple factorizations satisfies $2 + \sup(\Delta_1(x)) \le c_1(x)$, there are numerical semigroups with elements that have $\sup(\Delta_0(x)) = c_0(x) = c'_0(x)$. For example, in the same Alabio monoid, $Ab_6 = \langle 32, 48, 56, 76, 94, 153\rangle$, $c_0(459) = c'_0(459) = \sup(\Delta_0(459)) = 5$. However, it still holds that $\sup(\Delta_0(x)) \le c'_0(x), c_0(x)$ (see 2.3.20).

Finally, while every element x of a numerical semigroup with $c_1(x) \le 3$ has an $\mathscr{L}_1$ set that is an interval, the numerical semigroup $\langle 5, 6, 7\rangle$ has $c_0(30) = c'_0(30) = 2$, and $\mathscr{L}_0(30) = \{1, 3\}$. In general, the tightness of the inequality in the remark above leads to many examples of $c_0(x), c'_0(x) \in \{2, 3\}$ with noninterval $\mathscr{L}_0(x)$. However, since $c_0(x), c'_0(x) = 1$ implies $\max(\Delta_0(x)) \le 1$, this condition does guarantee that $\mathscr{L}_0(x)$ is an interval.

Remark 4. In [13] , it is shown that for all numerical semigroups $S = \langle a_1, ..., a_k \rangle$, if x is sufficiently large, $c_1(x) = c_1(x + L)$, where $L := \mathrm{lcm}(a_1, ..., a_k)$. The proof relies on the following maps from $Z(x)$ to $Z(x + L)$:

$$\phi_i : Z(x) \to Z(x + L); \phi(f) = f + (0, ..., \frac{L}{a_i}, ..., 0)$$

and the fact that they preserve 1-catenary-distance between factorizations. This is the only property of the 1-catenary degree used in the argument. The property holds because, for all factorizations $f, f' \in Z(x)$, $\gcd(\phi(f), \phi(f')) = \gcd(f, f') + (0, ..., \frac{L}{a_i}, ..., 0)$, so $\phi(f) - \gcd(\phi(f), \phi(f')) = f - \gcd(f, f')$ and $\phi(f') - \gcd(\phi(f), \phi(f')) = f' - \gcd(f, f')$. Since any further transformation on these identical vectors will yield identical results, any strong catenary distance, obtained by taking the t-norm of these vectors, is preserved, not just the 1-catenary distance obtained from the 1-norm.

However, the asymptotic behavior of the weak catenary degree is known only for the 0-norm; it remains open in general.

2.4 Arithmetic Semigroups

Definition. A semigroup $S = \langle a, ha + d, ha + 2d, ..., ha + xd \rangle$ where $x \leq a - 1$ and $\gcd(a, d) = 1$ is said to have its generators in *generalized arithmetic progression.*

In particular, when $h = 1$, the generators are said to be in *arithmetic progression.*

Example 2.4.1. $S = \langle 4,\ 12 + 3,\ 12 + 6,\ 12 + 9 \rangle = \langle 4, 15, 18, 21 \rangle$ has generators in generalized arithmetic progression, where $a = 4, h = 3, d = 3$.

The following results characterize the Δ_0 set of semigroups with generators in generalized arithmetic progression.

Theorem 2.4.2. For all numerical semigroups $S = \langle a, ha + d, ..., ha + xd \rangle$ with generators in generalized arithmetic progression, $\Delta_0(S) \subseteq \{1, 2\}$.

Proof. For some element $y \in S$, let the "partial support interval" $s(\hat{f})$ of a factorization $\hat{f} := (f_1, ..., f_n) \in Z_l(y)$ be the difference between indices of the largest and smallest nonzero entries of $(f_2, ..., f_n)$ (note the missing f_1). We will prove the following statement, which implies the theorem:

For all factorizations $\hat{f} \in Z(y)$, either $|\hat{f}|_0 \leq 3$, or there exists $\hat{f}'$ in $Z(y)$ such that $|\hat{f}'|_0 - |\hat{f}|_0 \in [-2, 2]$ and $s(\hat{f}') \leq s(\hat{f}) - 1$.

This suffices because it implies a sequence of factorizations in $Z(y)$ beginning with $\hat{f}$ that may only terminate at a factorization F with $|\hat{F}|_0 \leq 3$, or $|\hat{F}|_0 - l_0(y) \leq 2$. This sequence must terminate because the ith term has partial support interval at most $s(\hat{f}_0) - (i - 1)$, and partial support interval is bounded below by 0. Finally, all successive terms in this sequence differ in 0-norm by at most 2.

Proof of statement: Suppose $\hat{f} = \langle f_1, ..., f_n \rangle$ is a factorization of x with $|\hat{f}|_0 > 3$. Let i, j be the indices of the smallest and largest nonzero entries of $(f_2, ..., f_n)$. Because S has its later generators in arithmetic progression, $a_i + a_j = a_{i+1} + a_{j-1}$. Furthermore, because $|\hat{f}|_0 > 3, s(\hat{f}) > 2$, so $i+1 \neq j$ and $j-1 \neq i$. Setting $m := min(f_i, f_j)$, we may trade $\hat{f}$ for $\hat{f}' := (..., f_i - m, f_{i+1} + m, ..., f_{j-1} + m, f_j - m, ...)$ if $i+1 \neq j-1$ or $(..., f_i - m, f_{i+1} + 2m, f_j - m, ...)$ otherwise. Observe that $|\hat{f}'|_0 - |\hat{f}|_0 \in [-2, 2]$. Furthermore, either f_i or f_j has vanished, while no coordinates with indices outside of (i, j) increased. So $s(f') \leq s(f) - 1$, as needed.

QEΔ

Remark 5. This method of proof was devised before the minimal presentation of numerical semigroups with generators in generalized arithmetic progression was explicitly defined (see Section 5.1). This minimal presentation implies $M_0(S) = 2$, immediately proving the above result. We have elected to keep this proof because it better illustrates how the intricacies of the specific trade structure of semigroups with generators in arithmetic progression restrict the 0-delta set.

The previous theorem implies that the 0-delta set of an arithmetic progression numerical semigroup is $\{1\}$ or $\{1, 2\}$. The following result completes the characterization by ruling out the first possibility.

Theorem 2.4.3. Let $S = \langle a, ha+d, ha+2d, \dots, ha+xd\rangle$ be a numerical semigroup with generators in generalized arithmetic progression, and $x > 1$. Then $2 \in \Delta_0(S)$.

Proof. Let $S = \langle a, ha + d, ha + 2d, \dots, ha + xd\rangle$ be a numerical semigroup with generators in generalized arithmetic progression, and let $y = a + (ha + d) + (ha + 2d) \cdots + (ha + xd)$. Then $z = (h, 1, 1, \dots, 1) \in Z(y)$, and $|z|_0 = x + 1$. Furthermore, $z' = (0, 3, 0, \dots, 1) \in Z(y)$, with $|z'|_0 = x - 1$. We claim that there cannot exist a factorization of y with a 0-length of x.

Suppose for contradiction that there exists $z'' = (z''_1, z''_2, \dots, z''_{x+1}) \in Z(y)$ such that $|z''|_0 = x$. Then all but one coordinate of z'' is nonzero; let i be the coordinate such that $z_i = 0$. Then, subtracting $f = (h, 1, \dots, f_i = 0, \dots, 1)$ from both z'' and z, we have

$$\begin{aligned}(z''_1 - h, z''_2 - 1, \dots, z''_i = 0, \dots, z''_{x+1} - 1) \cdot (a, ha + d, \dots, ha + xd) \\ = (0, 0, \dots, z_i = 1, \dots, 0) \cdot (a, ha + d, \dots, ha + xd) \\ = (z''_1 - 1)a + (z''_2 - 1)(ha + d) + \cdots + (z''_{x+1} - 1)(ha + xd) \\ = \begin{cases} ha + (i-1)d & \text{if } i > 1 \\ a & \text{if } i = 1 \end{cases}\end{aligned}$$

so $ha + (i - 1)d$ is a linear combination of the other generators of S. If $i > 1$, this contradicts S being minimally generated. If $i = 1$, so that a is a linear combination of generators other than a, $a = n(ha) + m(d)$ for some $n, m \in \mathbb{N}$, a contradiction. Therefore $x \notin \mathscr{L}_0(y)$, but $x+1, x-1 \in \mathscr{L}_0(y)$, so $x + 1 - (x - 1) = 2 \in \Delta_0(y)$, and so $2 \in \Delta_0(S)$.

QEΔ

Corollary 2.4.4. For a semigroup S with generators in generalized arithmetic progression, $c_0(S) = 2$.

Proof. It follows from the minimal trade presentation, T, we have of generalized arithmetic sequences that $M_0(S) \leq 2$ since $M_0(T) = 2$. Suppose for contradiction that there exists a presentation of trades T' such that $M_0(T') = 1$, then there exists some composition of trades of support 1 such that we get $(0, 2, 0, ...0) \sim (h, 0, 1, 0, ..., 0)$, which would mean that $(0, 1, 0, ...) \sim (0, 0, 1, 0, ...0)$ and $(0, 1, 0, ...0) \sim (h, 0, ..., 0)$, which is many contradictions.

QEΔ

Corollary 2.4.4 and Theorem 2.4.3 together prove that semigroups with generators in generalized arithmetic progression satisfy $\Delta_0(S) = \{1, 2\}$.

2.5 Mallard Monoids

In this section we exhibit, for any embedding dimension, a tight example for the bound on $\max(\Delta_0(S))$ given by Lemma 2.1.1.

Theorem 2.5.1. For all natural numbers n, there exists an $(n+1)$-generated numerical semigroup S with $n \in \Delta_0(S)$.

Proof. Let p_n denotes the n^{th} prime, and define the sequences

$$a_k = p_{n-k},\ 0 \leq k \leq n-2 \quad \text{ and } \quad b_k = p_{n+1-k},\ 0 \leq k \leq n-2 \tag{3}$$

Note that $\forall\ 0 \leq i \leq n-2, b_i > a_i$, and $\gcd(b_i, a_i a_{i+1} \cdots p_2) = 1$ since b_i is the prime following a_i.

Let S be the numerical semigroup on the compound sequence generated by (a) and (b). That is,

$$\begin{aligned} s_0 &= a_0 a_1 \cdots a_{n-2} = p_n p_{n-1} \cdots p_2, \\ s_1 &= b_0 a_1 \cdots a_{n-2} = p_{n+1} p_{n-1} \cdots p_2, \\ &\vdots \\ s_{n-1} &= b_0 b_1 \cdots b_{n-2} = p_{n+1} p_n \cdots p_3 \end{aligned} \tag{4}$$

$$S = \langle s_0, s_1, \ldots, s_{n-1} \rangle \tag{5}$$

The Frobenius number of this semigroup is given by (adapted from Vadim's slides on numerical semigroups of compound sequences)

$$\begin{aligned} F(S) &= \sum_{j=1}^{n} s_j a_{j-1} - \sum_{j=0}^{n} s_j \\ &= (p_{n+1} p_{n-1} \cdots p_2 \cdot p_n + \cdots + p_{n+1} p_n \cdots p_3 \cdot p_2) - (s_0 + s_1 + \cdots + s_{n-1}) \\ &= (n-1) \prod_{2}^{n+1} p_i - (s_0 + s_1 + \cdots + s_{n-1}) \end{aligned} \tag{6}$$

Note that this this expression involves the addition of $(n-1)$ odd terms and the subtraction of n odd terms; as a result, $F(S)$ is odd. Furthermore, note that $F(S)$ is coprime to every prime p_i, $1 \leq i \leq n+1$, since p_i divides every term except s_{i-1}, so $F(S) \equiv s_{i-1} \mod p_i$.

We now define the semigroup

$$Mal_{n+1} = \langle 2s_0, 2s_1, \dots, 2s_{n-1}, F(S)\rangle \tag{7}$$

and claim that $x = 5F(S)$ has $\Delta_0(x) = 4$.

First, note that any factorization of $4F(S)$ involves the first n elements of Mal_{n+1}:

$$\begin{aligned}
4F(S) &= 4(n-1)\prod_2^{n-1} p_i - 4(s_0 + s_1 + \dots + s_{n-1}) \\
&= 2(n-2)\prod_2^{n+1} p_i + \left(2\prod_2^{n+1} p_i - 4s_0\right) + \left(2\prod_2^{n+1} p_i - 4s_1\right) + \dots + \left(2\prod_2^{n+1} p_i - 4s_{n-1}\right) \\
&= 2(n-2)\prod_2^{n+1} p_i + 2s_0(p_{n+1}-2) + 2s_1(p_n - 2) + \dots + 2s_{n-1}(p_2 - 2) \\
&= (n-2)\left(\frac{\prod_2^{n+1} p_i}{s_i}\right)\cdot 2s_i + (p_{n+1}-2, p_n - 2, \dots, p_2 - 2, 0)\cdot(2s_0, 2s_1, \dots, 2s_{n-1}, g(S))
\end{aligned} \tag{8}$$

Allowing the first term to be added to any of the other terms in the factorization, we have n different factorizations, each with the first n generators of Mal_{n+1} nonzero.

Suppose for contradiction that there exists a factorization of $4F(S)$ for which one but not all of $s_0, s_1, \dots, s_{n-1}$ is not a factor. If s_i is not in this factorization, then $p_{n+1-i} | 4F(S)$, since p_{n+1-i} is a factor of every generator except s_i. Then since $p_{n+1-i} \nmid 4$, we must have $p_{n+1-i} | F(S)$, which contradicts the previous assertion that $p_i \nmid F(S)$ for all $1 \le i \le n+1$.

Therefore factorizations of $4F(S)$ include either 1 or n generators of Mal_{n+1}. Furthermore, factorizations of $5F(S)$ include either 1 or $n+1$ generators of Mal_{n+1}, since $5F(S)$ is odd, requiring that any factorization include at least one multiple of $F(S)$, and the remaining $4F(S)$ can only be factored as $4F(S)$ exclusively or as a product of the other n generators. Thus the only factorizations of $5F(S)$ are

$$(0, 0, \dots, 0, 5) \sim Z_i(4F(S)) + (0, 0, \dots, 1) = (z_{i0}, z_{i1}, \dots, z_{i(n-1)}, 1) \tag{9}$$

where $Z_i(4F(S))$ is some factorization of $4F(S)$ with coefficients $z_{i0}, z_{i1}, \dots, z_{i(n-1)}, 0,\ z_i \neq 0$ for $0 \le i \le n-1$.

Altogether,

$$(L)_0(4F(S)) = \{1, n\} \implies \Delta_0(4F(S)) = \{n-1\}$$

and

$$(L)_0(5F(S)) = \{1, n+1\} \implies \Delta_0(5F(S)) = \{n\},$$

so $\{n-1, n\} \subset \Delta_0(Mal_{n+1})$.

QEΔ

Theorem 2.5.2. For all natural numbers n, there exists an $(n+1)$-generated numerical semigroup T with $\Delta_0(T) = [1, n] \cap \mathbb{N}$.

Proof. We will use the same construction as in Theorem 2.5.1 and show that in Mal_{n+1}, $\Delta_0(Mal_{n+1}) = [1, n] \cap \mathbb{N}$. From Lemma 2.1.1, Theorem 2.1.2, and Theorem 2.5.1, we know that $\{1, n-1, n\} \subseteq \Delta_0(Mal_{n+1}) \subseteq [1, n] \cap \mathbb{N}$. It remains to show that all integers $\{2, \dots, n-2\}$ are contained in $\Delta_0(Mal_{n+1})$.

Going forward, let

$$\Pi_i = \prod_{j=2}^{i} p_j, \quad \Pi_\star = \prod_{j=2}^{n+1} p_k \tag{10}$$

We claim that for $i \in \{2, \dots, n-1\}$,

$$n - i + 1 \in \Delta_0(x), \quad x = (2\Pi_i + 1)F(S) \tag{11}$$

To prove this, we begin by noting that this product will be odd, since the product of the first i primes (including 2) will be even, and $F(S)$ is always odd (see Theorem 2.5.1). Since $F(S)$ is the only odd generator of Mal_{n+1}, any factorization of x must include an odd multiple of $F(S)$.

To show that $i \in \Delta_0(x)$, we must show that there is a gap of size i in $\mathscr{L}_0(x)$.

To begin, clearly

$$(0, 0, \dots, 0, 2\Pi_i + 1) \tag{12}$$

is a factorization of x, so $1 \in \mathscr{L}_0(x)$.

Since any factorization of x must include an odd multiple of x, consider now the factorization containing only one such multiple.

$$\begin{aligned} x - F(S) &= 2\Pi_i F(S) \\ &= 2\Pi_i (n-1)\Pi_\star - 2\Pi_i(s_0 + s_1 + \dots + s_{n-1}) \\ &= 2(n-1)\Pi_i\Pi_\star - 2\Pi_i(s_0 + s_1 + \dots + s_{n-i}) - 2\Pi_i s_{n+1-i} - \dots - 2\Pi_i s_{n-1} \end{aligned} \tag{13}$$

Then, since $\forall\ 1 \leq j \leq i,\ p_j |\ \Pi_i$, and $\forall\ 0 \leq j \leq n-1,\ p_{n+1-j}s_j = \Pi_\star$,

$$\begin{aligned} x - F(S) &= 2(n-1)\Pi_i\Pi_\star - 2\Pi_i(s_0 + s_1 + \dots + s_{n-i}) - 2\Pi_i s_{n+1-i} - \dots - 2\Pi_i s_{n-1} \\ &= 2(n-1)\Pi_i\Pi_\star - 2\Pi_i(s_0 + s_1 + \dots + s_{n-i}) - 2\left(\frac{\Pi_i}{p_i}\right)\Pi_\star - \dots - 2\left(\frac{\Pi_i}{p_2}\right)\Pi_\star \\ &= 2\left((n-1)\Pi_i - \frac{\Pi_i}{p_i} - \dots - \frac{\Pi_i}{p_2}\right)\Pi_\star - 2\Pi_i(s_0 + s_1 + \dots + s_{n-i}) \\ &= 2\left((n-1)\Pi_i - \frac{\Pi_i}{p_i} - \dots - \frac{\Pi_i}{p_2} - n - i + 1\right)\Pi_\star + 2(n-i+1)\Pi_\star - 2\Pi_i(s_0 + s_1 + \dots + s_{n-i}) \\ &= 2\left((n-1)\Pi_i - \frac{\Pi_i}{p_i} - \dots - \frac{\Pi_i}{p_2} - n - i + 1\right)\Pi_\star \\ &\quad + 2s_0\left(\frac{\Pi_\star}{p_{n+1}} - \Pi_i\right) + 2s_1\left(\frac{\Pi_\star}{p_n} - \Pi_i\right) + \dots + 2s_{n-i}\left(\frac{\Pi_\star}{p_{i+1}} - \Pi_i\right). \end{aligned} \tag{14}$$

This can be written as a factorization of the form

$$\begin{aligned}&\left(\frac{\Pi_\star}{p_{n+1}} - \Pi_i,\ \frac{\Pi_\star}{p_n} - \Pi_i,\ \ldots\ ,\ \frac{\Pi_\star}{p_{i+1}} - \Pi_i,\ 0,\ \ldots\ ,\ 0,\ 1\right)\\ &+ 2s_m\left((n-1)\Pi_i - \frac{\Pi_i}{p_i} - \cdots - \frac{\Pi_i}{p_2} - n - i + 1\right)\left(\frac{\Pi_\star}{s_m}\right),\ 0 \le m \le i-1\end{aligned} \tag{15}$$

where the final term can be added to any of the first $n-i+1$ entries of the factorization. Therefore $n-i+2 \in \mathscr{L}_0(x)$.

Now, suppose for contradiction that there exists a factorization of $x - F(S) = 2\Pi_i F(S)$ for which some but not all of $s_0, s_1, \ldots, s_{n-i}$ are not factors. If $s_m,\ 0 \le m \le n-i$ is not in this factorization, then $p_{n+1-m} |\ 2\Pi_i F(S)$, since p_{n+1-m} is a factor of every generator except s_m. But then either $p_{n+1-m} |\ 2\Pi_i$, or $p_{n+1-m} |\ F(S)$, and clearly neither of these are possible. Therefore the factorization in Equation 15 demonstrates minimal support for the case where only one multiple of $F(S)$ is isolated.

We have shown how $\{1, n-i+1\} \subseteq \Delta_0(x)$ can be achieved, and proven that when 1 or $2\Pi_i + 1$ multiples of $F(S)$ are used in the factorization, these are the only lengths possible. Now consider the case when some other odd multiple of $F(S)$ is used in the factorization.

Note that $2\Pi_i$ is, by construction, the smallest multiple of i distinct primes. Writing x as $(2\Pi_i + 1 - k)F(S) + kF(S)$, $1 < k < 2\Pi_i + 1$, k odd, this means that $(2\Pi_i + 1 - k)F(S)$ can be divided by at most $i-1$ distinct primes. Since $(2\Pi_i + 1 - k)F(S)$ is even, this leaves at most $i-2$ distinct primes greater than 2.

Suppose for contradiction that a factorization with k multiples of $F(S)$ $(1 < k < 2\Pi_i + 1$, k odd) exists with less than $n-i+2$ factors. Such a factorization would include $F(S)$ as a factor, and so would have less than $n-i+1$ remaining factors from among the first n generators of Mal_{n+1}. This would require that at least $i-1$ generators have weights of 0. Supposing this to be the case, with $2s_{a_1}, \ldots, 2s_{a_{i-1}}$ all having weights of 0 in the factorization, it must be the case that $p_{n+1-a_1}, \ldots, p_{n+1-a_{i-1}}$ all divide $(2\Pi_i + 1 - k)F(S)$, since these primes divide all but their corresponding generators. But this implies that $(2\Pi_i + 1 - k)F(S)$ has $i-1$ distinct prime factors greater than 2, a contradiction.

Therefore, no factorization using an odd multiple of $F(S)$ between 1 and $2\Pi_i + 1$ achieves a factorization length of less than $n-i+2$. So $\{1, n-i+2\}$ is an unbroken interval in $(L)_0(x)$, and so $n-i+1 \in \Delta_0(x)$. Since this holds for all $i \in \{2, \ldots, n-1\}$, we have that $\{2, \ldots, n-1\} \subseteq \Delta_0(Mal_{n+1})$. Together with Theorem 2.1.2 and Theorem 2.5.1, this proves that $\Delta_0(Mal_{n+1}) = [1, n] \cap \mathbb{N}$.

QEΔ

Remark 6. For the family of semigroups used in Theorems 2.5.1 and 2.5.2, we will describe the (conjectured) minimal presentation. (Although this has not been rigorously proven, it is consistent with everything we know about the semigroups and matches the minimal presentations produced by Sage).

Let Mal_{n+1} be defined as in Theorem 2.5.1, having $n+1$ generators:

$$Mal_{n+1} = \langle t_0, t_1, \ldots, t_{n-1}, t_n \rangle = \langle 2s_0, 2s_1, \ldots, 2s_{n-1}, F(S) \rangle \tag{16}$$

Recall also that $2s_i$ is a product of the first $n+1$ primes less p_{n+1-i}. The minimal presentation for Mal_{n+1} can be expressed as follows:

$$
\begin{aligned}
&1.\ (0,0,\ldots,0,4) \sim (p_{n+1}-2, p_n-2, \ldots, p_2-2, 0) + (n-2)p_{n+1-i} \\
&\quad \text{where the final term is added to one of the first } i \text{ coefficients, } 0 \leq i \leq n-1 \\
&2.\ (p_{n+1}, 0, \ldots, 0, 0) \sim (0, p_n, \ldots, 0, 0) \\
&\quad \vdots \\
&\quad (0, \ldots, p_i, 0, \ldots, 0) \sim (0, \ldots, 0, p_{i-1}, \ldots, 0) \\
&\quad \vdots \\
&\quad (0, \ldots, p_3, 0, 0) \sim (0, \ldots, 0, p_2, 0) \\
&3.\ (1, 0, \ldots, 0, 2) \sim (0, p_{n-1}-1, \ldots, p_2-, 0) \\
&\quad (0, 1, \ldots, 0, 2) \sim (p_n-1, 0, \ldots, p_2-1, 0) \\
&\quad \vdots \\
&\quad (0, 0 \ldots, 1, 2) \sim (p_n-1, p_{n-1}-1, \ldots, 0, 0)
\end{aligned}
\tag{17}
$$

Example 2.5.3. As an example, we will construct the Mal_5.

To begin, let

$$(a) = p_4, p_3, p_2 = 7, 5, 3 \quad \text{and} \quad (b) = p_5, p_4, p_3 = 11, 7, 5 \tag{18}$$

as in Theorem 2.5.1 Equation 3.

Next, define the semigroup S with generators from the compound sequence formed using (a) and (b):

$$
\begin{aligned}
s_0 &= 7 \cdot 5 \cdot 3 = 105 \\
s_1 &= 11 \cdot 5 \cdot 3 = 165 \\
s_2 &= 11 \cdot 7 \cdot 3 = 231 \\
s_3 &= 11 \cdot 7 \cdot 5 = 385
\end{aligned}
\tag{19}
$$

$$S = \langle s_0, s_1, s_2, s_3 \rangle = \langle 105, 165, 231, 385 \rangle \tag{20}$$

(as described in Theorem 2.5.1 Equations 4 and 5.)

We calculate the Frobenius number of S using the equation described in Equation 6 of Theorem 2.5.1:

$$F(S) = 4(3 \cdot 5 \cdot 7 \cdot 11) - (105, 165, 231, 385) = 2579 \tag{21}$$

Finally, the monoid Mal_5 is formed by doubling each of the generators in S and appending $F(S)$:

$$Mal_5 = \langle 2s_0, 2s_1, 2s_2, 2s_3, F(S) \rangle = \langle 210, 330, 462, 770, 2579 \rangle \tag{22}$$

This monoid has the following minimal presentation (in agreement with Remark 6):

$$
\begin{aligned}
&1.\ (9,5,3,7,0) \sim (0,0,0,0,4)\\
\\
&2.\ (0,0,5,0,0) \sim (0,0,0,3,0)\\
&\quad\ (0,7,0,0,0) \sim (0,0,5,0,0)\\
&\quad\ (11,0,0,0,0) \sim (0,7,0,0,0)\\
\\
&3.\ (1,0,0,0,2) \sim (0,6,4,2,0)\\
&\quad\ (0,1,0,0,2) \sim (10,0,4,2,0)\\
&\quad\ (0,0,1,0,2) \sim (10,6,0,2,0)\\
&\quad\ (0,0,0,1,2) \sim (10,6,4,0,0)
\end{aligned}
\tag{23}
$$

Lastly, it can be shown computationally that $\Delta_0(Mal_5) = \{1,2,3,4\}$. In particular,

$$
\begin{aligned}
\Delta_0(5 \cdot 2579) &= \{4\}\\
\Delta_0(7 \cdot 2579) &= \{1,3\}\\
\Delta_0(31 \cdot 2579) &= \{1,2\}
\end{aligned}
\tag{24}
$$

(in agreement with Theorem 2.5.2).

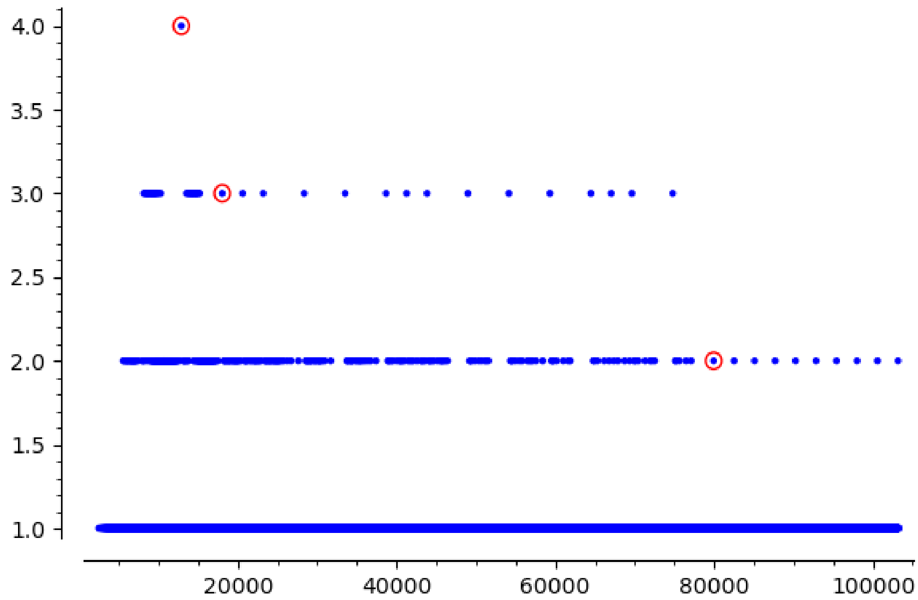


Figure 2: $\Delta_0(x)$ for $x \in Mal_5$ up to $40 * 2579$, with the points $(5 \cdot 2579, 4)$, $(7 \cdot 2579, 3)$, and $(31 \cdot 2579, 2)$ in red.

2.6 Alabio Monoids

All numerical semigroups shown thus far have had a Δ_0 set that was a perfect interval of natural numbers, with no "gaps." This inspired a long-standing conjecture that required a family of explicitly constructed counterexamples to disprove.

Theorem 2.6.1. There exist numerical semigroups S with $d \in (\min(\Delta_0(S)), \max(\Delta_0(S)))$ but $d \notin \Delta_0(S)$.

The proof of this is presented via a counterexample:

Example 2.6.2. We will construct the Alabio Monoid with $n = 8$ generators, which is the smallest such monoid to have a gap of size 3 in it's Δ_0 set.

To begin, let $a = 2^{n-2} = 2^6 = 64$ and $b = 3 \cdot 2^{n-3} = 3 \cdot 2^5 = 96$. The semigroup S is generated recursively:

$$\begin{aligned} s_0 &= a = 64 \\ s_1 &= b = 96 \\ s_2 &= s_0 + s_1/2 = 112 \\ s_3 &= s_1 + s_2/2 = 152 \\ s_4 &= s_2 + s_3/2 = 188 \\ s_5 &= s_3 + s_4/2 = 246 \\ s_6 &= s_4 + s_5/2 = 311 \end{aligned} \tag{25}$$

$$S = \langle s_0, s_1, s_2, s_3, s_4, s_5, s_6 \rangle = \langle 64, 96, 112, 152, 188, 246, 311 \rangle \tag{26}$$

Then, the final monoid Ab_8 is formed by the gluing

$$Ab_8 = 2S + \sum_0^6 s_i = \langle 128, 192, 224, 304, 376, 492, 622, 1169 \rangle \tag{27}$$

A minimal presentation of Ab_8 is:

$$\begin{aligned} &1.\ (2,0,0,0,0,0,0,0) \sim (0,3,0,0,0,0,0,0) \\ \\ &2.\ (2,1,0,0,0,0,0,0) \sim (0,0,2,0,0,0,0,0) \\ &\quad\ (0,2,1,0,0,0,0,0) \sim (0,0,0,2,0,0,0,0) \\ &\quad\ (0,0,2,1,0,0,0,0) \sim (0,0,0,0,2,0,0,0) \\ &\quad\ (0,0,0,2,1,0,0,0) \sim (0,0,0,0,0,2,0,0) \\ &\quad\ (0,0,0,0,2,1,0,0) \sim (0,0,0,0,0,0,2,0) \\ \\ &3.\ (1,1,1,1,1,1,1,0) \sim (0,0,0,0,0,0,0,2) \end{aligned} \tag{28}$$

It can be shown computationally that $\Delta_0(Ab_8) = \{1, 2, 3, 6, 7\}$. Notably, $4, 5 \notin \Delta_0(Ab_8)$.

This counterexample proves that the 0-delta set of a numerical semigroup need not be an interval. However, as the previous example also illustrates, with a sufficiently large Alabio monoid, the trade structure permits a multiple elements to be missing from the 0-delta set. Understanding this trade structure allows us to prove the existence of numerical semigroups with arbitrarily large gaps in their 0-delta set.

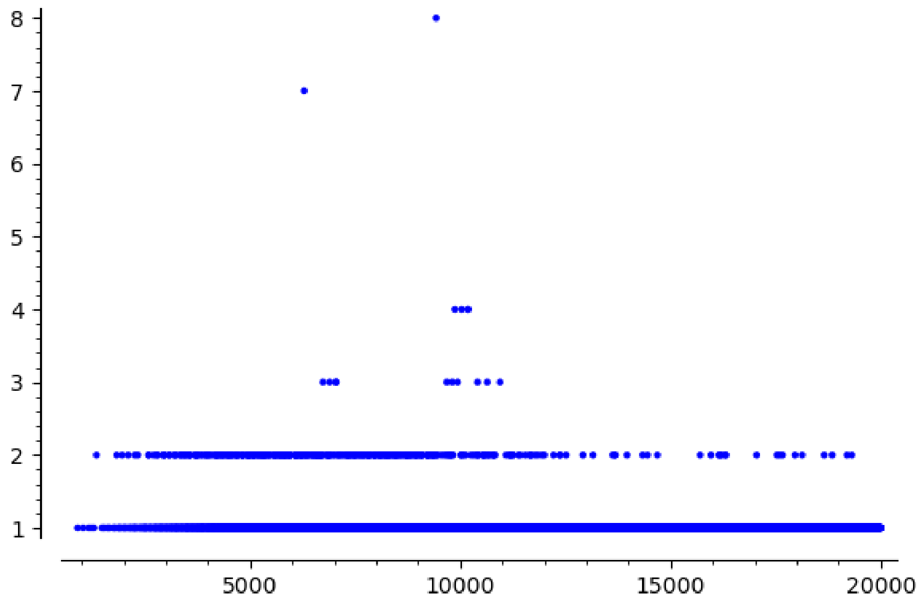


Figure 3: Elements of $\Delta_0(x)$ for $x \in Ab_8$ up to $x = 20,000$

Theorem 2.6.3. Let $Ab_n := \langle a_1, ..., a_n \rangle$ be the Alabio Monoid (constructed as in Example 2.6.2) with n generators. Then

$$1.\ (2, 0, \ldots, 0, 0) \sim (0, 3, \ldots, 0, 0)$$

$$\begin{aligned} 2.\ (2, 1, 0, 0, \ldots, 0, 0) &\sim (0, 0, 2, 0, \ldots, 0, 0) \\ (0, 2, 1, 0, \ldots, 0, 0) &\sim (0, 0, 0, 2, \ldots, 0, 0) \\ &\vdots \\ (0, 0, \ldots, 2, 1, 0, 0) &\sim (0, 0, \ldots, 0, 0, 2, 0) \end{aligned} \tag{29}$$

$$3.\ (1, 1, \ldots, 1, 0) \sim (0, 0, \ldots, 0, 2)$$

is a minimal presentation for Ab_n.

Proof. We will first show that $Ap(Ab_n, a_1)$ has unique presentation, with all factorization vectors having coordinates 0 or 1 (∞-norm at most 1). We'll do this by proving the following:

Lemma 2.6.4. If $\hat{f} = (f_1, ..., f_n)$ is a factorization with residue r mod m and $f_i \geq 2$ for some coordinate i, $\hat{f}$ is not a factorization of an element of $Ap(Ab_n, m)$.

Proof. Let x be an element of Ab_n with $x \equiv r$ mod m and a factorization $\hat{f}$ with $\hat{f}_\infty \geq 2$. For each such factorization, let $m(f)$ be the largest coordinate of f witnessing $\hat{f}_\infty \geq 2$.

Consider $\min(m(f) : f \in Z(x)) = k$ with the factorization $\hat{f} = (f_1, ..., f_n)$ witnessing. If $k = 1$, $x - a_1 \in S$, so $x \notin Ap(Ab_n, a_1)$. Else, given that $2a_2 = 3a_1$, $2a_i = 2a_{i-2} + a_{i-1}$ for all $i \in [2, n-1]$

and $2a_n = \sum_{i=1}^{n-1} a_i$, we may trade $\hat{f}$ for $\hat{f}' = (f_1', f_2', ..., f_k - 2\lfloor \frac{f_k}{2} \rfloor, f_{k+1}, ..., f_n)$, leaving all coordinates higher than f_k unchanged and changing f_k to 0 or 1.

If k was n, then $f_1' \geq 1$, so $\hat{f} \notin Ap(Ab_n, n)$. Else, a coordinate before k has increased by at least 2, so $m(\hat{f}') < m(\hat{f})$, a contradiction. QEΔ

Furthermore, $|Ap(Ab_n, a_1)| = a_1 = 2^{n-1}$–exactly the number of distinct factorization vectors with dimension n, ∞-norm at most 1, and leading coordinate 0. So not only does every element of $Ap(S, a_1)$ have a factorization as such a vector, every such vector is the *unique* factorization of an element of $Ap(Ab_n, a_1)$.

According to example 2.5 of an old REU paper [18], we may then find a minimal presentation of T by finding "outer Betti elements of the Kunz nilsemigroup of T", equivalently, factorization vectors $\hat{f} = (f_1, ..., f_n)$, not of elements of $Ap(Ab_n, a_1)$, such that for all positive coordinates i of $\hat{f}$, $\{\hat{f} - e_i\} = Z(x)$ for some $x \in Ap(Ab_n, a_1)$. As shown above, if $|\hat{f}|_\infty \leq 1$, $\hat{f}$ corresponds to an element of $Ap(Ab_n, a_1)$. So $f_i \geq 2$ for some $i \in [1, n]$. If $\hat{f}$ has another nonzero coordinate j, $|\hat{f} - e_j|_\infty \geq 2$, and if $f_i > 2$, $|\hat{f} - e_i|_\infty \geq 2$. So $\hat{f} = (0, ..., 0, 2, 0, ...,)$. The same paper also says that we may build a minimal presentation of Ab_n from one trade at each of the Ab_n-Betti-elements corresponding to these outer Betti elements. The trades in the theorem statement are a set of such trades, so they form a minimal presentation of T.

QEΔ

These trades allow two elements with n, $n-1$ in their Δ_0 set, respectively, but ensure that all other elements have "mid-length" factorizations that prevent high Δ_0 values.

Theorem 2.6.5. For all $n \geq 8$ and $k \in [3, 2 + \frac{n}{6}]$, the Alabio monoid $Ab_n := \langle a_1, ..., a_n \rangle$ satisfies $n - k \notin \Delta_0(Ab_n)$. Furthermore, $n-1, n-2 \in \Delta_0(Ab_n)$.

Proof. Let $x \in Ab_n$, with $n - k \in \Delta_0(x)$. If trades of the first and second type from 2.6.3 span $Z(x)$, $\max \Delta_0(x) \leq c'(x) \leq 2$. So x must have two factorizations differing only by the third type of trade. Let f, f' be two such factorizations with the largest 0-norm gap, $|f|_0 - |f'|_0$. If this gap is less than the target delta value of $n-k$, then $n - k \notin \Delta_0(x)$. [need more justification]. So $|f|_0 - |f'|_0 \geq n - k$, meaning $|f'|_0 \leq k$. Furthermore, $|f'|_0 = k$ iff $|f|_0 = n$, i.e., iff both f and f' have nonzero final coordinate.

We may then express $f \sim f'$ in the following form, splitting apart the minimal trade, final coordinate, and remaining coordinates:

$$\begin{aligned} &(1, 1, \ldots, 1, 0) \sim (0, 0, \ldots, 0, 2) \\ + &(0, 0, 0, 0, \ldots, 0, a) \sim (0, 0, 0, 0, \ldots, 0, a) \\ + &\hat{x}' \sim \hat{x}' \end{aligned} \tag{30}$$

Now $l := |\hat{x}'|_0 \leq k-1$. To arrive at a contradiction, we will now examine case-by-case the possibilities for the smallest coordinate of $\hat{x}'$, i.

Case 1: $i \leq \frac{n}{2}$

Here, we may perform on $\hat{f}$ the trade

$(0, 0, ..., 2_i, 1, 0, 1, 0, ..., 1_{i+1+2k}, 0, ..., 0) \sim (0, 0, ..., 2_{i+2+2k}, 0, ...)$, obtained by iterating trade type 2. This decreases $2 + k$ of the nonfinal coordinates of $\hat{f}$; between 0 and l of those are more than 1, depending on collisions between coordinates involved in this trade and coordinates of $\hat{x}$. This means the result has 0-norm between $|\hat{f}|_0 - (k+2)$ and $|\hat{f}|_0 - (k+2) + l$, a subset of $[n-3-k, n-2-k+l]$. Any 0-norm in this range is less than $n - k$ away from either l or n, which means this factorization prevents the target Δ_0.

Case 2: $i > \frac{n}{2}$.

In this case, $\hat{f}$ is of the form $(1, 1, 1, ..., f_i, ...,)$. Applying the trade

$$(0, 0, ..., 2_i, ...) \sim (0, ..., 2_{i-2-2k}, 1, ...1, 0, 1_{i-1}, 0_i, ...,)$$

changes this to $(0, ..., 3_{i-2-2k}, 2, 1, ..., 2_{i-1}, f_i - 2, ...)$. Applying many type 2 trades then changes this into $(0, ..., 3_{i-2-2k}, 0, 2, 0, ..., 4_{i-1}, f_i - 2, ...)$. Either way, k of the 1s have changed to 0. We then obtain a factorization f'' of length $n-1-k$ or $n-k$; in both cases, $|f''|_0 - l, n - |f''|_0 < n-k$, preventing the target Δ_0.

Case 3: $|\hat{x'}|$ has no nonzero coordinates, i.e., $\hat{x'} = \hat{0}$. If $a \leq 1$, f', f are the only factorizations of x, so $\Delta_0(x) = \{n\}$ or $\{n-1\}$. Else, we may perform the type 3 trade on f' to get a factorization of the form $(2, 2, 2, ..., a-2)$. The same iterated trade used in case 1 will change this to a factorization of the form $(0, 1, 2, 1, 2, 1, ...)$, which can again be changed into $(0, 1, 0, 0, 2, 0, 2, ...,)$. This gives a factorization of length approximately $\frac{n}{2}$, which also prevents the target Δ_0.

QEΔ

2.7 Conjectures and Open Work

Since 0-delta sets of a numerical semigroup S decay to $\{1\}$, each nonzero value of $\Delta_0(S)$ last appears at some element. Based on experimental data, we conjectured that at the element x of last appearance, the delta value in question is the largest value of $\Delta_0(x)$.

Conjecture 1. *Let S be a numerical semigroup with $i \in \Delta_0(S)$. Then $\max(\Delta_0(\max\{x : i \in \Delta_0(x)\})) = i$.*

The agreement of greater and lesser catenary degrees across a semigroup and the importance of Betti elements to that result leads to the following conjecture:

Conjecture 2. *At all Betti elements b of a numerical semigroup, $c_p(b) = c'_p(b)$*

We also have a conjecture analogous to one in [33] and mirroring our result that the maximal c'_0 values of a semigroup occur at a Betti element:

Conjecture 3. *The minimal c'_0 values of a semigroup occur at a Betti element.*

As mentioned, the asymptotic behavior of the lesser catenary degree is unknown for all norms except the 0-norm.

The relationship between gluings and trade structure grows more complex as embedding dimension grows. The methods used to classify the Δ_0 sets of numerical semigroups with embedding dimension 3 fail for any higher embedding dimension. Full, generic classifications for the Δ_0 set of such higher-dimensional numerical semigroups remain open.

Relatedly, it is known that the $M_p(S)$ bound is sometimes, but not always tight for numerical semigroups. While the $M_0(S)$ bound is tight for numerical semigroups constructed completely from monoscopic gluings, behavior for generic numerical semigroups, or even numerical semigroups constructed from more complex gluings, is unknown. The tightness of $M_p(S)$ for other p is completely unknown.

The effect of generic gluings on trade structure and Δ_0 set is also unknown; all proven results are, again, restricted to the case of monoscopic gluings.

3 Δ_∞

Recall from Section 1 that the ∞-length of a factorization is maximum coordinate in that factorization. Thus, larger elements, which can be represented with more copies of atoms, have larger ∞-length factorizations. So, unlike with the 0-norm, there is no easy upper bound on the maximal element of the ∞-delta set of an arbitrary semigroup. The lower bound, however, is identical.

Theorem 3.0.1. For all numerical semigroups S, $\min(\Delta_\infty(S)) = \gcd(\Delta_\infty(S)) = 1$.

Proof. Let $x \in S$ be non-∞-half-factorial. Let $f_1, f_2 : \mathscr{A}(S) \to \mathbb{N}_0$ be two functions encoding two factorizations of x with different ∞-lengths l_1, l_2 occuring at $f_1(a_1)$, $f_2(a_2)$. WLOG let $l_1 > l_2$. If $f_1 \geq f_2$ on the entire domain, then $f_1 - f_2$ is a nontrivial factorization of 0, a contradiction. So $f_2(a) > f_1(a)$ for some $a \in \mathscr{A}(S)$. Consider the element $x + (l_1 - f_2(a) - 1)a$. It has factorizations f_1', f_2' defined by $f_1'(i) = f_1(i) + (l_1 - f_2(a) - 1)\delta_{ia}$; $f_2'(i) = f_2(i) + (l_1 - f_2(a) - 1)\delta_{ia}$, where δ denotes the Kronecker delta. We will show that $|f_1'|_\infty = l_1$ and $|f_2'|_\infty = l_1 - 1$.

We have that $f_1'(a) = f_1(a) + (l_1 - f_2(a) - 1) < f_2(a) + (l_1 - f_2(a) - 1) = l_1 - 1 \leq l_1 = f_1'(a_1)$. Since $f_1(b) \leq l_1$ for all atoms b, $f_1'(b) \leq l_1$ for all atoms $b \neq a$. So $l_1 = |f_1'|_\infty$.

However, $f_2'(a) = f_2(a) + (l_1 - f_2(a) - 1) = l_1 - 1$. Since $f_2(b) \leq l_2 < l_1$ for all atoms b, $f_2'(b) \leq l_2 < l_1$ for all atoms $b \neq a$. So $l_1 - 1 = |f_2'|_\infty$.

QEΔ

Remark 7. Similarly to Theorem 2.1.2, this result also holds for all commutative, cancellative, atomic, non-∞-half-factorial monoids.

It is also natural, when working with the ∞ norm, to partition factorizations by maximal coordinate. The following notation encodes this.

Definition. Suppose $S = \langle a_1, ..., a_k \rangle$ is a semigroup and $i \in [1, k]$.

1. Let $Z(x, i)$ be the set of factorizations with the ith coordinate as a maximal coordinate.
2. Further let $\mathscr{L}_\infty(x, i) := \{|f|_\infty : f \in Z(x, i)\}$.
3. Let $l_\infty(x, i) := \min(\mathscr{L}_\infty(x, i))$; $L_\infty(x, i) := \max(\mathscr{L}_\infty(x, i))$.

3.1 Initial Results: Two Towers

As with the 0-norm, we begin with the simplest case, numerical semigroups with embedding dimension 2. The proof relies on the fact that for any two generated numerical semigroup $S = \langle a_1, a_2 \rangle$, $\{(a_2, 0) \sim (0, a_1)\}$ is a minimal presentation for S. Therefore, elements of the $Z(x, 1)$ and $Z(x, 2)$ factorization sets $x \in S$ are spaced only by multiples of a_2 and a_1, respectively. This property is illustrated in the following example:

Example 3.1.1. Included is a plot of the two ∞-length sets of 550 in $\langle 5, 11 \rangle$, which illustrates the dual "tower" structure of ∞-lengths in 2-generated numerical semigroups.

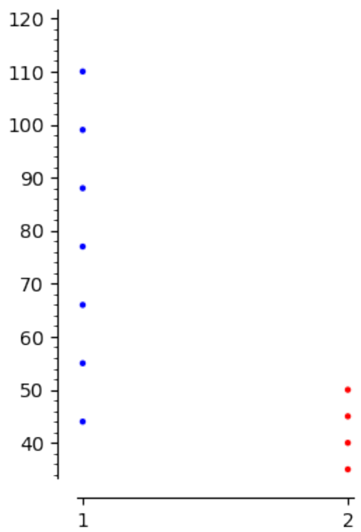


Figure 4: Elements of $Z(550, 1)$ and $Z(550, 2)$ in $\langle 5, 11 \rangle$

This characterization of $Z(x, 1)$ and $Z(x, 2)$ is sufficient to describe the Δ_∞ set of such semigroups. The first four results in the following Theorem describe the general content and periodicity of ∞-delta set elements, while the last two results elaborate on the elements of $\Delta_\infty(x)$ which vary across the period.

Theorem 3.1.2. Let $S = \langle a_1, a_2 \rangle$ be a numerical semigroup. Then

1. $\Delta_\infty(S) = [1, a_2]$

Furthermore, for $x \geq (a_2^3 + a_2)(a_1 + a_2)$,

2. $[1, a_1] \subseteq \Delta_\infty(x)$
3. $a_2 \in \Delta_\infty(x)$

4. $\Delta_\infty(x+a_2^2)=\Delta_\infty(x)$

Lastly, for $x \geq (a_1+1)a_2^2$,

5. For $n \in [a_1+2, a_2-1]$, if $n \in \Delta_\infty(x)$, $n-1 \in \Delta_\infty(x+a_2)$. Furthermore, if none of $[a_1+1, a_2-1]$ are in $\Delta_\infty(x-i)$, $1 \leq i \leq a_1+1$, then $a_2-1 \in \Delta_\infty(x)$.
6. More specifically, let $a_1^{-1} := a_1^{-1} \mod a_2$, $0 \leq a_1^{-1} < a_2$, and $a_2^{-1} := a_2^{-1} \mod a_1$, $0 \leq a_2^{-1} < a_1$. Then

$$\frac{x-(xa_2^{-1}-\lfloor\frac{xa_2^{-1}}{a_1}\rfloor a_1)a_2}{a_1}-\frac{x-(xa_1^{-1}-\lfloor\frac{xa_1^{-1}}{a_2}\rfloor a_2)a_1}{a_2} \equiv k \mod a_2,\ 1 \leq k \leq a_2 \implies k \in \Delta_\infty(x) \tag{31}$$

(This is really just an explicit way of saying $\frac{x-b}{a_1} - \frac{x-c}{a_2} \equiv k \mod a_2,\ 1 \leq k \leq a_2 \implies k \in \Delta_\infty(x)$, where $xa_2^{-1} \equiv b \mod a_1,\ 0 \leq b \leq a_1-1$, and $xa_1^{-1} \equiv c \mod a_2,\ 0 \leq c \leq a_2-1$.)

Proof. We begin by using the minimal trade $T=(a_2,0) \sim (0,a_1)$ to describe the factorization and length sets for $x \in S$, where x is large.

Let $f_0=(s_0,t_0) \in Z(x)$ be such that $s_0 \geq t_0$ and s_0-t_0 is minimal. Let $f_0'=(s_0',t_0') \in Z(x)$ be the result of applying T to f_0; by minimality, $s_0' < t_0'$. We may obtain the factorization subsets $Z(x,1) := f_0, f_1, f_2, ..., f_y$ and $Z(x,2) := f_0', f_1', ..., f_z'$ by exhaustively performing the trades $(a_2,-a_1)$ and $(-a_2,a_1)$, respectively. Then all factorizations $f_i=(s_i,t_i)$ in $Z(x,1)$ satisfy $|f_i|_\infty = s_i = s_0+ia_2$, while all factorizations $f_i'=(s_i',t_i')$ in $Z(x,2)$ satisfy $|f_i'|_\infty = t_i' = t_0'+ia_1$. The length set $\mathscr{L}_\infty(x)$ then equals $\{s_0+ia_2 : i \in [0,y]\} \cup \{t_0'+ia_1 : i \in [0,z]\}$. Denote this union by $\mathscr{L}_\infty(x,1) \cup \mathscr{L}_\infty(x,2)$ (these are our two "towers").

We will need the following lemma:

Lemma 3.1.3. For $x > (a_1+1)a_2^2$, $\max(\mathscr{L}_\infty(x)) \in \mathscr{L}_\infty(x,1)$.

Proof. Observe that $|f_y|_\infty, |f_z'|_\infty$ are maximal in $\mathscr{L}_\infty(x,1), \mathscr{L}_\infty(x,2)$. Furthermore, $x = f_z' \cdot (a_1,a_2) \geq |f_z'|_\infty \cdot a_2$ and $x = f_y \cdot (a_1,a_2) = |f_y|_\infty \cdot a_1 + t_y a_2$. By maximality of f_y, $t_y < a_2$, so $x < |f_y|_\infty \cdot a_1 + a_2^2$. Combining inequalities gives $|f_z'|_\infty \cdot a_2 < |f_y|_\infty \cdot a_1 + a_2^2$, or $|f_z'|_\infty < |f_y|_\infty \cdot \frac{a_1}{a_2} + a_2$. It then suffices to show that $|f_y|_\infty \cdot \frac{a_1}{a_2} + a_2 \leq |f_y|_\infty$, or that $a_2^2 \leq (a_2-a_1)|f_y|_\infty$. This holds because $a_2-a_1 \geq 1$ and $|f_y|_\infty = \frac{x-t_y a_2}{a_1} \geq \frac{x-a_2^2}{a_1} \geq a_2^2$. QE$\Delta$

Consider a gap between two consecutive lengths in $\mathscr{L}_\infty(x)$. If this gap involves two lengths in $\mathscr{L}_\infty(x,1)$, its size is a_2. Therefore, such gaps only determine whether $a_2 \in \Delta_\infty(x)$; all other elements of $\Delta_\infty(x)$ may be found by examining predecessors and successors of lengths in $\mathscr{L}_\infty(x,2)$. Accordingly, let $l' \in \mathscr{L}_\infty(x,2)$. By the lemma, l' always has a successor, l''. There are two cases to consider here.

If $l' = L_\infty(x,2)$, $l'' \in \mathscr{L}_\infty(x,1)$, meaning $l''-l' = s_0+ia_2-l'$ for some i. Since $l''-l' \leq a_2$, i is such that $l''-l'$ is the least positive representative of the residue class of s_0-l', modulo a_2.

Else, if $[l', l'+a_1]$ (meaning the interval between l' and the next length in $\mathscr{L}_\infty(x,2)$) contains an element of $\mathscr{L}_\infty(x,1)$, $l'' \in \mathscr{L}_\infty(x,1)$. Once again, this means that $l''-l'$ is the least positive representative of the residue class of s_0-l', modulo a_2. Otherwise, $l'' \in \mathscr{L}_\infty(x,2)$, meaning $l''-l' = a_1$. Note that in both subcases, $l''-l' \leq a_1$.

Now if $l' \neq \min(\mathscr{L}_\infty(x))$, l' has a predecessor, l. If $[l'-a_1, l']$ (the interval between l' and the (if it exists) previous length in $\mathscr{L}_\infty(x,2)$) contains an element of $\mathscr{L}_\infty(x,1)$, $l \in \mathscr{L}_\infty(x,1)$. Similar to the previous case, this means that $l'-l$ is the least positive representative of $l'-s_0$, modulo a_2. Otherwise, $l \in \mathscr{L}_\infty(x,2)$, meaning $l'-l = a_1$. Again, in both subcases, $l'-l \leq a_1$.

These characterizations of $Z(x), \mathscr{L}_\infty(x)$, and $\Delta_\infty(x)$ suffice to verify the four claims in the theorem.

1. Claim 1: Fix $d \in [1, a_2-1]$, let r be a natural number such that $r \equiv a_1^{-1} \mod a_2$ (possible because a_1, a_2 are coprime), and set $x := (a_2-d)ra_1a_2$. Then $f'_z = (0, (a_2-d)ra_1)$, meaning $L_\infty(x,2) = (a_2-d)ra_1$. Furthermore, $f_y = ((a_2-d)ra_2, 0)$, meaning $L_\infty(x,1) = (a_2-d)ra_2 > L_\infty(x,2)$ (so $L_\infty(x,2)$ has a successor). Since the successors of $L_\infty(x,2)$ are all elements of $\mathscr{L}_\infty(x,1)$ spaced a_2 apart, $L_\infty(x,1) - L_\infty(x,2) \equiv l' - L_\infty(x,2) \mod a_2$, where l' is the immediate successor of $L_\infty(x,2)$ in $\mathscr{L}_\infty(x)$. Then $L_\infty(x,1) - L_\infty(x,2) = (a_2-d)ra_2 - (a_2-d)ra_1 \equiv d \mod a_2$ meaning, by the above characterization, that the gap between $L_\infty(x,2)$ and l' is d. Therefore $d \in \Delta_\infty(x) \subseteq \Delta_\infty(S)$.

   Lastly, to show that $a_2 \in \Delta_\infty(S)$, let $x := a_1a_2^2$. Then $f'_z = (0, a_1a_2)$, meaning $L_\infty(x,2) = a_1a_2$, and $f_y = (a_2^2, 0)$, meaning $L_\infty(x,1) = a_2^2 \geq a_2(a_1+1) = a_1a_2 + a_2$. Performing the trade $(-a_2, a_1)$ on f_y gives the factorization $f_{y-1} = (a_2^2 - a_2, a_1)$, which has length $l' = a_2^2 - a_2 > L_\infty(x,2) = a_1a_2$. Thus the predecessor of $L_\infty(x,1)$ is part of $Z(x,1)$, and so $L_\infty(x,1) - l' = a_2$ implies $a_2 \in \Delta_\infty(x) \subseteq \Delta_\infty(S)$.

Claims 2, 3, and 4 apply to $x \geq (a_2^3 + a_2)(a_1 + a_2)$:

2. Claim 2: Fix $x \in S$, $d \in [1, a_1]$, and let r_1 be the least positive natural number with $r_1 \equiv a_1^{-1} \mod a_2$. Further let r_2 be the least positive natural number with $r_2 \equiv s_0 - d - t'_0 \mod a_2$.

   Let $f'_0 = (s'_0, t'_0)$ be the factorization of x with $t'_0 > s'_0$, $t'_0 - s'_0$ minimal. We know that such a factorization exists since $x > (a_2^3 + a_2)(a_1 + a_2) > \max(\mathrm{Ap}(S, a_2)) + a_2^2 = (a_2-1)a_1 + 2a_2^2$, so x can be factored as $x = (p_1, p_2) + (0, k)$ where $p_1a_1 + p_2a_2 \in \mathrm{Ap}(S, a_2)$, so that $p_1 < a_2$, and $k \geq a_2$. Therefore $Z(x,2)$ is nonempty, so we are ensured a factorization with $t'_0 > s'_0$ and minimal difference between the two coordinates.

   Then, since $t'_0 - s'_0$ is minimal, we can write $f'_0 = (s'_0, t'_0) = (0, t'_0 - s'_0) + (s'_0, s'_0)$. Since $t'_0 - s'_0 < a_1 + a_2$ (otherwise a $(a_2, -a_1)$ trade could be performed, contradicting minimality),

$$
\begin{aligned}
(a_2 + a_1)a_2 + s'_0(a_1+a_2) \geq x = (t'_0 - s'_0)a_2 + s'_0(a_1+a_2) &\geq (a_2^3 + a_2)(a_1+a_2) \\
(a_2 + a_1)a_2 + s'_0(a_1+a_2) \geq (a_2^3 + a_2)(a_1+a_2) & \\
\implies s'_0 \geq a_2^3 &
\end{aligned} \tag{32}
$$

   This means that trade $(-a_2, a_1)$ can be applied to (s'_0, t'_0) at least a_2^2 times, so $L_\infty(x,2) \geq t'_0 + a_2^2a_1$. Then $l' := t'_0 + r_1r_2a_1 < t'_0 + a_2^2a_1 \leq L_\infty(x,2)$, meaning $l' \in L_\infty(x,2)$. Furthermore, $l' \equiv t'_0 + (s_0 - d - t'_0)(a_1^{-1})(a_1) \mod a_2 \equiv s_0 - d \mod a_2$. Because $l' + d \equiv s_0 \mod a_2$, $l' + d \in L_\infty(x,1)$, and furthermore, because $d \leq a_1$, $l' + d \in [l', l' + a_1]$. So $l' + d$ is the successor of l' in $\mathscr{L}_\infty(x)$. So $l' + d - l' = d \in \Delta_\infty(x)$.

3. Claim 3: Because $x \geq (a_2^3 + a_2)(a_1 + a_2) > \frac{a_2^2 a_1 + a_2^3}{a_2 - a_1}$,

$$
\begin{aligned}
L_\infty(x,1) &= \frac{x - t_0' a_2}{a_1} \geq \frac{x - a_2^2}{a_1} \\
L_\infty(x,2) &= \frac{x - s_0' a_1}{a_2} \leq \frac{x}{a_2} \\
L_\infty(x,1) - L_\infty(x,2) &\geq \frac{x - a_2^2}{a_1} - \frac{x}{a_2} = \frac{(a_2 - a_1)x - a_2^3}{a_1 a_2} \geq \frac{(a_2 - a_1)\frac{a_2^2 a_1 + a_2^3}{a_2 - a_1} - a_2^3}{a_1 a_2} = a_2
\end{aligned}
\tag{33}
$$

Therefore since $|f_y|_\infty = L_\infty(x,1) > L_\infty(x,2)$ and $|f_{y-1}|_\infty = L_\infty(x,1) - a_2 \geq L_\infty(x,2)$, $|f_y|_\infty, |f_{y-1}|_\infty$ are successive lengths in both $\mathscr{L}_\infty(x)$ and $L_\infty(x,1)$. By the above characterization, $a_2 \in \Delta_\infty(x)$.

4. Claim 4: With $x \geq (a_2^3 + a_2)(a_1 + a_2)$, let $x' := x + a_2^2$. Since x is sufficiently large, by claims 2 and 3, $[1, a_1] \cup \{a_2\} \subseteq \Delta_\infty(x), \Delta_\infty(x + a_2^2)$. By the Δ_∞ characterization, all gaps between two elements of $L_\infty(x,1)$ or two elements of $L_\infty(x',1)$ and all gaps involving least one nonmaximal element of $\mathscr{L}_\infty(x,2)$ or $\mathscr{L}_\infty(x',2)$ lie in this common subset of $\Delta_\infty(x), \Delta_\infty(x')$.

   It thus suffices to show that the gap between $L_\infty(x,2))$ and its successor matches the gap between $L_\infty(x',2)$ and its successor. By the Δ_∞ characterization, this is equivalent to showing $L_\infty(x,2)) - L_\infty(x,1)) \equiv L_\infty(x',2)) - L_\infty(x',1)) \mod a_2$.

   Let $f_y(x)$ and $f_y(x')$ be the factorizations at which $L_\infty(x,1)$ and $L_\infty(x',1)$ occur, respectively. Likewise, let $f_z'(x)$ and $f_z'(x')$ be the factorizations at which $L_\infty(x,2)$ and $L_\infty(x',2)$ occur. Since $x' = x + a_2^2$, clearly $f_y(x') = f_y(x) + (0, a_2) = (s_y, t_y + a_2)$. Then, the trade $(a_2, -a_1)$ can be performed on this factorization an additional $\lfloor \frac{a_2}{a_1} \rfloor$ times compared to those used to span $f_y(x) \sim f_z'(x)$. So $L_\infty(x',1) = L_\infty(x,1) + \lfloor \frac{a_2}{a_1} \rfloor a_2$. Then $L_\infty(x',1) - L_\infty(x',2) = L_\infty(x,1)) + \lfloor \frac{a_2}{a_1} \rfloor a_2 - L_\infty(x,2) - a_2 \equiv L_\infty(x,1)) - L_\infty(x,2)) \mod a_2$.

Claims 5 and 6 use the distance between $L_\infty(x,1)$ and $L_\infty(x,2)$, and rely on the following assumption that $Z(x,1)$ and $Z(x,2)$ are non-empty, and that $L_\infty(x,1) > L_\infty(x,2)$. Then, we are guaranteed to have $f_y = (s_y, t_y) \in Z(x,1)$ such that $|f_y|_\infty = s_y = L_\infty(x,1)$ and $f_z' = (s_z', t_z') \in Z(x,2)$ such that $|f_z'|_\infty = t_z' = L_\infty(x,2)$, with $s_y > t_z'$.

We want to ensure that x is large enough to permit a factorization $f_\star = (s_\star, t_\star)$ with $s_\star > t_\star$, and a factorization $f_\star' = (s_\star', t_\star')$ such that $t_\star' > s_\star'$. Given the trades in our minimal presentation, if $t_\star \geq a_1$, we can perform a trade to decrease $t_\star$ and increase $s_\star$. We must ensure that when $t_\star < a_1$, $s_\star \geq t_\star$. Then we must have $x \geq (a_1 - 1)(a_1 + a_2)$ so that $x - (a_1 - 1)a_2 \geq (a_1 - 1)a_1$.

Similarly, we must ensure that when $s_\star' < a_2$, $t_\star' \geq s_\star'$. It suffices to have $x \geq (a_2 - 1)(a_1 + a_2)$; since $(a_2 - 1)(a_1 + a_2) > (a_1 - 1)(a_1 + a_2)$, this bound is sufficient to ensure that both $Z(x,1)$ and $Z(x,2)$ are nonempty.

Finally, Lemma 3.1.3 ensures that $L_\infty(x,1) > L_\infty(x,2)$. For the result of this Lemma to hold, we must have $x \geq (a_1 + 1)a_2^2$. Since $(a_1 + 1)a_2^2 > (a_2 - 1)(a_1 + a_2)$, this bound meets all necessary conditions.

5. Claim 5: As with claim 4, we will be comparing the gap between $L_\infty(x,2)$ and its successor to the gap between $L_\infty(x + a_2, 2)$ and its successor. (Unlike claim 4, we are not concerned with any lengths less than $L_\infty(x,2)$). If $L_\infty(x,1) - L_\infty(x,2) \equiv n \mod a_2$, $n \in [0, a_2]$,

then the gap between $L_\infty(x,2)$ and its successor (an element of $L_\infty(x,1)$) will also be n. Likewise for $L_\infty(x+a_2,2)$ and its successor. We will prove that if $L_\infty(x,1) - L_\infty(x,2) \equiv n \mod a_2$, then $L_\infty(x+a_2,1) - L_\infty(x+a_2,2) \equiv n-1 \mod a_2$. It suffices to show that $L_\infty(x+a_2,1) - L_\infty(x+a_2,2) \equiv L_\infty(x,1) - L_\infty(x,2) - 1 \mod a_2$.

Let $f_y(x)$ and $f_y(x+a_2)$ be the factorizations of x and $x+a_2$ respectively where $|f_y(x)|_\infty = L_\infty(x,1)$ and $|f_y(x+a_2)|_\infty = L_\infty(x+a_2,1)$. Likewise, let $f_z(x)$ and $f_z(x+a_2$ be the factorizations of x and $x+a_2$ where $|f_z(x)|_\infty = L_\infty(x,2)$ and $|f_z(x+a_2)|_\infty = L_\infty(x+a_2,2)$. Clearly, $f_z(x+a_2) = f_z(x) + (0,1)$, so $L_\infty(x+a_2,2) = L_\infty(x,2)+1$. By applying the same sequence of $(a_2,-a_1)$ trades to $f_z(x+a_2)$ which are required to span $f_z(x) \sim f_y(x)$, we can factor $x+a_2$ as $f_y(x) + (0,1) = (s_y(x), t_y(x)+1)$. (Recall that since $f_y(x) = (s_y(x), t_y(x))$, $|f_y(x)|_\infty = s_y(x) = L_\infty(x,1)$ and $t_y(x) < a_1$, else another trade could be applied to increase $s_y(x)$). Then $|f_y(x+a_2)|_\infty = s_y(x) = L_\infty(x,1)$ iff $t_y(x)+1 < a_1$, and $|f_y(x+a_2)|_\infty = s_y(x) + a_2 = L_\infty(x,1) + a_2$ iff $t_y(x)+1 = a_1$. In both cases, $L_\infty(x+a_2,1) \equiv L_\infty(x,1) \mod a_2$.

Thus $L_\infty(x+a_2,1) - L_\infty(x+a_2,2) \equiv L_\infty(x,1) - (L_\infty(x,2)+1) \mod a_2$. This means that for sufficiently large x with $L_\infty(x,1) - L_\infty(x,2) \equiv n \mod a_2$, $0 \le n \le a_2 - 1$, incrementing x by a_2 will produce a series of gaps $n-1, n-2, \ldots, n-(a_2-1), n, \ldots \pmod{a_2}$. This is sufficient to verify claim 5.

Remark 8. This result also verifies claim 1. Here, we increment x by a_2 to obtain elements with gaps of decreasing size; in the argument for claim 1, elements of the form $(a_2-d)ra_1a_2$ are used to produce gaps of size d (with $1 \le d \le a_2-1$ and $r \equiv a_1^{-1} \mod a_2$). Then, by decreasing d by one, we are producing an element that is ra_1a_2 greater than the previous example. Then since $r \equiv a_1^{-1} \mod a_2$, $ra_1 = ka_2+1$ for some $k \in \mathbb{N}$, so an increase of $ra_1a_2 = (ka_2+1)a_2$ which amounts (using the claim 5 characterization) to k full cycles through the residue classes, and an additional increase by a_2. So we see that the solution implemented by these claims use the same periodic properties of the gap between $L_\infty(x,2)$ and its predecessor.

Additionally, since claim 4 establishes that $\Delta_\infty(x)$ is periodic for x with period a_2^2, the assertion from claim 5 that an increase of a_2 increments one of the elements of the Δ_∞ set (an element which, in many cases, represents a gap of unique size in the length set), we see that a_2 cannot be a period for $\Delta_\infty(x)$. Thus no period which divides a_2^2 is a period for $\Delta_\infty(x)$, so a_2^2 is the smallest period possible.

6. Claim 6: Using the aforementioned assumptions, and letting $f_y(x) = (s_y, t_y)$ with $|f_y(x)|_\infty = s_y = L_\infty(x,1)$ and $f'_z(x) = (s'_z, t'_z)$ with $|f'_z(x)| = t'_z = L_\infty(x,2)$, we can express x as

$$x = s_y a_1 + t_y a_2 = s'_z a_1 + t'_z a_2, \quad 0 \le t_y < a_1, \quad 0 \le s'_z < a_2 \tag{34}$$

Furthermore, since $x \equiv t_y a_2 \mod a_1$, $t_y = x a_2^{-1} \mod a_1$ (where $a_2^{-1} := n$ such that $na_2 \equiv 1 \mod a_1$, $0 \le n < a_1$). Likewise, $x \equiv s'_z a_1 \mod a_2 \implies s'_z = x a_1^{-1} \mod a_2$ (where $a_1^{-1} := m$ such that $ma_1 \equiv 1 \mod a_2$, $0 \le m < a_2$). Since $0 \le t_y \le a_1$ and $0 \le s'_z < a_2$, we can calculate t_y and s'_z explicitly as remainders modulo a_1 and a_2, respectively:

$$t_y = x a_2^{-1} - \left\lfloor \frac{x a_2^{-1}}{a_1} \right\rfloor a_1 \qquad s'_z = x a_1^{-1} - \left\lfloor \frac{x a_1^{-1}}{a_2} \right\rfloor a_2 \tag{35}$$

Then,

$$
\begin{aligned}
L_\infty(x,1) - L_\infty(x,2) = s_y - t'_z &= \frac{x - t_y a_2}{a_1} - \frac{x - s'_z a_1}{a_2} \\
&= \frac{x - (x a_2^{-1} - \lfloor \frac{x a_2^{-1}}{a_1} \rfloor a_1) a_2}{a_1} - \frac{x - (x a_1^{-1} - \lfloor \frac{x a_1^{-1}}{a_2} \rfloor a_2) a_1}{a_2} \\
&\equiv k \mod a_2, \quad 1 \le k \le a_2
\end{aligned}
\tag{36}
$$

(k could also be calculated explicitly in the same fashion as t_y and s'_z, excepting that a_2 be used in place of the zero-residue).

Thus, using only the semigroup element x and the generators a_1 and a_2, the gap produced between the maximum-length $Z(x,2)$ factorization and its predecessor can be precisely calculated. Claim 5 is made redundant by this argument, since it can be seen that substituting $x + a_2$ for x in equation 36 causes k to decrease by 1.

Furthermore, for $x \ge (a_2^3 + a_2)(a_1 + a_2)$, claims 2 and 3 apply, which combine with this result to allow us to explicitly describe $\Delta_\infty(x)$: $\Delta_\infty(x) = [1, a_1] \cup \{a_2\} \cup k$ (where k may or may not be one of the elements already listed).

QEΔ

We also prove a result for Δ_∞ that mirrors Lemma 2.1.4 for Δ_0.

Corollary 3.1.4. For all numerical semigroups $\langle a_1, ..., a_k \rangle$ and $x \in S$, if x is sufficiently large, $1 \in \Delta_\infty(x)$.

Proof. Theorem 3.1.2 covers the case where $k = 2$.

Else, let $m := \max\{|y|_\infty : y \in Ap(S, a_1, a_2)\}$. If x is large, x may be expressed as $k(a_1 + a_2) + y$, where $y \in Ap(S, a_1, a_2)$. So x has a factorization of the form $(k + y_1 - m, k + y_2 - m, 0, ..., 0) + (m, m, y_3, ..., y_k)$, where $(y_1, ..., y_k)$ is a factorization of y. The element e corresponding to $(k + y_1 - m, k + y_2 - m, 0, ..., 0)$ lies in $g\langle \frac{a_1}{g} \frac{a_2}{g} \rangle$, where $g = \gcd(a_1, a_2)$, so by Theorem 3.1.2, e has two factorizations of the forms $(e_1, e_2, 0, ...,)$ and $(e'_1, e_1 - 1, ..., 0)$, with ∞-norms $e_1, e_1 - 1$, respectively. Therefore, x has two factorizations $(e_1 + m, e_2 + m, y_3, ..., y_k) \sim (e'_1 + m, e_1 - 1 + m, ..., 0)$. Because $m \ge y_i$ for all $i \in [1, k]$, and $e_1 + m \ge e_2 + m$; $e'_1 + m \le e_1 - 1 + m$; these factorizations have ∞-norms $e_1 + m, e_1 - 1 + m$, respectively. So $1 \in \Delta_\infty(x)$. QEΔ

3.2 Atlantis and Periodicity

The following example motivates subsequent results.

Example 3.2.1. Let $S := \langle 14, 17, 21 \rangle$. Consider a very large multiple of $14 + 17 + 21 = 52$, $x := 52n$. By construction, $(n, n, n) \in Z(x)$. Furthermore, since we may trade $3 \cdot 14$ for $2 \cdot 21$, $(n + 3, n, n - 2)$, $(n + 6, n, n - 4)$, etc. are in $Z(x)$. Similarly, $14 \cdot 17 = 17 \cdot 14$, so $(n + 17, n - 14, n)$, $(n + 34, n - 28, n)$, etc. are in $Z(x)$. So $n + 3, n + 6, ...,$ and $n + 17, n + 34, ...$ are all lengths in $\mathscr{L}_\infty(x, 1)$. Additionally, by mixing and matching these two trades, we can get other lengths, such as $n + 17 + 3$, in $\mathscr{L}_\infty(x, 1)$.

Since $\langle 3, 17\rangle$ is itself a cofinite numerical semigroup, we eventually get a run of consecutive integers in $\mathscr{L}_\infty(x, 1)$ that lasts as long as trades from the other coordinates are available.

We consider whether the same is true for lengths in $\mathscr{L}_\infty(x, 2)$. Like with the first coordinate, we can apply trades the trades $(17, 0, 0) \sim (0, 14, 0)$ and $(0, 0, 17) \sim (0, 21, 0)$ to obtain factorizations in $\mathscr{L}_\infty(x, 2)$ with ∞-lengths of $n+14, n+28, \ldots$ and $n+21, n+42, \ldots$. However, unlike in the first tower, $\gcd(14, 21) = 7$, so even when combining these trades, one can only trade out of or into the second coordinate in multiples of 7. So 7 is the tightest that lengths in $\mathscr{L}_\infty(x, 2)$ can be clustered together.

The following theorem generalizes and formalizes these ideas, describing the structure of the many towers which can be distinguished through the murky water of the length sets of factorizations.

Definition. Let $S := \langle a_1, ..., a_k\rangle$ be a numerical semigroup with $x \in S$. For all $i \in [1, k]$,

1. Let $g_i := \gcd(\{a_j : i \neq j, j \in [1, k]\})$.
2. Let $\Sigma := \sum\limits_{j=1}^{k} a_j$

Theorem 3.2.2. Atlantis

Let $S := \langle a_1, ..., a_k\rangle$ be a numerical semigroup. There exists bounds b_i, B_i such that $b_i \equiv B_i \equiv 0 \mod g_i$ and for all large x, if $l \in [l_\infty(x, i) + b_i, L_\infty(x, i) - B_i]$ and $l \equiv l_\infty(x, i) \mod g_i$, then $l \in \mathscr{L}_\infty(x, i)$. Furthermore, $l \in \mathscr{L}_\infty(x, i)$ only if $l \equiv l_\infty(x, i) \mod g_i$.

Proof. Fix $i \in [1, k]$. Let $\Sigma_i := \sum\limits_{i\neq j} \frac{a_j}{g_i}$, and let $S_i := \langle\{\frac{a_j}{g_i} : j \neq i\}\rangle$. Set $m := \max(l_\infty(a) : a \in \mathrm{Ap}(S_i, \Sigma_i))$, $b_i := \frac{1}{a_1}(\Sigma_i g_i m + g_i)$ and $B_i := \frac{g_i(F(S_i)+1)}{a_i}$. Suppose $l \in [l_\infty(x, i) + b_i, L_\infty(x, i) - B_i]$ and $l \equiv l_\infty(x, i) \mod g_i$.

Since $a_i l \equiv a_i l_\infty(x, i) \mod g_i$, $x - a_i l \equiv x - a_i l_\infty(x, i) \equiv 0 \mod g_i$. Furthermore, $a_i l \leq a_i(L_\infty(x, i) - \frac{g_i}{a_i}(F(S_i) + 1))$, so $a_i l + g_i(F(S_i) + 1) \leq L_\infty(x, i)a_i \leq x$ gives $x - a_i l \geq g_i(F(S_i) + 1)$, meaning $x - a_i l \in g_i S_i$. So $x - a_i l$ has a factorization in S with ith coordinate 0. To produce a factorization in $Z(x, i)$ with ∞-norm l, it thus suffices to show that $x - a_i l$ has such a factorization in S with ∞-norm at most l.

As an element of gS_i, $x - a_i l$ may be expressed as $g_i(q\Sigma_i + a)$, where $a \in \mathrm{Ap}(S_i, \Sigma_i)$. So $x - a_i l$ has a factorization in S with ith coordinate 0 and ∞-norm at most $q + m$. Since $q \leq \frac{x - a_i l}{g_i \Sigma_i}$, this is at most $\frac{x - a_i l}{g_i \Sigma_i} + m$, or $\frac{x - a_i l + m\Sigma_i g_i}{g_i \Sigma_i}$. Applying the lower bound of $l_\infty(x, i) + b_i$ to l bounds this above by $\frac{x - a_i l_\infty(x,i) - a_i b_i + m\Sigma_i g_i}{g_i \Sigma_i} = \frac{x - a_i l_\infty(x,i) - 1}{g_i \Sigma_i}$.

Let $f := (f_1, ..., f_k)$ be a factorization of x witnessing $|f|_\infty = f_i = l_\infty(x, i)$. Since $l_\infty(x, i)a_i + \sum\limits_{j\neq i} f_j a_j = x$ and $f_j \leq l_\infty(x, i)a_i$ for all $j \neq i$, $x - a_i l_\infty(x, i) = \sum\limits_{j\neq i} f_j a_j \leq \sum\limits_{j\neq i} l_\infty(x, i)a_j = l_\infty(x, i)(g_i\Sigma_i)$. So $\frac{x - a_i l_\infty(x,i) - g_i}{g_i \Sigma_i} \leq \frac{l_\infty(x,i)(g_i\Sigma_i) - g_i}{g_i \Sigma_i} < l_\infty(x, i) < l$, as desired. This proves the first claim.

Suppose $l \in \mathscr{L}_\infty(x, i)$. Then $x - la_i \in g_i S_i$ and $x - l_\infty(x, i) \in g_i S_i$, so $x - la_i \equiv x - l_\infty(x, i)a_i \equiv 0 \mod g_i$. If a_i, g_i are not coprime, S is not cofinite. So a_i, g_i are coprime, meaning a_i is invertible $\mod g_i$, so $l \equiv l_\infty(x, i) \mod g_i$.

Example 3.2.3. Included is a plot of the three length sets of 2000 in $\langle 14, 17, 21\rangle$. Note that the top and bottom of $\mathscr{L}_\infty(x, 1)$ and $\mathscr{L}_\infty(x, 3)$ are sparse, but that both towers quickly become and remain dense. Also note the spacing of the lengths in $\mathscr{L}_\infty(x, i)$–7 apart, which, as expected, is $\gcd(14, 21)$. Said spacing begins immediately, since $F(S_2) = F(2, 3) = 1$.

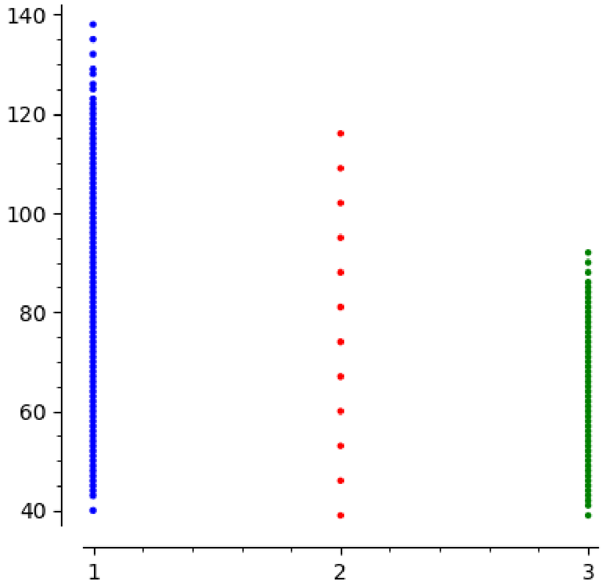


Figure 5: Elements of $\mathscr{L}_\infty(2000, 1)$ (blue), $\mathscr{L}_\infty(2000, 2)$ (red), and $\mathscr{L}_\infty(2000, 1)$ (green), in $\langle 14, 17, 21\rangle$.

Remark 9. For a 2-generated numerical semigroups $\langle a_1, a_2\rangle$, $g_1 = a_2$, $g_2 = a_1$, and $S_1, S_2 = \langle 1\rangle$. As expected, lengths in $\mathscr{L}_\infty(x, 1)$ are spaced apart by a_2, and lengths in $\mathscr{L}_\infty(x, 2)$ are spaced apart by g_1.

While the size of $\mathscr{L}_\infty(x)$ is roughly proportional to x, the Atlantis Theorem (3.2.2) shows that $\mathscr{L}_\infty(x)$ exhibits irregular behavior only in small, constant-bounded regions. This renders the examination of the ∞-delta set of large elements of a numerical semigroup extremely tractable.

Said delta sets are, in fact, periodic. While the proof in its entirety is complicated, its general structure is captured by the following diagram:

In $[\max(l_\infty(x, 1)+b_1, l_\infty(x, 2)+b_2), L_\infty(x, 2)-B_2]$, every length is at most $\min(g_1, g_2)$ away from a different length. This means that all gaps arising from lengths in this region are at most $\min(g_1, g_2)$. When x is large, the dense parts of towers 1 and 2 overlap above all other towers; when x is even larger, these towers do so for long enough that all gaps in $[1, \min(g_1, g_2)]$ are present between consecutive lengths in this region (this argument is reminiscent of that in Theorem 3.1.2 (part 2) for the 2-generated case). Finally, by the Atlantis Theorem (3.2.2), the majority of the portion of tower 1 that lies above tower 2 behaves regularly, with lengths spaced apart by g_1. All in all, these portions of $\mathscr{L}_\infty(x)$ consistently contribute exactly the set $[1, \min(g_1, g_2)] \cup \{g_1\}$ to $\Delta_\infty(x)$.

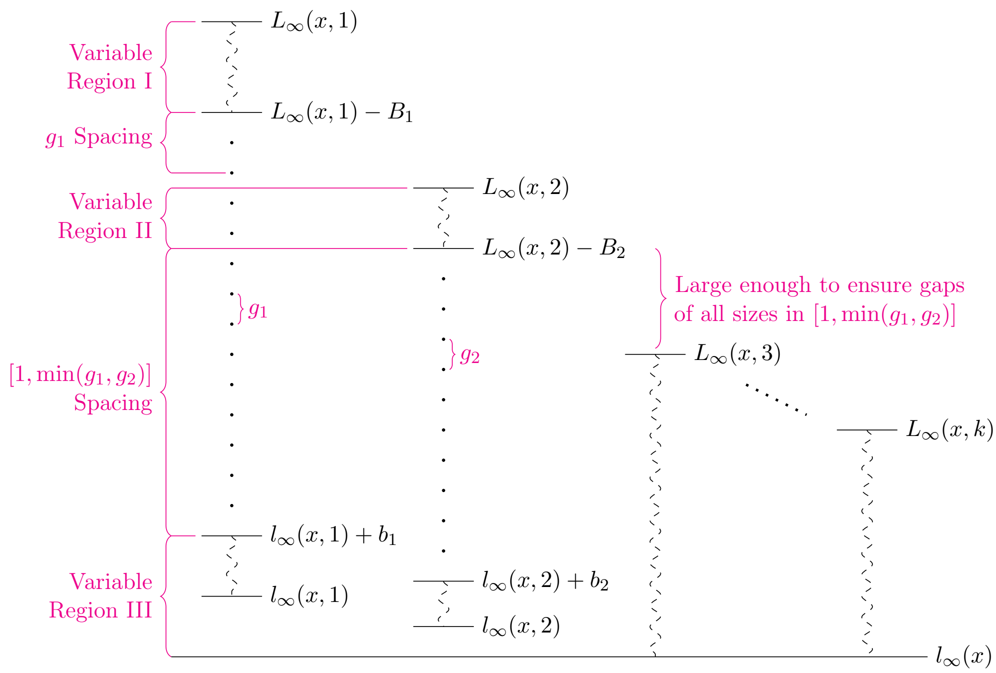


Figure 6: "Tower Diagram" of length set elements for $x > T_p$

This leaves three irregular regions of constant size to account for, labeled in the diagram as "Variable region 1", "Variable region 2", and "Variable region 3". Proving that all three of these are quasilinear completes the proof that the $\Delta_\infty(x)$ is periodic.

The following Lemmasaurus illustrates where each component fits into the logical progression toward this result:

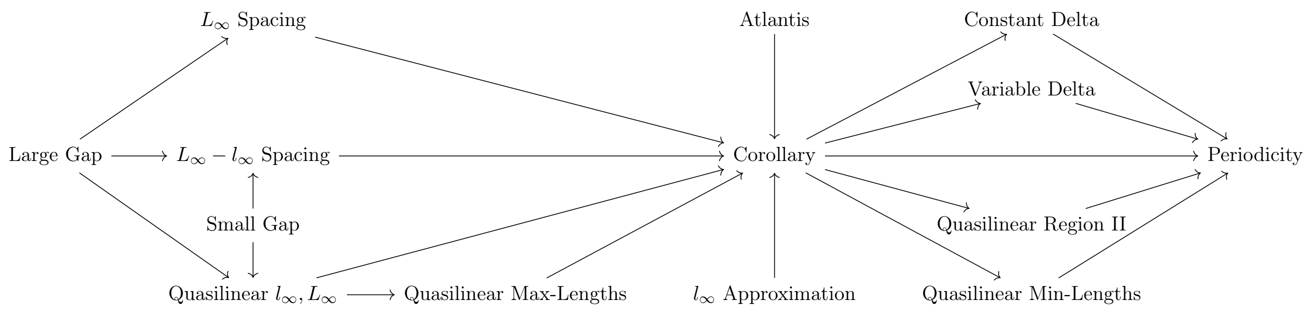


Figure 7: Structure of the results used to prove periodicity of $\Delta_(x)$.

The Small gap, Large gap, $L_\infty - l_\infty$-spacing, L-spacing, l_∞ approximation, Quasilinear L_∞, l_∞, and Quasilinear Max-Lengths lemmas are preliminary lemmas describing the behavior of $l_\infty(x,i)$, $L_\infty(x,i)$, and a few other significant lengths for all elements x exceeding various bounds. Corollary 1 unifies those bounds. The Constant Delta Lemma details the argument that $[1, \min(g_1, g_2)] \cup \{g_1\} \subset \Delta_\infty(x)$ for such large x. The Variable Delta Lemma shows that two consecutive lengths must fall into small, bounded regions of $[l_\infty(x), L_\infty(x)]$ to produce delta values outside the subset described by the Constant Delta Lemma. The Quasilinear Region II and Quasilinear Min-Lengths Lemma, together with results from the Quasilinear Max-Lengths Lemma, prove that those subsets of lengths are quasilinear with respect to x. Together, these five lemmas show that the ∞-delta set is periodic for large x.

We begin with the preliminary lemmas.

The first lemma establishes that the lower regions of each length set are within constant distance of each other, which, among other things, is necessary to show that the size of the irregular region near the bottom of the $\mathscr{L}_\infty$ set (collectively formed by the irregular regions near the bottoms of the individual towers) is truly bounded by a constant.

Lemma 3.2.4. Small Gap

Let $S := \langle a_1, ..., a_k \rangle$ be a numerical semigroup. For all $i \in [1, k]$, there exists p_i and T_{sg} such that for all $x > T_{sg}$, there exists a factorization $f := (f_1, ..., f_k)$ in $Z(x, i)$ with $|f|_\infty = l_\infty(x, i)$ and $|f|_\infty - \min(f_1, ..., f_k) \leq p_i$. Furthermore, $l_\infty(x) \in [l_\infty(x, i) - p_i, l_\infty(x, i)]$

Proof. Let $p_i := 2a_k + 1$, and suppose, for contradiction, that among all factorizations f witnessing $l_\infty(x, i)$, that the smallest gap $g = \min\{|f|_\infty - \min(f_1, ..., f_k) \mid f \text{ a factorization of } x\}$ exceeds p_i. Let f be a factorization of x witnessing $l_\infty(x, i)$ and g with the fewest coordinates equal to $l_\infty(x, i) - g$, and let f_i, f_s, f_t be the largest, second-largest, and smallest coordinates of f, respectively. Suppose $f_i - f_t > p_i$. Either $f_i - f_s > a_k$ or $f_s - f_t > a_k$. If the latter, trading at $a_s a_t$ decreases f_s by at most a_k and increases f_t by at most a_k. The number of coordinates holding the former value of f_t thus decreases; contradicting the choice of f. Else, the former holds, and trading at $a_i a_t$ decreases f_i by at most a_k and increases f_t by at most a_k. Since $f_i - a_k \geq f_s$, this new factorization is still in $Z(x, i)$. It thus has ∞-norm $f_i - a_k$, so it violates minimality of $l_\infty(x, i)$. Therefore, there exists a factorization f with $|f|_\infty = l_\infty(x, i)$ in $Z(x, i)$ with $|f|_\infty - \min(f_1, ..., f_k) \leq p_i$.

Furthermore, let $f' := (f'_1, ..., f'_k)$ witness $l_\infty(x)$. Because $(l_\infty(x, i) - p_i) \sum_{j=1}^{k} a_j \leq x \leq \sum_{j=1}^{k} f'_j a_j \leq l_\infty(x) \sum_{j=1}^{k} a_j$, $l_\infty(x, i) - p_i \leq l_\infty(x)$. So because $l_\infty(x) \leq l_\infty(x, i)$ as well, the second claim is true.

QEΔ

The second lemma approximates $L_\infty(x, i)$ for arbitrary i. Among other things, this helps to approximate lengths in $[L_\infty(x, i) - B_i, L_\infty(x, i)]$.

Lemma 3.2.5. Large Gap

Let $S = \langle a_1, ..., a_k \rangle$ be a numerical semigroup. For all $x \in S$, $\frac{x - k a_k^2}{a_i} < L_\infty(x, i) \leq \frac{x}{a_i}$.

Proof. Let $f := (f_1, ..., f_k)$ witness $L_\infty(x, i)$. If $f_j \geq a_i$ for any $j \neq i$, one could perform the trade at $a_i a_j$ on f to violate the maximality of f. So $f_j < a_i$ for all $j \neq i$.

So $x = \sum_{j=1}^{k} f_j a_j = L_\infty(x, i) a_i + \sum_{j \neq i} f_j a_j$, which, by the above constraint on f_j and the fact that $a_j \leq a_k$ for all j, is less than $L_\infty(x, i) a_i + \sum_{j \neq i} a_j^2 < L_\infty(x, i) a_i + k a_k^2$. So $L_\infty(x, i) > \frac{x - k a_k^2}{a_i}$.

Also, $L_\infty(x,i)a_i \leq L_\infty(x,i)a_i + \sum\limits_{j\neq i} f_j a_j = x$, so $L_\infty(x,i) \leq \frac{x}{a_i}$.

QE∆

The third lemma performs a similar approximation of $l_\infty(x,i)$, which will similarly prove useful in approximating lengths in Variable Region 3.

Lemma 3.2.6. l_∞ Approximation

Let $S = \langle a_1, \ldots, a_k \rangle$ be a numerical semigroup. There exists $c > 0$ such that for all $x \in S$,

$$l_\infty(x) \leq \left\lfloor \frac{x}{\Sigma} \right\rfloor + c$$

Proof. Let $S = \langle a_1, \ldots a_k \rangle$ be a numerical semigroup. Define

$$c = \max(\{\min(\mathscr{L}_\infty(p)) \mid p \in \mathrm{Ap}(S, \Sigma)\}$$

Now, set $x \in S$. Then $x = p + r\Sigma$, where $p \in \mathrm{Ap}(S, \Sigma)$ and $r = \lfloor \frac{x}{\Sigma} \rfloor \geq 0$. x can then be factored as $f_p + f_r = (p_1, \ldots, p_k) + (r, \ldots, r)$, where f_p is a factorization of p with $|f_p|_\infty = \min(\mathscr{L}_\infty(p))$. Then $|f_p + f_r|_\infty = |f_p|_\infty + r = \min(\mathscr{L}_\infty(p)) + r \leq c + \lfloor \frac{x}{\Sigma} \rfloor$, as desired. QE∆

We require the tops and bottoms of towers to be fully disjoint, with ample room for structure between.

Lemma 3.2.7. $L_\infty - l_\infty$-Spacing

Let $S = \langle a_1, ..., a_k \rangle$ be a numerical semigroup. For all positive integers s, there exists G_s such that $x > G_s$ implies $L_\infty(x,i) - l_\infty(x,j) > s$ for all i, j.

Proof. By Lemma 3.2.5, $L_\infty(x,i) > \frac{x - ka_k^2}{a_i}$, and by Lemmas 3.2.4 and 3.2.6, $l_\infty(x,j) \leq l_\infty(x) + p_j \leq \frac{x}{\Sigma} + c + p_j$. Combining inequalities gives $L_\infty(x,i) - l_\infty(x,j) > \frac{x - ka_k^2}{a_i} - (\frac{x}{\Sigma} + c + p_j)$, a linear function of x with leading coefficient $\frac{1}{a_i} - \frac{1}{\Sigma}$. Since this coefficient is positive, the lower bound on $L_\infty(x,i) - l_\infty(x,j)$ is unbounded above and monotonically increasing, meaning it will eventually exceed s and never dip below it again. This completes the proof. QE∆

Finally, we also require that the tops of towers occur in ascending order, with additional room sometimes required between tops of towers. The following lemma enforces this requirement.

Lemma 3.2.8. L-spacing

Let $S = \langle a_1, ..., a_k \rangle$ be a numerical semigroup. For all positive s, there exists T_s such that $x > T_s$ implies $L_\infty(x,i) - L_\infty(x,i+1) > s$ for all $i \in [1, k-1]$.

Proof. By Lemma 3.2.5, $L_\infty(x,i) > \frac{x-ka_k^2}{a_i}$ and $L_\infty(x,i+1) \leq \frac{x}{a_{i+1}}$. Combining both inequalities gives that $L_\infty(x,i)-L_\infty(x,i+1) > \frac{x-ka_k^2}{a_i} - \frac{x}{a_{i+1}}$. This is a linear (affine?) function of x with leading coefficient $\frac{1}{a_i} - \frac{1}{a_{i+1}}$, which is positive. As in Lemma 3.2.7, the lower bound on $L_\infty(x,i)-L_\infty(x,i+1)$ is unbounded above and monotonically increasing, completing the proof.

QE∆

These final two preliminary lemmas help define the boundaries of a mapping used to prove the general quasilinearity of Variable Regions 1, 2, and 3. They do so by showing that the edges of those regions, $l_\infty(x)$, $L_\infty(x,1)$, and $L_\infty(x,2)$, are themselves quasilinear.

Lemma 3.2.9. Quasilinear L_∞, l_∞

Let $S = \langle a_1, ..., a_k \rangle$ be a numerical semigroup and let $i \in [1,k]$. There exists T_q such that $x > T_q$ implies:

1. $l_\infty(x+\Sigma, i) = l_\infty(x,i) + 1$
2. $L_\infty(x+a_i, i) = L_\infty(x,i) + 1$.

Proof. Let $T_q = \max(T_{sg}, (p_i+1)\Sigma, ka_k^2 + a_i a_k)$, where T_{sg} and p_i are from Lemma 3.2.4.

1. Let $f \in Z(x,i)$ witness $l_\infty(x,i)$. Since $f+(1,...,1)$ is a factorization of $Z(x+\Sigma)$, $l_\infty(x,i)+1 \in \mathscr{L}_\infty(x+\Sigma, i)$, so $l_\infty(x+\Sigma,i) \leq l_\infty(x,i)+1$.

   Suppose $l_\infty(x+\Sigma,i) < l_\infty(x,i)+1$. Observe that $l_\infty(x+\Sigma,i) \geq l_\infty(x+\Sigma)$, which, by Lemma 3.2.6, implies $l_\infty(x+\Sigma,i) \geq \frac{x+\Sigma}{\Sigma}$, and, by assumption, $\frac{x+\Sigma}{\Sigma} \geq \frac{T_q}{\Sigma} \geq p_i+1$, so $l_\infty(x+\Sigma,i) \geq p_i+1$. Furthermore, by Lemma 3.2.4, there exists a factorization f' witnessing $l_\infty(x+\Sigma,i)$ with minimal coordinate at least $l_\infty(x+\Sigma,i) - p_i \geq p_i + 1 - p_i = 1$. So $f' - (1,...,1)$ has no negative coordinates, meaning that it is a factorization of x. Additionally, $f' - (1,...,1)$ also has its maximal coordinate at entry i and has ∞-norm $l_\infty(x+\Sigma,i) - 1$, violating minimality of $l_\infty(x,i)$. By contradiction, $l_\infty(x+\Sigma,i) = l_\infty(x,i)+1$.

2. Let $f \in Z(x,i)$ witness $L_\infty(x,i)$. Since $f + (0,...,1_i,...,0)$ is a factorization in $Z(x+a_i,i)$ with ∞-norm $L_\infty(x,i)+1$, $L_\infty(x+a_i,i) \geq L_\infty(x,i)+1$.

   Suppose $L_\infty(x+a_i,i) > L_\infty(x,i)+1$, with $f' = (f_1,...,f_k)$ a factorization of $x+a_i$ witnessing $L_\infty(x+a_i,i)$. As in Lemma 3.2.5, if $f_j \geq a_k$ for any $j \neq i$, the trade at $a_j a_i$ violates the maximality of f'. Furthermore, by Lemma 3.2.5, $L_\infty(x+a_i,i) > \frac{x+a_i-ka_k^2}{a_i}$, which, by assumption, is at least $\frac{ka_k^2+a_ka_i+a_i-ka_k^2}{a_i} = a_k+1$. So $(f_1,...,f_i-1,...,f_k)$ is a factorization of x with $f_i - 1 \geq a_k$ and $f_j \leq a_k$, $j \neq i$. This factorization still has maximal coordinate at entry i, and contradicts maximality of $L_\infty(x,i)$. Thus, $L_\infty(x+a_i,i) = L_\infty(x,i)+1$.

QE∆

Lemma 3.2.10 extends the argument from Lemma 3.2.9 to small intervals near the tops of towers.

Lemma 3.2.10. Quasilinear Max-Length Intervals

Let $S := \langle a_1, \ldots, a_k \rangle$ be a numerical semigroup. For all $i \in [1, k]$, let $M_i(x) := [L_\infty(x, i) - B_i, L_\infty(x, i)] \cap \mathscr{L}_\infty(x, i)$. Then there exists T_m such that $x > T_m$ implies $M_i(x + a_i) = \{l + 1 \mid l \in M_i(x)\}$.

Proof. Let $x' := x + a_i$. We will define a bijection between the largest lengths of x and x'. Let $\phi : M_i(x) \to M_i(x')$ be a function between the largest lengths of x, x' defined by $\phi(l) = l + 1$.

To show that this map is well defined, we need to show that for all $l \in M_i(x)$, $l + 1 \in M_i(x')$ (i.e., that $l + 1 \in \mathscr{L}_\infty(x', i)$ and $l + 1 \in [L_\infty(x', i) - B_i, L_\infty(x', i)]$). $l + 1 \in \mathscr{L}_\infty(x', i)$ since for all $f \in Z(x, i)$, $f' = f + (\ldots, 1_i, \ldots)$ is a factorization of x' that also has maximal coordinate i, meaning that $|f'|_\infty = |f|_\infty + 1 \in \mathscr{L}_\infty(x', 1)$. $l + 1 \in [L_\infty(x', i) - B_i, L_\infty(x', i)]$ because Lemma 3.2.9 gives $L_\infty(x', i) = L_\infty(x, i) + 1$, so $L_\infty(x, i) - B_i \leq l \leq L_\infty(x, i)$ implies

$$L_\infty(x', i) - B_i = L_\infty(x, i) + 1 - B_i \leq l + 1 \leq L_\infty(x, i) + 1 = L_\infty(x', i) \tag{37}$$

Moreover this map, as a translation by a constant, is injective. So it only remains to show that ϕ is surjective.

Suppose l' is in the codomain of ϕ. In order to show that $l' - 1$ is in the domain of ϕ, we must exhibit a factorization of x with maximum coordinate i and length $l' - 1$.

Let $B := \max\{B_j a_j \mid j \in [1, k]\}$ and set $T_m := k a_k^2 + B + \frac{k a_k^2 + B}{a_1} a_k$, so that $x > T_m \implies x' > T_m + a_i$. We claim that for $x' > T_m + a_i$, any factorization $f' \in Z(x', i)$ with length l' satisfies $f'_i > f'_j$ for all coordinates $j \neq i$. Let $r := x' - l' a_i$. Since $L_\infty(x', i) - B_i \leq l' \leq L_\infty(x', i)$,

$$x' - L_\infty(x', i) a_i \leq r \leq x' - (L_\infty(x', i) - B_i) a_i. \tag{38}$$

Furthermore, Lemma 3.2.5 gives that $\frac{x' - k a_k^2}{a_i} < L_\infty(x', i) \leq \frac{x'}{a_i}$. Combining inequalities, we have

$$0 \leq r < x' - \frac{x' - k a_k^2}{a_i} a_i + B_i a_i = k a_k^2 + B_i a_i, \tag{39}$$

which has upper and lower bounds constant for all x. Applying Lemma 3.2.5 again, the maximum-length factorization of r using any coordinate j is bounded above by $\frac{k a_k^2 + B_i a_i}{a_j}$. The maximum-length factorization of r which can be achieved using coordinates other than i is thus $\max\{\frac{k a_k^2 + B_i a_i}{a_j} \mid j \neq i\} \leq \frac{k a_k^2 + B_i a_i}{a_1}$ (less than in the case that $r = 1$, equal otherwise). Therefore, as long as $l' > \frac{k a_k^2 + B_i a_i}{a_1}$, x' will have a maximum length factorization with i as the largest coordinate such that $f'_i > f'_j$ for all $j \neq i$. This is guaranteed as long as $x' \geq k a_k^2 + B_i a_i + (\frac{k a_k^2 + B_i a_i}{a_1} + 1) a_i$, which is true since $x' \geq T_m + a_i = k a_k^2 + B + \frac{k a_k^2 + B}{a_1} a_k + a_i \geq k a_k^2 + B_i a_i + (\frac{k a_k^2 + B_i a_i}{a_1} + 1) a_i$.

Thus, with $x' \geq T_m + a_i$, let $f' \in Z(x', i)$ be a factorization satisfying $|f'|_\infty = l'$, $f'_i > f'_j$ for all $j \neq i$. Then $f = f' - (\ldots, 1_i, \ldots)$ satisfies $f_i \geq f_j$ for all $j \neq i$ (so f has a maximum i^{th} coordinate) and $|f|_\infty = |f'|_\infty - 1 = l' - 1$. Thus $l' - 1$ is in the domain of ϕ, so ϕ is surjective and therefore bijective.

QEΔ

Remark 10. The surjectivity argument in 3.2.10 is weaker than necessary, since the method of selecting T_m ensures that *all* factorizations of x' with maximal i^{th} coordinate and sufficiently large ∞-length have a maximal coordinate strictly greater than all other coordinates. However, it would suffice to show that there exists just *one* factorization of x' that meets this criterion. This proof could probably be re-written and improved upon (with an improved bound) using a Machu Picchu-style argument (see Section 3.3) to show that elements of the "penthouse" portion of the length set towers are all of the form $L_\infty(x, i) - p$ for some integer p. From there, the fact that $L_\infty(x + a_i, i) = L_\infty(x, i) + 1$ provides a concise argument for bijectivity of the map ϕ above.

We need all of these structural properties and some additional conditions to hold simultaneously. The following corollary collects these properties and provides a single lower bound for which they all hold.

Corollary 3.2.11. Bounds Required for Δ_∞ Periodicity

For all numerical semigroups S, there exists T_p such that if $x > T_p$:

1. Lemma 3.2.8 applies to give:
   a) $L_\infty(x, 1) - B_1 - 2g_1 > L_\infty(x, 2)$
   b) $L_\infty(x, 2) - B_2 - g_1 g_2 - 4g_2 - g_1 > L_\infty(x, 3)$
   c) $L_\infty(x, i) \leq L_\infty(x, 3)$ for $i \geq 3$.
2. Lemma 3.2.7 applies to give $L_\infty(x, 3) > l_\infty(x, 1) + b_1 + g_1, l_\infty(x, 2) + b_2 + g_2$
3. Theorem 3.2.2 applies.
4. Lemmas 3.2.4, 3.2.5, 3.2.9, 3.2.10 apply.
5. $\frac{(\frac{x}{\Sigma}+c)a_k - \max(p_1+b_1, p_2+b_2)a_k}{\Sigma} \geq 1 + a_k$.

Where c is from Lemma 3.2.6, p_1, p_2 are from Lemma 3.2.4, and b_1, b_1, B_1, B_2 are from Theorem 3.2.2.

Proof. Lemmas 3.2.4, 3.2.5, 3.2.8, 3.2.9, 3.2.7, 3.2.10 and Theorem 3.2.2 all give bounds that are constant across S, so we may set T_q larger than all of them. This leaves the final inequality to satisfy. Since the left hand side of that inequality is a linear function of x with a positive leading coefficient ($\frac{1}{\Sigma^2}$) and the right hand side is a constant, we may satisfy this inequality by making T_q large as well.

QEΔ

We are now ready to proceed with the major components leading directly into the main periodicity result. We begin by examining a set of values that consistently appear in the ∞-delta set of large x.

Lemma 3.2.12. Constant Delta

Let $S := \langle a_1, ..., a_k \rangle$ be a numerical semigroup. For all $x > T_p$ (from Corollary 3.2.11) $[1, \min(g_1, g_2)] \cup \{g_1\} \subseteq \Delta_\infty(x)$.

Proof. Let x be large, and let l be the second-smallest length such that

$$l \in (L_\infty(x,3) + g_1, L_\infty(x,2) - B_2 - g_1 g_2 - g_2] \cap \mathscr{L}_\infty(x,2).$$

Since Corollary 3.2.11 (part 1(b)) gives $L_\infty(x,3) < L_\infty(x,2) - B_2 - g_1 g_2 - 4g_2 - g_1 < L_\infty(x,2) - B_2 - g_1 g_2 - g_2$, the size of the right hand interval is at least $3g_2$ and is wholly contained in $[l_\infty(x,2) + b_2, L_\infty(x,2) - B_2]$. By Theorem 3.2.2, there exist two lengths in that interval in $\mathscr{L}_\infty(x,2)$, so l exists. Furthermore, by Corollary 3.2.11 (parts 1(b) and (c)), l exceeds $L_\infty(x,3), ..., L_\infty(x,k)$. By Corollary 3.2.11 (part 3), we can apply Theorem 3.2.2 to define $L := \{l, l+g_2, l+2g_2, ..., l+g_1 g_2\} \subseteq \mathscr{L}_\infty(x,2)$. If g_1, g_2 are not coprime, $\gcd(a_1, ..., a_k) > 1$. So g_1, g_2 are coprime, meaning L represents all residue classes modulo g_1. Let l' be a length in L where the residue class of $l' - l_\infty(x,1)$ modulo g_1 has a least positive representative r that is at most $\min(g_1, g_2)$.

We claim that $l' - r$ is the predecessor of l' in $\mathscr{L}_\infty(x)$. Since $l' - r > l - g_1 > L_\infty(x,3)$, either $l' - r \in \mathscr{L}_\infty(x,1)$ or $l' - r \in \mathscr{L}_\infty(x,2)$. Note that $l' \in [l, l + g_1 g_2] \subseteq (L_\infty(x,3), L_\infty(x,2))$. By Corollary 3.2.11 (part 1(a)), $(L_\infty(x,3), L_\infty(x,2)) \subset [l_\infty(x,1) + b_1 + g_1, L_\infty(x,1) - B_1]$. Furthermore, $l' - r > L_\infty(x,3) > l_\infty(x,1) + b_1 + g_1$ as a result of Corollary 3.2.11 (part 2). Altogether, we have $l' - r \in [l_\infty(x,1) + b_1, L_\infty(x,1) - B_1]$. These bounds, combined with the fact that $l' - r \equiv l_\infty(x,1)$ mod g_1, give (by Corollary 3.2.11 part 3; Theorem 3.2.2) that $l' - r$ is an element of $\mathscr{L}_\infty(x,1)$. Then, since $l' - (l' - r) = r \leq g_1$, and all elements of $\mathscr{L}_\infty(x,1)$ bounded by $[l_\infty(x,1) + b_1, L_\infty(x,1) - B_1]$ are spaced exactly g_1 apart (see Theorem 3.2.2, applied through Corollary 3.2.11 part 3), there can be no other element of $\mathscr{L}_\infty(x,1)$ between l' and $l' - r$; thus $l' - r$ is the predecessor of l' in $\mathscr{L}_\infty(x,1)$.

Now, recall that $l' \in \mathscr{L}_\infty(x,2)$ and $l' \in [l_\infty(x,2) + b_2, L_\infty(x,2) + B_2]$. So the predecessor of l' in $\mathscr{L}_\infty(x,2)$ is $l' - g_2 \leq l' - r$. Therefore, $l' - r$ is the predecessor of l' in $\mathscr{L}_\infty(x)$, meaning the gap between l' and its predecessor is exactly r. So for all $r \in [1, \min(g_1, g_2)]$, $r \in \Delta_\infty(x)$.

Furthermore, let the largest element of $\mathscr{L}_\infty(x,1)$ less than $L_\infty(x,1) - B_1$ be l'. By Corollary 3.2.11, and Theorem 3.2.2, $l' - g_1 \in L_\infty(x,1)$ and is larger than $L_\infty(x,2)$. So the predecessor of l' is $l' - g_1$, so $g_1 \in \Delta_\infty(x)$.

QE∆

As previously stated, this result classifies gaps contained within the majority of $\mathscr{L}_\infty(x)$. The following lemma formalizes which regions remain.

Lemma 3.2.13. Variable Deltas

Let $S := \langle a_1, ..., a_k \rangle$ be a numerical semigroup. For all $x > T_p$ and consecutive lengths $l, l' \in \mathscr{L}_\infty(x)$, $l < l'$, either

1. $l, l' \geq L_\infty(x,1) - B_1$,
2. $l' - l = g_1$,
3. $l \in [L_\infty(x,2) - B_2, L_\infty(x,2)]$,
4. $l' - l \leq \min(g_1, g_2)$, or
5. $l, l' \leq \max(l_\infty(x,1) + b_1, l_\infty(x,2) + b_2)$.

Proof. Recall that $L_\infty(x,1) - B_1$, $L_\infty(x,2) - B_2$, $l_\infty(x,1) - b_1$, and $l_\infty(x,2) - b_2$ are factorizations of x. Then, either...

1. $l \geq L_\infty(x,1) - B_1 \implies l' > L_\infty(x,1) - B_1$, so $l, l' \geq L_\infty(x,1) - B_1$,
2. $l \in (L_\infty(x,2), L_\infty(x,1) - B_1) \implies l' \in (L_\infty(x,2), L_\infty(x,1) - B_1]$,
3. $l \in [L_\infty(x,2) - B_2, L_\infty(x,2)]$,
4. $l \in [\max(l_\infty(x,1) + b_1, l_\infty(x,2) + b_2), L_\infty(x,2) - B_2)$ $\implies l' \in (\max(l_\infty(x,2) + b_2, l_\infty(x,1) + b_1), L_\infty(x,2) - B_2]$, or
5. $l < \max(l_\infty(x,1) + b_1, l_\infty(x,2) + b_2) \implies l' \leq \max(l_\infty(x,1) + b_1, l_\infty(x,2) + b_2)$, so $l, l' \leq \max(l_\infty(x,1) + b_1, l_\infty(x,2) + b_2)$.

These cases correspond to the five cases defined in the lemma. Cases 1, 3, and 5 need no further development; we will consider cases 2 and 4.

For case 2, if $l, l' \in (L_\infty(x,2), L_\infty(x,1) - B_1]$, $l, l' \in \mathscr{L}_\infty(x,1)$, and by Theorem 3.2.2, $l - l' = g_1$.

Next suppose, as in case 4, that $l, l' \in [\max(l_\infty(x,2)+b_2, l_\infty(x,1)+b_1), L_\infty(x,2)-B_2]$ are consecutive lengths in $\mathscr{L}_\infty(x)$, with $l < l'$. By Theorem 3.2.2, all values in this range that are congruent to $l_\infty(x,1) \mod g_1$ or $l_\infty(x,2) \mod g_2$ also lie in $\mathscr{L}_\infty(x)$. So $l - l' \leq \min(g_1, g_2)$. QE$\Delta$

Remark 11. These two lemmas show that the only gaps that vary $\Delta_\infty(x)$ for $x > T_p$, where T_p is from Corollary 3.2.11 (i.e., gaps potentially outside $[1, \min(g_1, g_2)] \cup \{g_1\}$) are...

- Gaps in $[L_\infty(x,1) - B_1, L_\infty(x,1)]$ (gaps between near-max lengths in $\mathscr{L}_\infty(x,1)$ - Variable Region I),
- Gaps in $[L_\infty(x,2) - B_2, L_\infty(x,2)]$, along with $L_\infty(x,2)$ and its successor (gaps between/after near-max lengths in $\mathscr{L}_\infty(x,2)$ - Variable Region II), and
- Gaps in $[l_\infty(x), \max(l_\infty(x,2) + b_2, l_\infty(x,1) + b_1)]$ (gaps between near-min lengths - Variable Region III).

(These regions correspond to items 1, 3, and 5 of Lemma 3.2.13, respectively).

The quasilinearity of Variable Region I follows from Lemma 3.2.10. It remains to show that Variable Regions II and III are quasilinear.

Lemma 3.2.14. Quasilinearity of Variable Zone II

Let $S := \langle a_1, ..., a_k \rangle$ be a numerical semigroup. For all x such that $x > T_p$, let $V(x) := [L_\infty(x,2) - B_2, L_\infty(x,2)] \cap \mathscr{L}_\infty(x)$. Then $V(x + a_1 g_1 a_2) = \{l + a_1 g_1 \mid l \in V(x)\}$. Furthermore, the gap between $L_\infty(x,2)$ and its successor is equal to the gap between $L_\infty(x + a_1 g_1 a_2, 2)$ and its successor.

Proof. Let $x > T_p$ and $x' := x + a_1 g_1 a_2$. Since $x > T_p$, Corollary 3.2.11 (part 1) gives that $L_\infty(x,i) < L_\infty(x,2) - B_2$ for all $i > 2$, so we know that $l \in V(x)$ implies that l is an element of $\mathscr{L}_\infty(x,1)$ or $\mathscr{L}_\infty(x,2)$. Likewise, $l' \in V(x')$ is an element of $\mathscr{L}_\infty(x',1)$ or $\mathscr{L}_\infty(x',2)$.

We will provide a bijection in the form of a translation between lengths in $V(x)$ and $V(x')$. Let $\phi : V(x) \to V(x')$ be defined as $\phi(l) = l + a_1 g_1$. To show that this map is well defined, let $l \in V(x)$. Corollary 3.2.11 allows us to apply Lemma 3.2.9, by which $L_\infty(x',2) = L_\infty(x,2) + a_1 g_1$, so $L_\infty(x,2) - B_2 \leq l \leq L_\infty(x,2)$ implies

$$L_\infty(x',2) - B_2 = L_\infty(x,2) - B_2 + a_1 g_1 \leq l + a_1 g_1 \leq L_\infty(x,2) + a_1 g_1 = L_\infty(x',2). \qquad (40)$$

So $l + a_1g_1 \in [L_\infty(x',2) - B_2, L_\infty(x',2)]$. If $l \in \mathscr{L}_\infty(x,2)$, Lemma 3.2.10 gives that $l + a_1g_1 \in \mathscr{L}_\infty(x',2)$. So $\phi(l) = l + a_1g_1 \in V(x')$. Otherwise, if $l \in \mathscr{L}_\infty(x,1)$ (and $l \notin \mathscr{L}_\infty(x,2)$),

$$l_\infty(x,1) + b_1 < L_\infty(x,2) - B_2 \leq l \leq L_\infty(x,2) < L_\infty(x,1) - B_1 \tag{41}$$

(by Corollary 3.2.11 parts 1 and 2), so Lemma 3.2.2 gives that $l \equiv l_\infty(x,1) \mod g_1$. Then, x, $l_\infty(x,1)$, l, $l+a_1g_1$, x', and $l_\infty(x',1)$ are all equivalent modulo g_1, and Lemma 3.2.2 ensures that since $l+a_1g_1 \equiv l_\infty(x',1) \mod g_1$ and $l+a_1g_1 \in [l_\infty(x',1)+b_1, L_\infty(x',1)-B_1]$, $l+a_1g_1 \in \mathscr{L}_\infty(x',1)$, so $\phi(l) = l + a_1g_1 \in V(x')$.

We have shown that ϕ is well defined; additionally, since ϕ is a translation by a constant, it is injective. It remains to show that ϕ is surjective. The argument is essentially the same as the argument for well-definition:

Suppose $l' \in V(x')$. We need to show that $l' - a_1g_1 \in V(x)$. As in the well-definition argument, Corollary 3.2.11 allows us to apply Lemma 3.2.10, by which $L_\infty(x',2) - B_2 \leq l' \leq L_\infty(x',2)$ implies

$$L_\infty(x,2) - B_2 = L_\infty(x',2) - B_2 - a_1g_1 \leq l' - a_1g_1 \leq L_\infty(x',2) - a_1g_1 = L_\infty(x,2), \tag{42}$$

so $l' - a_1g_1 \in [L_\infty(x,2) - B_2, L_\infty(x,2)]$. If $l' \in \mathscr{L}_\infty(x',2)$, then Lemma 3.2.10 also gives that $l' - a_1g_1 \in \mathscr{L}_\infty(x,2)$, so $l' - a_1g_1 \in V(x)$. Otherwise, if $l' \in \mathscr{L}_\infty(x',1)$ (and $l' \notin \mathscr{L}_\infty(x',2)$), Lemma 3.2.2 gives $l' \equiv l_\infty(x',1) \mod g_1$. Then, x', $l_\infty(x',1)$, l', $l' - a_1g_1$, x, and $l_\infty(x,1)$ are all equivalent modulo g_1, so Lemma 3.2.2 ensures that since $l' - a_1g_1 \equiv l_\infty(x) \mod g_1$ and $l' - a_1g_1 \in [l_\infty(x,1) + b_1, L_\infty(x,1) - B_1]$, $l' - a_1g_1 \in \mathscr{L}_\infty(x,1)$, so $l' - a_1g_1 \in V(x)$. Therefore there exists $l' - a_1g_1 \in V(x)$ such that $\phi(l' - a_1g_1) = l' \in V(x')$. So ϕ is surjective, and therefore bijective. Thus $\phi(l) = l + a_1g_1$ is a bijective function from $V(x) \to V(x')$, so $V(x') = \{l + a_1g_1 \mid l \in V(x)\}$.

The last thing to prove is that the gap between $L_\infty(x,2)$ and it's successor is the same as the gap between $L_\infty(x + a_1g_1a_2, 2)$ and it's successor:

Let $x' := x + a_1g_1a_2$. The successor L' of $L_\infty(x',2)$ is in $\mathscr{L}_\infty(x',1)$, since, by Corollary 3.2.11 (part 1), $L_\infty(x',2)$ is larger than all other lengths besides those in $\mathscr{L}_\infty(x',1)$. Also from Corollary 3.2.11 (part 1(a)), $L_\infty(x',2) < L_\infty(x',1) - B_1 - 2g_1$, so by Theorem 3.2.2, $\mathscr{L}_\infty(x',1)$ contains all lengths congruent modulo g_1 to $l_\infty(x',1)$ in $[L_\infty(x',2), L_\infty(x',2) + g_1]$. So L' is the smallest number after $L_\infty(x',2)$ congruent to $l_\infty(x',1)$ modulo g_1, meaning $L' - L_\infty(x',2)$ is the least positive representative of the residue class of $l_\infty(x',1) - L_\infty(x',2)$ modulo g_1. Since $l_\infty(x',1) \equiv L_\infty(x',1) \mod g_1$ and, by Lemma 3.2.9, $L_\infty(x',1) = L_\infty(x,1) + g_1a_2$ and $L_\infty(x',2) = L_\infty(x,2) + a_1g_1$, this residue is equivalent to $L_\infty(x,1) + g_1a_2 - L_\infty(x,2) - a_1g_1 \mod g_1$. Since g_1a_2 and a_1g_2 are both multiples of g_1, this is equivalent to $L_\infty(x,1) - L_\infty(x,2) \mod g_1$, the gap between $L_\infty(x,2)$ and its successor. So this gap is identical in x, x'.

QEΔ

Lemma 3.2.15. Quasilinearity of Min-Length Intervals (Variable Region III)

Let $S := \langle a_1, ..., a_k \rangle$ be a numerical semigroup. For all x such that $x > T_p$, let $V(x) := [l_\infty(x), \max(l_\infty(x,2) + b_2, l_\infty(x,1) + b_1)] \cap \mathscr{L}_\infty(x)$. Then $V(x + \Sigma) = \{l + 1 \mid l \in V(x)\}$.

Proof. Let $x' := x + \Sigma$. We will define a bijection between the sets of smallest lengths of x, x'. Let $\rho : V_1(x) \to V_1(x')$ be defined by $\rho(l) = l + 1$.

To show that this map is well-defined, we need to show that $l+1 \in V_2(x')$, i.e., that $l+1 \in \mathscr{L}(x)$ and $l'+1 \in [l_\infty(x'), \max(l_\infty(x,2)+b_2, l_\infty(x,1)+b_1)]$. Because for all $f \in Z(x)$, $f+(1,...,1)$ is a factorization of x' with ∞-norm $|f|_\infty + 1$, for all $l \in \mathscr{L}_\infty(x)$, $l+1 \in \mathscr{L}_\infty(x')$. Furthermore, Corollary 3.2.11 allows us to apply Lemma 3.2.9, which states that $l_\infty(x',1) = l_\infty(x,1)+1$ and $l_\infty(x',2) = l_\infty(x,2)+1$. Thus if $l \le \max(l_\infty(x,2)+b_2, l_\infty(x,1)+b_1)$ is in the domain, $l+1 \le \max(l_\infty(x',2)+b_2, l_\infty(x',1)+b_1)$ is in the codomain. Moreover, this map, as a translation, is injective. So it only remains to show that ρ is surjective.

Suppose l is in the codomain of ρ, with the factorization $f \in Z(x',i)$ witnessing, where $f := (f_1,...,f_k)$ is such a factorization with the fewest entries equal to 0. We will show that $f-(1,...,1)$, is a valid factorization of x that yields a preimage of l. If f has no entries equal to 0, $f-(1,...,1)$ is a vector in $\mathbb{N}^k$, and $f-(1,...,1)$ is a valid factorization of x whose length $l-1$ is a preimage of l. Else, we will derive a contradiction by finding a large entry of f (not at coordinate i) that will allow us to decrease the number of small entries in f. Let $m := \max\{f_j : i \neq j\}$ occur at coordinate n. Since f is a factorization of x',

$$x' = \sum_{j=1}^{k} f_j a_j \le l a_i + \sum_{j \neq i} m a_j = l a_i + m\Bigg(\sum_{i \neq j} a_j\Bigg) \le l a_k + m\Bigg(\sum_{i \neq j} a_j\Bigg).$$

By assumption l is at most $\max(l_\infty(x',1)+b_1, l_\infty(x',2)+b_2)$, and by the p_1, p_2 bounds on $l_\infty(x')$ given by Lemma 3.2.4, this is at most $l_\infty(x') + \max(p_1+b_1, p_2+b_2)$. Then

$$x' \le l a_k + m\Bigg(\sum_{i \neq j} a_j\Bigg) \le (l_\infty(x') + \max(p_1+b_1, p_2+b_2))a_k + m\Bigg(\sum_{i \neq j} a_j\Bigg).$$

So $x' - l_\infty(x')a_k - \max(p_1+b_1, p_2+b_2)a_k \le m(\sum_{i \neq j} a_j) \le m\Sigma$. By Lemma 3.2.6, $l_\infty(x') \le \frac{x'}{\Sigma} + c$. Substituting $\frac{x}{\Sigma} + c$ for $l_\infty(x')$, we obtain

$$m \ge \frac{(\frac{x'}{\Sigma} + c)a_k - \max(p_1+b_1, p_2+b_2)a_k}{\Sigma} \ge 1 + a_k,$$

where the second inequality follows from Corollary 3.2.11 (part 5). If there exists a coordinate, j, with 0 as an entry, we can then trade at $a_n a_j$ to eliminate the 0 at that coordinate without dropping m below 1. This, as mentioned, contradicts the choice of f by producing a factorization with fewer 0 coordinates. So f cannot have any entries less than 1. So $f-(1,...,1)$ gives a preimage of l with respect to ρ, so ρ is surjective. QEΔ

Finally, having proven that all of $\mathscr{L}_\infty(x)$ is either predictably-spaced or quasilinear, the periodicity of $\Delta_\infty(x)$ can be proven.

Theorem 3.2.16. Periodicity

Let $S := \langle a_1,...,a_k\rangle$ be a numerical semigroup and let $p := \operatorname{lcm}(a_1 g_1 a_2, \Sigma)$. Let $x \in S$ be such that $x > T_p$. Then $\Delta_\infty(x+p) = \Delta_\infty(x)$.

Proof. Suppose l, l' are consecutive lengths in $\mathscr{L}_\infty(x+p)$, with $l < l'$. Let $V_1(x+p) = [L_\infty(x+p,1) - B_1, L_\infty(x+p)] \cap \mathscr{L}_\infty(x+p)$ (Variable Region I), $V_2(x+p) = [L_\infty(x+p,2) - B_2, L_\infty(x+p,2)] \cap \mathscr{L}_\infty(x+p)$ (Variable Region II), and $V_3(x+p) = [l_\infty(x+p), \max(l_\infty(x+p,1)+b_1, l_\infty(x+p,2)+b_2)] \cap \mathscr{L}_\infty(x+p)$ (Variable Region III). Let $V_1(x)$, $V_2(x)$, $V_3(x)$ be defined likewise.

By Lemma 3.2.13, either $l' - l \in [1, \min(g_1, g_2)] \cup \{g_1\}$ or one of the following is true:

1. $l, l' \geq L_\infty(x+p,1) - B_1 \implies l, l' \in V_1(x+p)$,
2. $l \in [L_\infty(x+p,2) - B_2, L_\infty(x+p,2)]$, or
3. $l, l' \leq \max(l_\infty(x+p,1) + b_1, l_\infty(x+p,2) + b_2) \implies l, l' \in V_3(x+p)$.

The second case can be further divided into two distinct possibilities by the fact that $L_\infty(x+p,2)$ is a factorization of $x+p$, so if $l < L_\infty(x+p,2)$, $l' \in V_2(x+p)$:

2. $l \in [L_\infty(x+p,2) - B_2, L_\infty(x+p,2)]$ implies...
   a) $l, l' \in V_2(x+p)$, or
   b) $l = L_\infty(x+p,2)$

A similar classification holds for consecutive lengths in x. Furthermore, by Lemma 3.2.12, $[1, \min(g_1, g_2)] \cup \{g_1\} \subseteq \Delta_\infty(x), \Delta_\infty(x+p)$. So to show that $\Delta_\infty(x) = \Delta_\infty(x+p)$, it then suffices to show that:

1. If l, l' are consecutive in $V_1(x+p)$, $l - \frac{p}{a_1}, l' - \frac{p}{a_1}$ are consecutive in $V_1(x)$, and if l, l' are consecutive in $V_1(x)$, $l + \frac{p}{a_1}, l' + \frac{p}{a_1}$ are consecutive in $V_1(x+p)$. This is given by Lemma 3.2.10 (through Corollary 3.2.11), since p is a multiple of a_1.
2. a) If l, l' are consecutive in $V_2(x+p)$, $l - \frac{p}{a_1 g_1 a_2}(a_1 g_1), l' - \frac{p}{a_1 g_1 a_2}(a_1 g_1)$ are consecutive in $V_2(x)$, and if l, l' are consecutive in $V_2(x)$, $l + \frac{p}{a_1 g_1 a_2}(a_1 g_1), l' + \frac{p}{a_1 g_1 a_2}(a_1 g_1)$ are consecutive in $V_2(x+p)$. This is given by Lemma 3.2.14, since p is a multiple of $a_1 g_1 a_2$.
   b) The gap between $L_\infty(x+p,2)$ and its successor equals the gap between $L_\infty(x,2)$ and its successor. This is also given by Lemma 3.2.14, since p is a multiple of $a_1 g_1 a_2$.
3. If l, l' are consecutive in $V_3(x+p)$, $l - \frac{p}{\Sigma}, l' - \frac{p}{\Sigma}$ are consecutive in $V_3(x)$, and if l, l' are consecutive in $V_3(x)$, $l + \frac{p}{\Sigma}$, $l' + \frac{p}{\Sigma}$ are consecutive in $V_3(x+p)$. This is given by Lemma 3.2.15, since p is a multiple of Σ.

Therefore $\Delta_\infty(x+p) = \Delta_\infty(x)$.

QEΔ

3.3 Machu Picchu

One wonders, after seeing the periodicity result, whether any of the three variable regions may be more precisely characterized. The following section does so for Variable Region 1, as well as the "g_1 Spacing" region, by describing the behavior of the upper regions of each length tower for large elements of a numerical semigroup. We call the main result "Machu Picchu" because it describes the very top of the ∞-length set for sufficiently large elements in any numerical semigroup.

We first describe a specific element of Variable Region 1 for elements which are are multiples of a_1 and sufficiently large. The appropriate size is detailed in the proof of the theorem.

Theorem 3.3.1. Let, $x \in S = \langle a_1, ..., a_n \rangle$, $L'_\infty(x) = \max(\mathscr{L}_\infty(x) - \{L_\infty(x)\})$, and $\delta(x) = L_\infty(x) - L'_\infty(x) \in \Delta_\infty(x)$. Furthermore, let $M_\delta(S) = \frac{\beta}{a_1}$, where β is the smallest Betti element divisible by a_1. Then for sufficiently large x, $\delta(x) \leq M_\delta(S)$ with equality if $a_1 | x$, implying $M_\delta(S) \in \Delta_\infty(S)$.

Proof. Let $f_1 = (k, 0, 0, ...0)$ and f_2 be factorizations of β such that $f_1 \cdot f_2 = 0$. Let $x \in S$ such that $x = q_1 a_1$. Writing $x = q_i a_i + r_i$, also assume x is sufficiently large such that $q_1 - M_\delta(S) > q_i$ for each $i \in [2, n]$. It follows that $(q_1, 0, 0, ..., 0) \sim (q_1 - M_\delta(S), 0, 0, ..., 0) + f_2$. Because $c_i < q_i$ for the ith component c_i of f_2, we have $q_1, (q_1 - M_\delta(S)) \in \mathscr{L}_\infty(x)$.

Now suppose, for contradiction, there exists $q_1 - j \in \mathscr{L}_\infty(x)$ with $q_1 - M_\delta < q_1 - j < q_1$. Therefore, there is a factorization $(q_1 - j, \alpha_2, ..., \alpha_n)$ of x. Note that $\gcd((q_1, 0, 0, ..., 0), (q_1 - j, \alpha_2, ..., \alpha_n)) = (q_1 - j, 0, 0, ..., 0)$. It follows that the trade $(j, 0, ..., 0) \sim (0, \alpha_2, ..., \alpha_n)$ occurs at a Betti element $\beta' = j a_1$. Because $j < M_\delta(S)$, $\beta' < \beta$, contradicting our choice of β in the defintion of $M_\delta(S)$. Thus, $q_1 = L_\infty(x)$ and $q_1 - M_\delta(S) = L'_\infty(x)$, so for $x \in S$ with $a_1 | x$, $\delta(x) = M_\delta(S)$.

Noting that $f_1 \sim f_2$ is a trade that can be applied for sufficiently large x with $a_1 \nmid x$, we see $\delta(x) < M_\delta(S)$ in general. QEΔ

Noting that $2a_1$ cannot be a Betti element, since otherwise a_1 would not be the smallest generator, we have the following corollary.

Corollary 3.3.2. For all numerical semigroups S, $\max(\Delta_\infty(S)) \geq M_\delta(S) \geq 3$

As a preliminary to the main result, we give a quick proof that the gaps between elements in Variable Region 1 and the "g_1 Spacing" region, are gaps in the overall Δ_∞-set.

Theorem 3.3.3. Let $x \in S = \langle a_1, ..., a_k \rangle$ and let $Z'(x, 1) = \{z \in Z(x, 1) : |z|_\infty > L_\infty(x, 2)\}$. Then $\Delta_\infty(Z'(x, 1)) \subseteq \Delta_\infty(x)$.

Proof. Let $\mathscr{L}'_\infty(x, 1) = \{\ell \in \mathscr{L}_\infty(x, 1) : \ell > L_\infty(x, 2)\}$. This is the set of lengths of factorizations in $Z'(x, 1)$. We must show that $\mathscr{L}'_\infty(x, 1)$ is a consecutive subset of $\mathscr{L}_\infty(x)$, i.e., that if $\ell' > \ell'' > \ell$ for some $\ell, \ell' \in \mathscr{L}'_\infty(x, 1)$ and $\ell'' \in \mathscr{L}_\infty(x)$, then $\ell'' \in \mathscr{L}'_\infty(x, 1)$ as well. Because $\ell'' > \ell > L_\infty(x, 2)$, $\ell'' > L_\infty(x, 2)$. Furthermore, since $L_\infty(x, 2) \geq L_\infty(x, i)$ for $i \in [2, k]$, we have $\ell'' \notin \mathscr{L}_\infty(x, i)$ for $i \in [2, k]$. Thus, $\ell'' \in \mathscr{L}_\infty(x, 1)$, implying $\ell'' \in \mathscr{L}'_\infty(x, 1)$. We conclude $\Delta_\infty(Z'(x, 1)) \subseteq \Delta_\infty(x)$. QE$\Delta$

We are now ready to present the main result characterizing Variable Region 1 and the "g_1 Spacing" region.

Theorem 3.3.4. Machu Picchu

Let $x \in S = \langle a_1, ..., a_k \rangle$ such that $x \equiv r \mod a_1$ where $r \in \mathrm{Ap}(S)$ as well as sufficiently large so that $L_\infty(x, 2) > L_\infty(x, i)$ for $i \in [3, k]$. Furthermore, let $S' = \langle a_2, ..., a_k \rangle$, $I = [0, L_\infty(x) - L_\infty(x, 2))$, and $T = \frac{((r + a_1\mathbb{Z}) \cap S') - r}{a_1} \cap I$. Then $\Delta_\infty(Z'(x, 1)) = \Delta(T)$.

Proof. We will show there is a bijection $\varphi : \mathscr{L}'_\infty(x, 1) \to T$ such that, for $\ell, \ell' \in \mathscr{L}'_\infty(x, 1)$, $|\ell - \ell'| = |\varphi(\ell) - \varphi(\ell')|$, i.e, φ preserves distances. Our choice of φ is $\varphi(\ell) = L_\infty(x) - \ell$. We must show $\varphi(\ell) \in T$, φ is bijective, and $|\ell - \ell| = |\varphi(\ell) - \varphi(\ell')|$.

We first show $\varphi(\ell) \in T$. Because $\ell > L_\infty(x, 2)$, $\varphi(\ell) \in I$, so we only need to show $\varphi(\ell) \in \frac{((r + a_1\mathbb{Z}) \cap S') - r}{a_1}$. Note that $x = L_\infty(x) a_1 + r$ and for every $\ell \in \mathscr{L}'_\infty(x, 1)$, there exists a factorization $(\ell, c_2, ..., c_k)$ of x. That is, $L_\infty(x) a_1 + r = \ell a_1 + c_2 a_2 + \cdots + c_k a_k$ for some $c_2, ..., c_k \in \mathbb{N}_0$, so

$a_1(L_\infty(x)-\ell)+r = c_2a_2+\cdots+c_ka_k$. It follows that $r+a_1\varphi(\ell) \in (r+a_1\mathbb{Z})\cap S'$. Since $\frac{r+a_1\varphi(\ell)-r}{a_1} = \varphi(\ell)$, we have $\varphi(\ell) \in T$.

We now show φ is a bijection. For any $\ell \in \mathscr{L}'_\infty(x,1)$, $\varphi^{-1}(L_\infty(x)-\ell) = \{\ell\}$, so φ is injective. To see φ is surjective, let $n \in \mathbb{Z}$ such that $n \in I$ and $r+na_1 \in (r+a_1\mathbb{Z})\cap S'$. Thus, $na_1+r = c'_2a_2+\cdots+c'_ka_k$ for some $c'_2, ..., c'_k \in \mathbb{N}_0$. Noting that $L_\infty(x)-(L_\infty(x)-n) = n$, let $\hat{\ell} = L_\infty(x)-n$. We will show $\hat{\ell} \in \mathscr{L}'_\infty(x,1)$, implying $\varphi(\hat{\ell}) = n$. Since $n = L_\infty(x)-\hat{\ell}$, $L_\infty(x)a_1 - \hat{\ell}a_1 + r = c'_2a_2+\cdots+c'_ka_k$. Hence, $L_\infty(x)a_1 + r = \hat{\ell}a_1 + c'_2a_2\cdots+c'_ka_k$, meaning $(\hat{\ell}, c'_2, ..., c'_k)$ is a factorization of x. Because $n = L_\infty(x)-\hat{\ell} < L_\infty(x)-L_\infty(x,2)$, we have $\hat{\ell} > L_\infty(x,2)$. It follows that $|(\hat{\ell}, c'_2, ..., c'_k)|_\infty = \hat{\ell}$. Hence, $\hat{\ell} \in \mathscr{L}'_\infty(x,1)$, so φ is surjective and thus a bijection.

To see φ preserves distances, note that $|\ell-\ell'| = |-\ell+\ell'| = |(L_\infty(x)-\ell)-(L_\infty(x)-\ell')| = |\varphi(\ell)-\varphi(\ell')|$. We conclude that, because there exists a distance preserving bijection between $\mathscr{L}'_\infty(x,1)$ and T, $\Delta_\infty(Z'(x,1)) = \Delta(T)$. QE$\Delta$

Corollary 3.3.5. $\Delta(T) \subseteq \Delta_\infty(x)$.

3.4 Arithmetic Semigroups

The following theorem characterizes the ∞-delta set of semigroups with generators in arithmetic progression. The idea of the proof is to find a gap in the factorizations of an element that we want to show is in the ∞-delta set, then prove that there cannot be any factorizations with ∞-length between the lengths of the two factorizations giving us the desired gap.

Theorem 3.4.1. For a semigroup with generators in arithmetic progression, $S = \langle a, a+d, a+2d, ..., a+xd\rangle$, $\Delta_\infty(S) = \{1, 2, ..., \lfloor\frac{a-1}{x}\rfloor+1+d\}$.

Proof. We have the following type II trades for S, $T_1 : e_{r+1} + (\lfloor\frac{a-1}{x}\rfloor)e_x \sim (\lfloor\frac{a-1}{x}\rfloor+1+d)e_0, T_2 : e_{r+2} + (\lfloor\frac{a-1}{x}\rfloor)e_x \sim (\lfloor\frac{a-1}{x}\rfloor+d)e_0 + e_1, T_3 : e_{r+3} + (\lfloor\frac{a-1}{x}\rfloor)e_x \sim (\lfloor\frac{a-1}{x}\rfloor+d)e_0 + e_2, ..., T_{x-r} : (\lfloor\frac{a-1}{x}\rfloor+1)e_x \sim (\lfloor\frac{a-1}{x}\rfloor+d)e_0 + e_{x-r-1}$. We know that for $y \in \Delta_\infty(S)$, we must have that $y \le \lfloor\frac{a-1}{x}\rfloor+1+d$ since $y \le M(T)$.
Let the black lines denote T_1, and the purple lines denote some composition of type I trades. Consider $(a+\frac{a-1-r}{x}+d+1)e_0+(m)e_1$ for $0 \le m \le \lfloor\frac{a-1}{x}\rfloor+d$, we are looking for the largest infinity length strictly smaller than $a+\frac{a-1-r}{x}+d+1$.

$$(a+\tfrac{a-1-r}{x}+d+1)e_0+(m)e_1 \quad\text{———}\quad ae_0+(m)e_1+e_{r+1}+(\tfrac{a-1-r}{x})e_x$$

$$((r+1)\tfrac{a-1-r}{x}+1+r)e_0+((x-r-1)(\tfrac{a-1-r}{x})+m)e_1+(1+\tfrac{a-1-r}{x})e_{r+1}$$

$$(\tfrac{a-1-r}{x}+1)e_0+(m+a)e_1$$

We can disregard applications of type I trades to the initial factorization as that will give us inifnity lengths greater than or equal to $a+\frac{a-1-r}{x}+d+1$. This is the case since initially the only trade we have is applying the same type I trade twice: $(a+\frac{a-1-r}{x}+d+1)e_0+(m)e_1 \sim$

$(a+\frac{a-1-r}{x}+d+2)e_0+(m-2)e_1+e_2 \sim (a+\frac{a-1-r}{x}+d+3)e_0+(m-4)e_1+2e_2$. At this point we can now apply type I trades decreasing the number of copies we have of the second or third generator, but it is clear that we cannot get fewer than $a+\frac{a-1-r}{x}+d+1$ copies of the first generator by applying type I trades.

Suppose we can apply some compostion of trades of the forms T_i and T_j^{-1} to a factorization obtained by applying type I trades to the initial factorization. Suppose we apply strictly more trades of the form T_i. Then applying trades of the form T_i will increase the number of copies we have of the first generator by at least $\frac{a-1-r}{x}+d$ and applying trades of the form T_j^{-1} will decrease the number of copies we have of the first generator by at most $\frac{a-1-r}{x}+d+1$. In order to apply T_i such that we only increase the number of copies of the first generator by $\frac{a-1-r}{x}+d$, we must have at least one copy of a generator between the $r+2$th and $x-r-1$th generators obtained by applying type I trades to move one or more copies of e_{r+1} to one of those generators or applying type I trades to move something from the second generator to one of those generators(we can disregard this case as it would increase the number of copies we have of the first generator by at least 2). So, we must apply T_k^{-1} for $k \geq 3$, in which case we need to apply type I trades in order to get at least one copy of e_{k-1}, which would increase the number of copies of the first generator by at least one. So, we would have that $(a+\frac{a-1-r}{x}+d+1)e_0+me_1 \sim ae_0+me_1+e_{r+1}+(\frac{a-1-r}{x})e_x \sim (a-\frac{a-1-r}{x}-d+2^{k-3})e_0+(m-2^{k-2})+e_{r+1}+e_{r+k}+2(\frac{a-1-r}{x})e_x \sim (a-\frac{a-1-r}{x}-d+2^{k-3})e_0+(m-2^{k-2})+e_{r+2}+e_{r+k-1}+2(\frac{a-1-r}{x})e_x \sim (a+\frac{a-1-r}{x}+d+2^{k-3})e_0+(m-1^{k-2})+e_{k-2}$, which gives us infinity length greater than or equal to that of the initial factorization.

If we apply one more "inverse" trade than other trades, we have two cases. The first is that we get the factorization (perhaps after applying type I trades) $ae_0+me_1+e_{r+1}+(\frac{a-1-r}{x})e_x$; this case will be adressed later. The other case is that we get, perhaps after apply type I trades to move as many copies of generators that aren't the first or last into the first generator then everything else into at most two other generators, $(a+q_1\cdot 2^{\ell_1-1}+q_2\cdot 2^{\ell_2-1}-k)e_0+(m-q_1\cdot 2^{\ell_1-1}-q_2\cdot 2^{\ell_2-1})e_1+q_1e_{\ell_1}+q_2e_{\ell_2}+(\frac{a-1-r}{x})e_x \sim (\ell_2\frac{a-1-r}{x}+q_1\cdot 2^{\ell_1-1}+q_2\cdot 2^{\ell_2-1}-k)e_0+((x-\ell_2)\frac{a-1-r}{x}+m-q_1\cdot 2^{\ell_1-1}-q_2\cdot 2^{\ell_2-1})e_1+q_1e_{\ell_1}+(q_2+\frac{a-1-r}{x})e_{\ell_2} \sim (\frac{a-1-r}{x}+q_1\cdot 2^{\ell_1-1}+q_2\cdot 2^{\ell_2-1}-k-q_2(\ell_2-1)-q_1(\ell_1-1))e_0+(a-1-r+m-q_1\cdot 2^{\ell_1-1}-q_2\cdot 2^{\ell_2-1}+q_2\ell_2+q_1\ell_1)e_b$ where $k \leq q_1+q_2$.
If we apply p more"inverse" trades than the other trade, then by the preceding reasoning we would get the factorization $(\frac{a-1-r}{x}+k')e_0+(a-1-r+m-q_1\cdot 2^{\ell_1-1}-q_2\cdot 2^{\ell_2-1}+q_2(\ell_2-p)+q_1(\ell_1-p))e_b$, so we can disregard such cases.

If we apply the same number of trades of both kinds, we will end up, perhaps after applying some type I trades, at the initial factorization. This is the case since $T_i \circ T_j^{-1} = e_{j-1}+e_{r+i} \sim e_{i-1}+e_{r+j}$, which is just the composition of some type I trades.

To get the largest number of copies of any generator using type I trades after applying T_1 to the initial factorization, we have to pull things out of the outer (leftmost, rightmost) generators and into some inner generator. This is the case since the other options are increasing the number of copies of the first or last generator, and it is clear that both of those options will give us smaller infinity lengths.

Suppose we can apply some composition of trades of the form T_i, T_j^{-1} to some factorization obtained from applying type I trades to $ae_0+me_1+e_{r+1}+(\frac{a-1-r}{x})e_x$. Then, if we apply more trades of the form T_i than the other kind, we will either get the initial factorization or get the factorization, perhaps after applying some type I trades, $(a+\frac{a-1-r}{x}+d+1+q_1\cdot 2^{\ell_1-2}+q_2\cdot 2^{\ell_2-2})e_0+(m-q_1\cdot$

$2^{\ell_1-1} - q_2 \cdot 2^{\ell_2-1})e_1 + q_1 e_{\ell_1} + q_2 e_{\ell_2}$ where $q_1, q_2, q \geq 0$. Since this will give us infinity length greater than or equal to that of the initial factorization, we can disregard this case.

If we apply $p+1$ more trades of the form T_j^{-1} than those of the form T_i, we will get the factorization, perhaps after applying some type I trades, $(a - p\frac{a-1-r}{x} - pd - p' + q_1 \cdot 2^{\ell_1-2} - q_2 \cdot 2^{\ell_2-2})e_0 + (m - q_1 \cdot 2^{\ell_1-1} - q_2 \cdot 2^{\ell_2-1})e_1 + q_1 e_{\ell_1} + q_2 e_{\ell_2} + (p+1)(\frac{a-1-r}{x})e_x$ where $p' \leq p+1$. Clearly, applying type I trades to that factorization will give us an infinity length strictly smaller than $m+a$.

If we apply the same number of trades of both kinds, we will end up, perhaps after applying some type I trades, at the initial factorization. This is the case since $T_i \circ T_j^{-1} = e_{j-1} + e_{r+i} \sim e_{i-1} + e_{r+j}$, which is just the composition of some type I trades.

Thus, we can disregard further application of type II trades to $ae_0 + me_1 + e_{r+1} + (\frac{a-1-r}{x})e_x$ and the factorizations obtained by applying type I trades to.

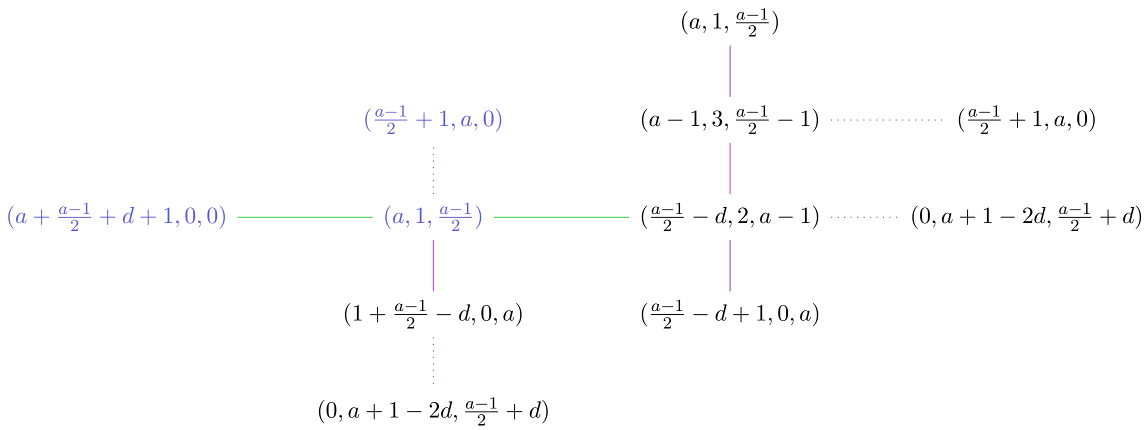
QE∆

Example 3.4.2. Consider the 3-generated case: $= \langle a, a+d, a+2d \rangle$. A presentation for the trades of when $2|a$ is $T_1 : (0,2,0) \sim (1,0,1), T_4 : (\frac{a-2}{2} + 1 + d, 0, 0) \sim (0, 0, \frac{a-2}{2} + 1)$. In the following diagrams, let applications of T_1 be denoted by purple lines, and applications of T_2 be denoted by red lines; dotted lines denote multiple applications of a trade. Consider $(a + \frac{a-2}{2} + d + 1, m, 0)$ for $0 \leq m \leq \frac{a-2}{2} + d$.

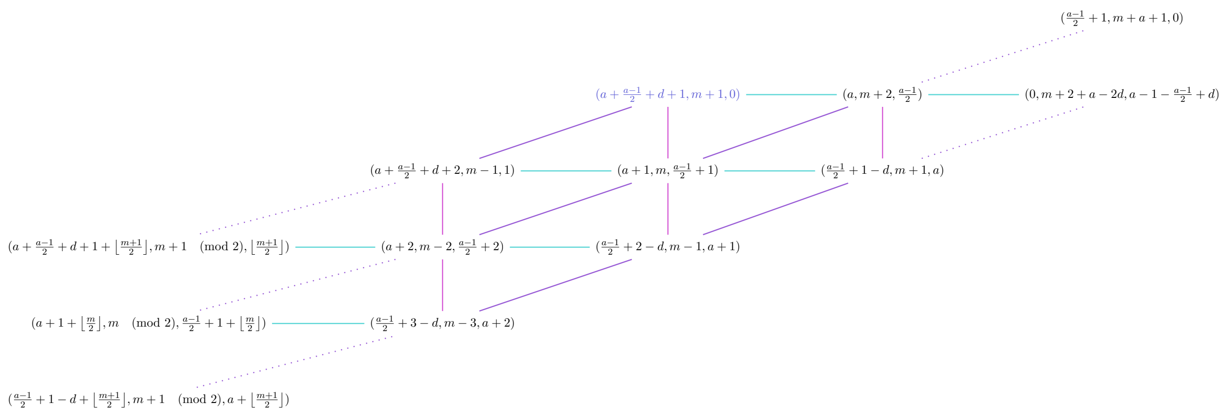
$$\begin{tikzcd}
(a+\frac{a-2}{2}+d+1+\lfloor\frac{m}{2}\rfloor, m+1 \pmod 2, \lfloor\frac{m}{2}\rfloor) & (a+\lfloor\frac{m}{2}\rfloor, m \pmod 2, \lfloor\frac{m}{2}\rfloor+\frac{a-2}{2}+1) & (\frac{a-2}{2}+1-d+\lfloor\frac{m}{2}\rfloor, m \pmod 2, a+\lfloor\frac{m}{2}\rfloor) \\
(a+\frac{a-2}{2}+d+1, m, 0) \arrow[u, dash, dotted] \arrow[r, dash] & (a, m, \frac{a-2}{2}+1) \arrow[u, dash, dotted] \arrow[d, dash, dotted] \arrow[r, dash] & (\frac{a-2}{2}+1-d, m, a) \arrow[u, dash, dotted] \arrow[d, dash, dotted] \\
 & (\frac{a-2}{2}+1, m+a, 0) & (0, m+a-2d, \frac{a-2}{2}+1+d)
\end{tikzcd}$$

We want the largest infinity length smaller than $a+\frac{a-2}{2}+d+1$, applying T_1 to $(a+\frac{a-2}{2}+d+1, m, 0)$ will give us strictly larger infinity lenghts, so we can ignore those factorizations. The factorizations with the largest infinity lengths will have a zero in the first or third entry, or a zero or 1 in the second entry, because otherwise we could apply T_1 to get a factorization with larger inifinity length. And so, it follows from the preceding diagram that the next largest infinity length is $m+a$ since $m \leq \frac{a-2}{2} + d$, thus we can get gaps of length $a + \frac{a-2}{2} + d + 1 - (m+a) = \frac{a-2}{2} + d + 1 - m$.

A presentation for the trades of when $2 \nmid a$ is $T_1 : (0,2,0) \sim (1,0,1), T_2 : (0, 0, \frac{a-1}{2} + 1) \sim (\frac{a-1}{2} + d, 1, 0), T_3 : (\frac{a-1}{2} + d + 1, 0, 0) \sim (0, 1, \frac{a-1}{2})$. In the following diagrams, let applications of T_3 be denoted by green lines, and applications of T_2 be denoted by pink lines.



It follows from the preceding diagram that in the case that $d \leq \frac{a-1}{2}+1$, the second largest infinity length is a, so we have a gap of $a+\frac{a-1}{2}+d+1-a = \frac{a-1}{2}+d+1$ in the Δ_∞ set of the element corresponding to these factorizations. In the case that $d > \frac{a-1}{2}$, only the blue factorizations are possible, and so the second largest infinity length is a, thus we have a gap of $a+\frac{a-1}{2}+d+1-a = \frac{a-1}{2}+d+1$ in the Δ_∞ set of the element corresponding to these factorizations.
Now consider $(a+\frac{a-1}{2}+d+1, m+1, 0)$ for $0 \leq m < \frac{a-2}{2}+d$.



We want to find the largest inifinity length strictly smaller than $a+\frac{a-1}{2}+d+1$. We can ignore applications of T_1 to $(a+\frac{a-1}{2}+d+1, m+1, 0)$ as that gives us strictly larger infinity lengths. We can disregard the further applications of T_3 and T_2 after the initial application of T_2 to $(a+\frac{a-1}{2}+d+1, m+1, 0)$ as further applications of type II trades will increase the number of multiples of $\frac{a-1}{2}$ in the rightmost generator and so will decrease the number of times we can pull out one copy of the smallest generator into the inner generator we are trying to get the most copies of. And so, the next largest inifinity length is $m+a+1$, therefore we can get gaps of length $a+\frac{a-1}{2}+d+1-(m+a+1) = \frac{a-1}{2}+d-m$.

Corollary 3.4.3. For a semigroup S with generators in arithmetic progression, $C_\infty(S) = \lfloor \frac{a-1}{x} \rfloor + 1 + d$.

Proof. It follows from the minimal trade presentation, T, we have of arithmetic sequences that $M_\infty(S) \le \left\lfloor \frac{a-1}{x} \right\rfloor + 1 + d$ since $M_\infty(T) = \left\lfloor \frac{a-1}{x} \right\rfloor + 1 + d$. Suppose for contradiction that there exists a presentation of trades such that $M_\infty(T) < \left\lfloor \frac{a-1}{x} \right\rfloor + 1 + d$, then since $\Delta_\infty(S)$ is bounded by $M_\infty(S)$ and we have that by Theorem 3.4.1 $\left\lfloor \frac{a-1}{x} \right\rfloor + 1 + d \in \Delta_\infty(S)$ this is a contradiction. QEΔ

3.5 Gaps in ∞-Delta Sets

Unlike with the 0-delta set, it is fairly straighforward to show that the ∞-delta set of a numerical semigroup may have an arbitrarily large gap. We may, roughly, attribute this to the fact that the maximal ∞-delta value is more loosely restricted than the maximal 0-delta value.

Theorem 3.5.1. Let $S = \langle 3, 3m+1, 3m+2 \rangle$ where $m \ge 3$, then $\Delta_\infty(S)$ has gap of $m-2$.

Proof. Suppose there exists $x \in S$ such that $m + 2 + \ell \in \Delta_\infty(x)$ for $0 \le \ell \le m-3$. Since S is a semigroup with maximal embedding dimension, a minimal presentation for its trade structure is $T_1 : (2m+1, 0, 0) \sim (0, 1, 1), T_2 : (m+1, 1, 0) \sim (0, 0, 2), T_3 : (m, 0, 1) \sim (0, 2, 0)$. Suppose for contradiction that there exists $x \in S$ such that $2 + m + \ell \in \Delta_\infty(x)$ where $0 \le \ell \le m-3$. Then x must have a factorization $(k+m+2+\ell, b_1, c_1)$ or $(a_1, k+m+2+\ell, c_1)$ or $(a_1, b_1, k+m+2+\ell)$ with infinity length $k+m+2+\ell$, where $k \ge 0$. If the factorization we have is $(k+m+2+\ell, b_1, c_1)$, then we must have that the next largest infinity length is k, but this is impossible since we can apply T_2 or T_3 to the factorization to get gaps of $m+1$ and m which means that the next largest factorization will give us a gap of at most m. If we cannot apply T_2 or T_3, then $b_1 = c_1 = 0$, so if we can apply T_1, we get a gap of $2m$ or $2m+1$, and any trades we apply after this cannot give us larger infinity lengths since we would necessarily be decreasing the number of copies of the first generator as we only have one copy each of the second and third generators, and if the first generator is smaller than the infinity length of $(k-m+1+\ell, 1, 1)$ then we must have 0 copies of it.

Suppose we have the factorization $(a_1, k+m+2+\ell, c_1)$, the trades we have only allow us to decrease the number of copies of the second generator by $0, 1$, or 2. So, it follows that the next largest factorization has infinity length greater than or equal to $k+m+\ell$.

Suppose we have the factorization $(a_1, b_1, k+m+2+\ell)$, the trades we have only allow us to decrease the number of copies of the third generator by $0, 1$, or 2. So, it follows that the next largest factorization has infinity length greater than or equal to $k+m+\ell$.

We can get a gap of $2m$ by considering the factorization $(2m+1, 0, 0) \sim (0, 1, 1)$, and a gap of $m+1$ by considering the factorization $(m+3, 1, 0) \sim (2, 0, 2)$. QE$\Delta$

3.6 ∞-Catenary Degree and Compound Sequences

In this section, we examine catenary degree when $t = \infty$, with a special eye towards numerical semigroups on compound sequences.

As is the case with $t = 0$, there are two notions of distance in the case where $t = \infty$. Let $X = (x_1, ..., x_n)$ and let $Y = (y_1, ..., y_n)$. We define $c_\infty(S)$ to use the norm $d_\infty(X, Y) = ||X|_\infty - |Y|_\infty|_\infty = |\max\{x_i : i \in [1, n]\} - \max\{y_j : j \in [1, n]\}|$. We define $c'_\infty(S)$ to use the norm $d'_\infty(X, Y) = |X - Y|_\infty = \max\{|x_i - y_i| : i \in [1, n]\}$. As is the case with $t = 0$, $c_\infty(S) = c'_\infty(S)$, while $c_\infty(x) \le c'_\infty(x)$ by the reverse triangle inequality.

We first provide a lower bound on $c_\infty(S)$ taken from the Machu Picchu section.

Theorem 3.6.1. Let $S = \langle a_1, ..., a_n\rangle$. Then $c_\infty(S) \geq M_\delta(S)$.

Proof. Let $x \in S$ such that $a_1|x$ and x is sufficiently large such that $\delta(x) = M_\delta(S)$. Let x_0 be a factorization of x with $|x_0|_\infty = L_\infty(x)$. Let x_n be another factorization of x with $|x_n|_\infty \neq L_\infty(x)$. Note that, for any factorizations of x, say x_j and x_k, if $|x_j|_\infty = L_\infty(x)$ and $|x_k|_\infty \neq L_\infty(x)$, then $d_\infty(x_j, x_k) \geq M_\delta(S)$. In any factorization sequence between x_0 and x_n, with variable n, there must exist such a pair of consecutive factorizations. Thus, for every factorization sequence between x_0 and x_n, there exists some $i \in [0, n-1]$ such that $d_\infty(x_i, x_{i+1}) \geq M_\delta(S)$. It follows that $c_\infty(x) \geq M_\delta(S)$. QEΔ

Note that this bound is generally not sharp. $M_\delta(S)$ may not be a sharp lower bound of $\max(\Delta_\infty(S))$, which is turn a lower bound of $c_\infty(S)$. We now provide the $\infty-$catenary degree of 2-generated numerical semigroups.

Theorem 3.6.2. Let $S = \langle a_1, a_2\rangle$. Then $c_\infty(S) = a_2$.

Proof. Let $x \in S$ such that x does not have unique factorization and let $x_0, x_n \in \mathbb{R}^2$ be distinct factorizations of x. Note that, for any distinct factorizations of x, say, x_i and x_j, $d'_\infty(x_i, x_j) = ka_2$ for some $k \in \mathbb{N}$. Thus, $c'_\infty(x) \geq a_2$.

Now, write x_0 as (a, b). Then $x_n = (a+ta_2, b-ta_1)$ for some $t \in \mathbb{Z}-\{0\}$. Without loss of generality, assume $t > 0$. Then there exists a factorization sequence $(x_0, x_1, ..., x_n)$ where $x_i = (a+ia_2, b-ia_1)$, implying $d'_\infty(x_i, x_{i+1}) = a_2$ for all $i \in [0, n-1]$. Therefore, $c'_\infty(x) \leq a_2$, so $c_\infty(x)' = a_2$ for all $x \in S$. We conclude $c'_\infty(S) = c_\infty(S) = a_2$. QEΔ

We now move to numerical semigoups on compound sequences. Historically, this follows from the study of numerical semigroups on geometric sequences. These are of the form $\langle a^q, a^{q-1}b, a^{q-2}b^2, ..., b^q\rangle$ with $\gcd(a, b) = 1$. This construction gives partial geometric sequences of the type taught in an introductory calculus class which also form valid generators of a numerical semigroup, and were studied in [35] and [37]. More recently, Claire Kiers, a student at a previous SDSU mathematics REU, generalized this concept to compound sequences in [31]. Compound sequences were then studied further in [36].

We now define numerical semigroups on compound sequences.

Definition. Take $k, a_1, a_2, \ldots, a_{k-1}, b_1, b_2, \ldots, b_{k-1} \in \mathbb{Z}^+$ such that

1. $1 < a_i < b_i$ for all $i \in [1, k-1]$;
2. $\gcd(a_i, b_j) = 1$ for all $i \geq j$.

The *compound sequence* generated by the above values is the finite sequence

$$(a_1 \cdots a_{k-1}, b_1a_2 \cdots a_{k-1}, ..., b_1 \cdots b_{k-2}a_{k-1}, b_1 \cdots b_{k-1})$$

with length k. In other words, we consider $(a_1, ..., a_{k-1})$ and $(b_1, ..., b_{k-1})$ as sequences, start with the product $a_1 \cdots a_{k-1}$, and replace each a_i with b_i one at a time until we are left with the final term $b_1 \cdots b_{k-1}$. A *numerical semigroup on a compound sequence*, which we will abbreviate as

NSCS following the convention of [31], is simply the numerical semigroup generated by compound sequence. Note that a compound sequence is a geoemtric sequence as described above if $a_i = a_j$ and $b_i = b_j$ for all $i, j \in [1, k-1]$. The standard minimal presentation for NSCSs as stated in [31] is given in the proof of the first theorem of this section.

The following theorem partially characterizes the Δ_∞-set of NSCSs.

Theorem 3.6.3. Let S be a semigroup with generators in a compound sequence,

$$S = \langle a_1a_2 \cdots a_{k-1}, b_1a_2a_3 \cdots a_{k-1}, ..., b_1b_2 \cdots b_{k-2}a_{k-1}, b_1b_2 \cdots b_{k-1}\rangle$$

Then $\{1, 2, ..., b_1\} \subseteq \Delta_\infty(S) \subseteq \{1, 2, ..., \max b_i\}$.

Proof. S has a minimal presentation of trades $T_1 : (b_1, 0, ..., 0) \sim (0, a_1, 0, ..., 0), T_2 : (0, b_2, 0, ..., 0) \sim (0, 0, a_2, 0, ..., 0), ...T_q : (0, .., 0, b_q, 0) \sim (0, ..., 0, 0, a_q)$. Consider the element corresponding to the factorization $(a_1 + b_1 - k, 0, ..., 0)$ for $k \leq a_1$, it has factorizations $(a_1 + b_1 - k, 0, ..., 0) \sim (a_1 - k, a_1, 0, 0)$, in the case that $a_1 \leq b_2$, we can apply T_2 until we have fewer than b_2 copies of the second generator but since $a_2 < b_2$ the infinity length of the factorizations obtained by applying T_2 will be strictly smaller than a_1. Further applications of T_i for $i > 2$ would give us strictly smaller infinity length by the same reasoning. Since the largest infinity length strictly smaller than $a_1 + b_1 - k$ is a_1, it follows that $\{b_1 - a_1, b_1 - a_1 + 1, ..., b_1\} \subseteq \Delta_\infty(S)$.

Now consider the element corresponding to the factorization $(b_1, k', 0, ..., 0)$ for $1 \leq k' < b_1 - a_1$, it is possible that we can apply T_1^{-1} for some k', in which case we will get infinity length(s) strictly larger than b_1. We want to find the largest infinity length of this element that is strictly smaller than b_1. The only other trade we can apply is T_1, when we do so we get that $(b_1, k', 0, ..., 0) \sim (0, a_1 + k', 0, ...0)$. If $b_2 \leq a_1 + k'$, we can apply T_2 to $(0, a_1 + k', 0, ...0)$ until we have fewer than b_2 copies of the second generator, but since $a_2 < b_2$ the infinity length of the factorizations obtained by applying T_2 will be strictly smaller than $a_1 + k'$, and further applications of T_i for $i > 2$ would give us strictly smaller infinity length by the same reasoning. And so, we have that $\{1, 2, ..., b - a_1 - 1\} \subseteq \Delta_\infty(S)$.

Since $\Delta_\infty(x) \leq M(T)$ for arbitrary $x \in S$, it is clear that we cannot have anything larger than $\max b_i$ in $\Delta_\infty(S)$. QEΔ

This is a general description of the Δ_∞-set for all NSCSs. Note that if $b_1 \geq b_i$ for all $i \in [1, k-1]$, we know the full Δ_∞-set.

Corollary 3.6.4. Let S be an NSCS with $b_1 \geq b_i$ for all $i \in [1, k-1]$. Then $\Delta_\infty(S) = \{1, ..., b_1\}$.

This result gives easy corollaries for certain subfamilies of NSCSs. One such subfamily is numerical semigroups of geometric sequences.

Corollary 3.6.5. Let $S = \langle a^q, a^{q-1}b, ..., ab^{q-1}, b^q\rangle$ be a semigroup with generators in geometric progression. Without loss of generality let $b > a$. Then $\Delta_\infty(S) = \{1, 2, ..., b\}$.

Another such subfamily is supersymmetric semigroups.

Definition. Let $p_1, ..., p_k \in \mathbb{Z}^+$ with $p_1 > p_2 > \cdots > p_k$ and $\gcd(p_i, p_j) = 1$ for all $i, j \in [1, k]$. Let $B = p_1 \cdots p_k$. Then $S = \langle B/p_1, ..., B/p_k \rangle$ is called a *supersymmetric semigroup.*

A supersymmetric semigroup is a special case of NSCSs with $b_1 = p_1, a_1 = b_2 = p_2, a_2 = b_3 = p_3, ..., a_k = p_k$. Since $p_1 > p_2 \cdots > p_k$, the above theorem applies.

Corollary 3.6.6. Let S be a supersymmetric semigroup. Then $\Delta_\infty(S) = \{1, ..., p_1\}$.

These results also give us catenary degree for these semigroups.

Theorem 3.6.7. Let S be an NSCS with $b_1 \geq b_i$ for all $i \in [1, k-1]$. Then $c_\infty(S) = b_1$.

Proof. This follows immediately from $b_1 = \max(\Delta_\infty(S)) \leq c_\infty(S) = M_\infty(S) \leq b_1$. QEΔ

Corollary 3.6.8. Let S be a geometric sequence numerical semigroup. Then $c_\infty(S) = b$.

Corollary 3.6.9. Let S be a supersymmetric numerical semigroup. Then $c_\infty(S) = b$.

We are also able to slightly improve the general bound on $\max(\Delta_\infty(S))$ if there exists $i \in [2, k-1]$ such that $b_1 < b_i$, and do this by bounding catenary degree first.

Theorem 3.6.10. Let S be a numerical semigroup with generators in a compound sequence such that $M_B := \max\{b_i : i \in [1, k-1]\} > b_1$. Then $c_\infty(S) \leq M_B - 1$.

Proof. Because $b_1 \neq M_B$, there exists $j \in [2, k-1]$ such that $M_B = b_j$ and $M_B \neq b_{i-j}$. Let $b_j e_j$ and $a_j e_{j+1}$ be two factorizations of the Betti element $\beta = b_1 \cdots b_j a_j \cdots a_{k-1}$. So $t_1 : b_j e_j \sim a_j e_{j+1}$ is a trade in the typical minimal presentation of S. Since $b_j = \max\{b_i : i \in [1, k-1]\}$, $b_j > a_{j-1}$. Thus, $t_2 : b_{j-1} e_{j-1} + (b_j - a_{j-1}) e_j \sim a_j e_{j+1}$ is another trade at β which may be used instead of t_1 in a minimal presentation of S. Let T be the minimal presentation with t_1 replaced by t_2. By definition of compound sequence numerical semigroups, $b_j - a_{j-1} < b_j - 1$ and $a_j < b_j$, while $b_{j-1} < b_j$ by choice of j. Thus, $M_\infty(T) \leq b_j - 1 = M_B - 1$, implying $c_\infty \leq M_B - 1$. QEΔ

Corollary 3.6.11. Let S be a compound sequence numerical semigroup such that $M_B := \max\{b_i : i \in [1, k-1]\} > b_1$. Then $\max(\Delta_\infty(S)) \leq M_B - 1$.

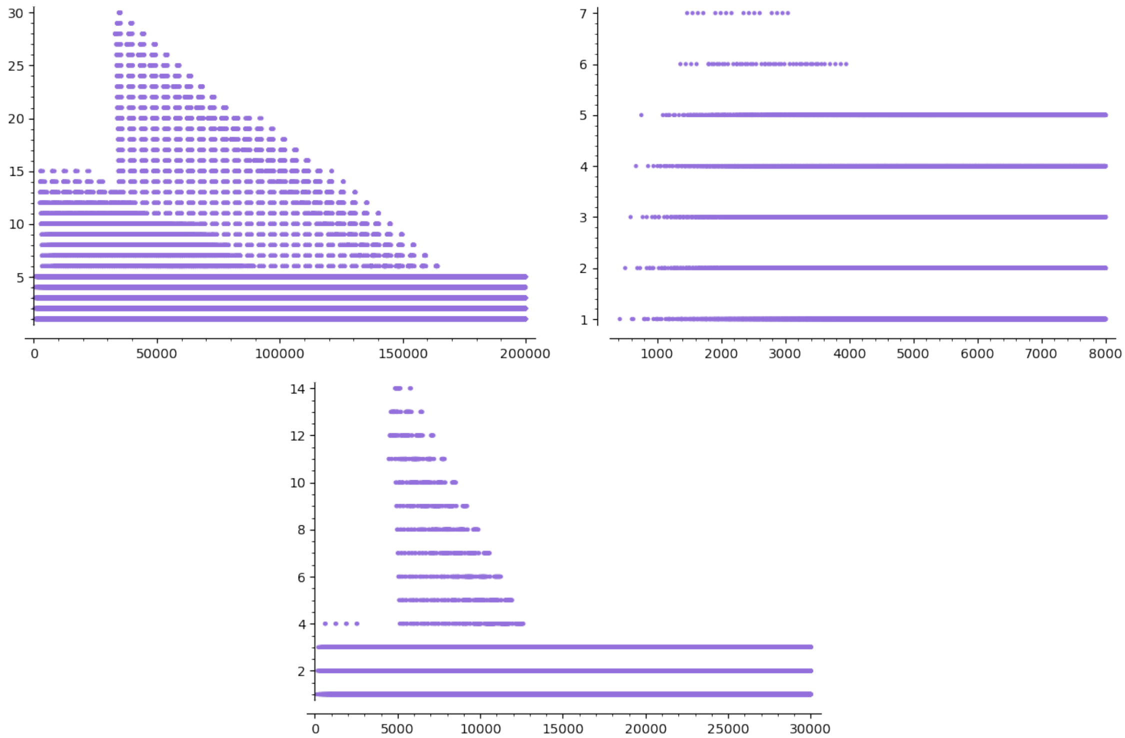


Figure 8: Top Left: $a = \{3, 3, 7\}$; $b = \{5, 23, 41\}$; $S = \langle 63, 105, 805, 4715\rangle$. Top Right: $a = \{4, 3, 7\}$; $b = \{5, 11, 8\}$; $S = \langle 84, 105, 385, 440\rangle$. Bottom: $a = \{3, 3, 7\}$; $b = \{2, 11, 29\}$; $S = \langle 63, 42, 154, 638\rangle$

3.7 Conjectures and Open Work

Conjecture 4. Let S be a semigroup with generators in a compound sequence, then $\Delta_\infty(S)$ is an interval.

Conjecture 5. Let S be a semigroup with generators in a compound sequence, $S = \langle a_1 a_2 \cdots a_q, a_2 a_3 \cdots a_q b_1, ..., a_q b_1 b_2 \cdots b_{q-1}, b_1 b_2 \cdots b_q\rangle$, then for all $q \in \Delta_\infty(S)$ where $q > b_1$ and $q > 3$, q is in $\Delta_\infty(x)$ for only a finite number of $x \in S$.

Conjecture 5 is primarily motivated by the plots in Figure 8.

The structure of Variable Region 3 from the Periodicity theorem is completely unknown. We surmised but disproved that said region might eventually be a full interval of natural numbers, or at least have maximally dense spacing (as restricted by the g_i).

The bounds at which various lemmas and results relating to Atlantis and Periodicity are all unknown, but should be straightforward (if tedious) to derive.

The effect of gluings on delta sets and $M_\infty(S)$ is poorly understood.

Small elements of numerical semigroups that precede the bounds on Atlantis theorems exhibit irregular delta and length set behavior that is unknown.

There is no classification theorem for ∞-delta sets of 3-generated numerical semigroups analagous to that of 0-delta sets.

4 Δ_t

The t-norm, aside from the special cases $t \in \{0, 1, \infty\}$, lacks a simple, discrete interpretation. Investigations into those norms then naturally take on a more analytic flavor. As such, the results in this section may upset the more discerning mathematical palate and/or result in some mild indigestion.

We begin, as usual, by examining the humble 2-generated numerical semigroup, which already exhibits some unruly behavior. Even the plots are daunting.

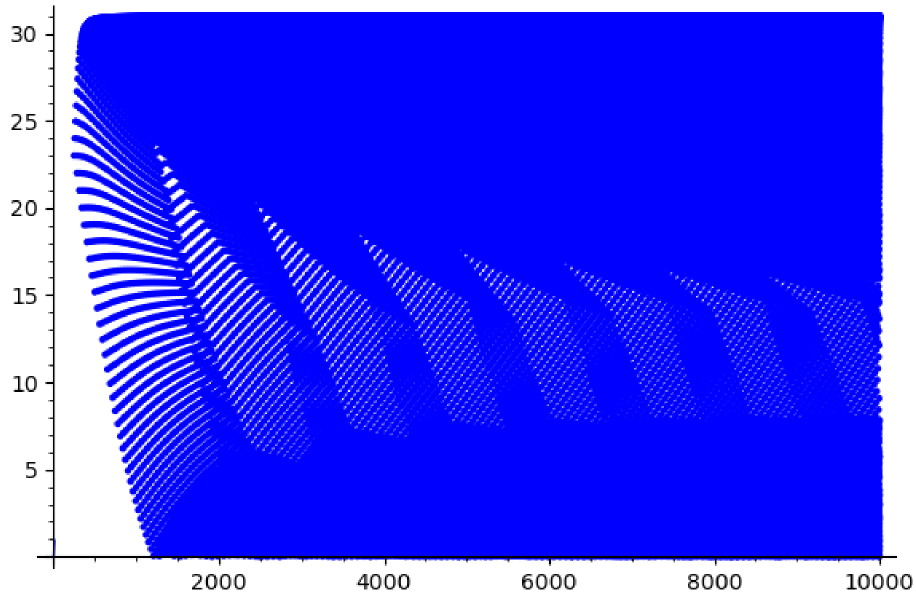


Figure 9: Δ_3 sets of $\langle 8, 31 \rangle$

4.1 Fermat's Last Lemma

The following result is folklore:

Lemma 4.1.1. For all natural numbers $n > 2$, there do not exist natural numbers a, b, c such that $a^n + b^n = c^n$.

Proof. Left for the reader. QEΔ

The previous lemma finds its chief importance in the suggestion that rational lengths under the t norm are rare. Subsequent results confirm that suspicion.

Let $S := \langle a_1, a_2 \rangle$ be a numerical semigroup, and let $t > 2$ be an integer.

Proposition 2. For all $x \in S$, $|\mathscr{L}_t(x) \cap \mathbb{Q}| = |\mathscr{L}_t(x) \cap \mathbb{N}| \leq 2$.

Proof. Let $l \in \mathscr{L}_t(x)$, with the factorization $\hat{f} = (f_1, f_2)$ witnessing. Suppose $l = \frac{m}{n} \in \mathbb{Q}$. Then $l^t = f_1^t + f_2^t$, meaning $m^t = (nf_1)^t + (nf_2)^t$, which, by Lemma 4.1.1, implies $nf_1 = 0$ or $nf_2 = 0$. So $f_1 = 0$ or $f_2 = 0$, meaning f equals $(\frac{x}{a_1}, 0)$ or $(0, \frac{x}{a_2})$. So the possibilities for l are $\{\frac{x}{a_1}, \frac{x}{a_2}\} \cap \mathbb{N}$. QE$\Delta$

The following result is from [6].

Lemma 4.1.2. Let T be a finite set of irrational radicals, and let r be a linear combination over $\mathbb{Q}$ of elements in T. Then $r \in \mathbb{Q}$ only if there are two elements of T that are linearly dependent over $\mathbb{Z}$.

This leads to a result about $\Delta_t(x)$ similar to that about $\mathscr{L}_t(x)$.

Proposition 3. For all $x \in S$, $|\Delta_t(x) \cap \mathbb{Q}| = |\Delta_t(x) \cap \mathbb{Z}| \leq 1$. Furthermore, if $d \in \Delta_t(x) \cap \mathbb{Q}$, $d < a_2$. Finally, the set $\{x : |\Delta_t(x) \cap \mathbb{Q}| = 1\}$ is finite.

Proof. Let l, l' be consecutive in $\mathscr{L}_t(x)$ with $l' - l$ rational. Since $\mathbb{Q}$ is a field, l', l are both rational or both irrational. The former case corresponds to at most one rational element of $\Delta_t(x)$, as by Proposition 2, $\mathscr{L}_t(x)$ has at most two rational lengths. In the latter case, by Lemma 4.1.2, $ml' = nl$, where m, n are integers. So $l' - l = (1 - \frac{m}{n})l$, which, by field closure of $\mathbb{Q}$, is irrational.

So $\Delta_t(x)$ has at most one rational element, arising when $\mathscr{L}_t(x)$ has the maximal number of rational lengths, two. These two lengths, $\frac{x}{a_1}$, $\frac{x}{a_2}$, must be integers and must be consecutive in $\mathscr{L}_t(x)$.

Suppose $x = a_1 a_2$. x then has two factorizations, $(a_2, 0)$ and $(0, a_1)$, meaning $\Delta_t(S) = \{a_2 - a_1\}$.

Else, $x > a_1 a_2$. The factorization $f = (\frac{x}{a_1} - a_2, a_1)$ of x then has two nonzero coordinates. Since $a_2 > a_1$, $\frac{x}{a_1} > |f|_1$, which, by Holder's Inequality, is at least $|f|_t$. So $\frac{x}{a_2}$ must also exceed $|f|_t$ to be the predecessor of $\frac{x}{a_1}$. In this case, since $|f|_t > \frac{x}{a_1} - a_2$, $\frac{x}{a_2} > \frac{x}{a_1} - a_2$ as well, so $\frac{x}{a_1} - \frac{x}{a_2} < a_2$. Furthermore, $\frac{x}{a_2} > |f|_t > \frac{x}{a_1} - a_2$, which gives a constant upper bound on x. So there are finitely many elements $x \in S$ with rational elements in $\Delta_t(x)$.

QEΔ

Example 4.1.3. Let $S := \langle 2, 3 \rangle$. As explained, rational values may only occur in the delta sets of elements that are multiples of 6. We will list the factorization sets of several such multiples:

$$Z(6) = \{(3,0), (0,2)\}$$
$$Z(12) = \{(6,0), (3,2), (0,4)\}$$
$$Z(18) = \{(9,0), (6,2), (3,4), (0,6)\}$$

.

Since $|(6,2)|_t > |(0,6)|_t$ for $t > 1$, 18 already does not have a rational value in its delta set. In any larger multiple of 6, the two rational lengths will be even further apart, meaning no rational values exist in delta sets of elements beyond 18.

This result leads one naturally to question the cardinality of $\Delta_t(S) \cap \mathbb{R}/\mathbb{Q}$. Since S is countable and $\Delta_t(x)$ is countable for all $x \in S$, said cardinality is at most countable. The following results confirms that it is countably infinite. They rely on the observation that a negligible second coordinate contributes a negligible amount to the t-norm of a vector.

Lemma 4.1.4. For all nonnegative c, d, ϵ and real $t > 1$, there exists N such that if $k > N$, $c \leq d$, $|(k,c)|_t \in [k, k+\epsilon]$.

Proof. Let $N := \frac{d^t}{t\epsilon^{t-1}}$, and let $k > N$. We have that $(|(k,c)|_t)^t \leq (|(k,d)|_t)^t = k^t + d^t = k^t + tk\frac{d^t}{tk} = k^t + tk(\frac{d^t}{tk})^{\frac{t-1}{t-1}}$. These are the first two terms of the binomial expansion of $(k + (\frac{d^t}{tk})^{\frac{1}{t-1}})^t$, all of whose terms are positive, so $(|(k,d)|_t)^t < (k + (\frac{d^t}{tk})^{\frac{1}{t-1}})^t$. Taking t^{th} roots gives $|(k,c)|_t \leq |(k,d)|_t < k + (\frac{d^t}{tk})^{\frac{1}{t-1}}$, which, by the lower bound on k, is at most $k+\epsilon$. The observation that $|(k,c)|_t \geq k$ as well completes the proof. QEΔ

Example 4.1.5. Below are some 3-norms of 2-dimensional vectors over $\mathbb{N}$ with negligible second coordinate:

$$|(10001, 5)|_3 = 10001.0000004166$$
$$|(10002, 5)|_3 = 10002.0000004165$$
$$|(10003, 5)|_3 = 10003.0000004164$$

This means that the t-norm of factorizations with small second coordinate is very close to the ∞-norm of those same factorizations. This phenomenon is explored in more detail in the following section.

4.2 Accumulation Points

Proposition 4. a_2 is an accumulation point of $\Delta_t(S)$.

Proof. Pick $\epsilon < a_2$ and let N be such that for all $k > N$, $c \leq a_2^2$, $|(k,c)_t| \in [k, k+\epsilon]$ (N exists, by Lemma 4.1.4). Let $n > N + a_2$ and let $x = na_1$. We will show that the factorizations $(n, 0)$ and $(n - a_2, a_1)$ lead to consecutive lengths in $\mathscr{L}_t(x)$.

Suppose (s,t) is a factorization of x with $s < n$. By the trade structure of S, $t = \frac{a_1}{a_2}(n-s)$. There are two cases:

1. Suppose $s < n - a_2^2$. Then $t + s = \frac{a_1}{a_2}(n-s) + s = \frac{a_1}{a_2}n - \frac{(a_2-a_1)s}{a_2} < \frac{a_1}{a_2}n - \frac{(a_2-a_1)(n-a_2^2)}{a_2} = n - \frac{(a_2-a_1)(a_2^2)}{a_2} \leq n - a_2$. By Holder's inequality, $|(s,t)|_t \leq t + s \leq n - a_2 < |(n-a_2, a_1)|_t$.
2. Else, let $f_0 := (n, 0)$, $f_1 := (n - a_2, a_1)$,...$f_{a_1} := (n - a_1a_2, a_1^2)$. By assumption, $|f_i|_t \in [n - ia_2, n - ia_2 + \epsilon]$ for all $i \in [0, a_1]$, so $|f_1|_t > |f_j|_t$ for all $j > 1$, $j \in [0, a_1]$.

So $(n, 0), (n - a_2, a_1)$ lead to consecutive lengths in $\mathscr{L}_t(x)$. Moreover, by Lemma 4.1.4, $(n - a_2, a_1) \in [n - a_2, n - a_2 + \epsilon]$. So $|(n, 0)|_t - |(n - a_2, a_1)|_t \in [a_2 - \epsilon, a_2]$.

So for all $\epsilon \in [0, a_2)$, there exists x such that $d \in [a_2 - \epsilon, a_2]$ and $d \in \Delta_t(x) \subseteq \Delta_t(S)$. So a_2 is an accumulation point of $\Delta_t(S)$.

QEΔ

Corollary 4.2.1. $\Delta_t(S)$ has countably infinite irrational points.

We include again the plot from the beginning of the section, in which both of these results are apparent.

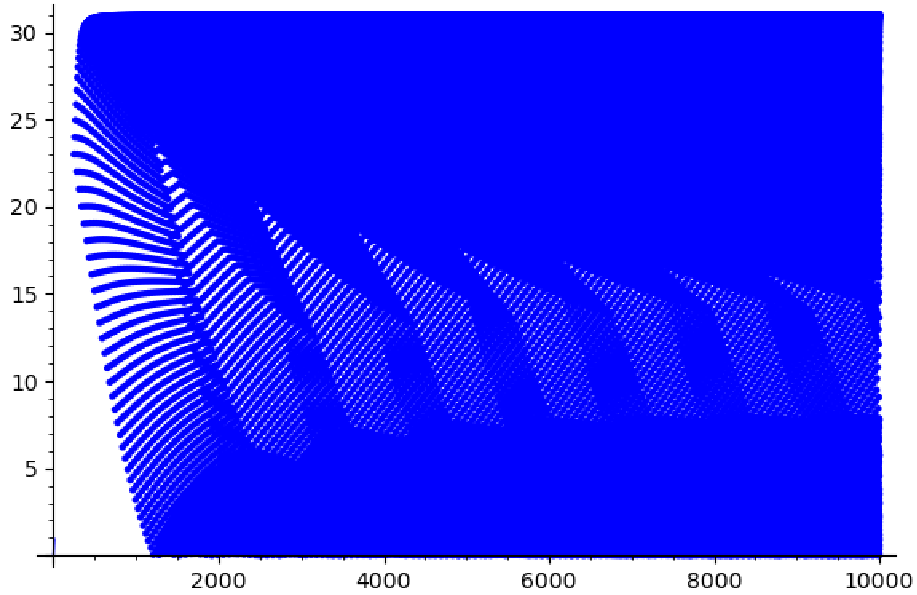


Figure 10: Δ_3 sets of $\langle 8, 31 \rangle$

This accumulation point is an integer. By Lemma 3, all integer values of $\Delta_t(S)$ are less than a_2, so it lies outside of $\Delta_t(S)$. These observations inspire the following conjectures:

1. All of the accumulation points of $\Delta_t(S)$ are integers.
2. All of the accumulation points of $\Delta_t(S)$ lie outside of $\Delta_t(S)$.
3. All of the accumulation points of $\Delta_t(S)$ are elements of $\Delta_\infty(S)$, and vice versa.

The approach from Lemma 4.1.4 and Proposition 4 extends to prove a more general result. It relies on the more general observation that a vector with negligible coordinates in every entry but one has a t-norm that is nearly that of the nonnegligible entry. We can prove this using the previous approximation result. Said approximation implies that factorizations near the top of $\mathscr{L}_\infty(x, 1)$ have t-norms and ∞-norms that nearly agree.

Theorem 4.2.2. Penthouse Party Let $S = \langle a_1, ..., a_k \rangle$ be a numerical semigroup. Suppose there are infinite $x \in S$ such that $d \in \Delta_\infty(x)$ and $l' - l = d$, for some consecutive l', l in $\mathrm{Ma_1chuP1cchu}(x)$ (also known as $[L_\infty(x,1) - B_1, L_\infty(x,1)] \cap \mathscr{L}_\infty(x)$). Then d is an accumulation point of $\Delta_t(S)$ for all $t > 1$.

Proof. Let $\epsilon \in (0, g_1)$, let $s := \max\{\frac{a+b}{a_2} : a \in Ap(S, a_1); b \leq B_1\}$, and let N be large enough such that if $k > N$, $c < \max(s,, (k,s)_t \in [k, k+\epsilon]$ (see Lemma 4.1.4). Suppose x is large enough that $L_\infty(x) - B_1 - g_1 > \frac{x}{a_2} > N$ and that $f \in Z(x)$. We will show:

1. If $|f|_\infty$ is in $\mathrm{Ma_1chuP1cchu}(x)$, then $|f|_t$ is in $[|f|_\infty, |f|_\infty + \epsilon]$.
2. Otherwise, $|f|_t < L_\infty(x,1) - B_1$.

1. Let $f = (f_1, ..., f_k) \in Z(x)$. Suppose f satisfies $f_1 = |f|_\infty \geq L_\infty(x,1) - B_1 - g_1$. Let $s' := |(f_2, ..., f_k)|_t$. Then $s' \leq |(f_2, ..., f_k)|_1$, by Holder's inequality. By Theorem 3.3.4 of **[444]**, $x - L_\infty(x,1) \in Ap(S, a_1)$, meaning $x - |f|_\infty \in \{a + b : a \in Ap(S, a_1); b \leq B_1\}$. So $s' \leq |(f_2, ..., f_k)|_1 \leq \frac{x - |f|_\infty}{a_2} \leq s$. Therefore, by Lemma 4.1.4, $|(f_1, s')|_t \in [f_1, f_1 + \epsilon]$. Since $|(f_1, s')|_t = |f|_t$, this means $|f|_t \in [f_1, f_1 + \epsilon]$.
2. Else, $f_1 < L_\infty(x,1) - B_1 - g_1$. Because $(\sum_{j=2}^{k} f_j)^t \geq \sum_{j=2}^{k} f_j^t$, the inequality $|(f_1, \sum_{j=2}^{k} f_j)|_t \geq |f|_t$ holds. Furthermore, $\frac{x - f_1 a_1}{a_2} \geq \sum_{j=2}^{k} f_j$. So $|(f_1, \frac{x - f_1 a_1}{a_2})|_t \geq |f|_t$. By a theorem of **[ateam]**, $|f_1, \frac{x - f_1 a_1}{a_2}|_t \leq |L_\infty(x,1) - B_1 - g_1, \frac{x - L_\infty(x,1) + B_1}{a_2}|_t$ (Proposition 4.2, characterization of t-norm of vectors on a line in $\mathbb{R}^2$). Since $\frac{x - L_\infty(x,1) + B_1}{a_2} \leq s$, by Lemma 4.1.4, $|L_\infty(x,1) - B_1 - g_1, \frac{x - L_\infty(x,1) + B_1}{a_2}|_t \in [L_\infty(x,1) - B_1 - g_1, L_\infty(x,1) - B_1 - g_1 + \epsilon]$. So $|f|_t \leq |L_\infty(x,1) - B_1 - g_1, \frac{x - L_\infty(x,1) + B_1}{a_2}|_t \leq L_\infty(x,1) - B_1 - g_1 + \epsilon$, which, since $\epsilon < g_1$, is less than $L_\infty(x,1) - B_1$.

Now suppose $|f|_\infty, |f'|_\infty$ are consecutive in $\mathrm{Ma_1chuP1cchu}(x)$. As shown above, $|f|_t \in [|f|_\infty, |f|_\infty + \epsilon]$ and $[|f'|_t, |f'|_t + \epsilon]$. Let l be the largest length in $[|f|_\infty, |f|_\infty + \epsilon] \cap \mathscr{L}_t(x)$ and let l' be the smallest length in $[|f'|_\infty, |f'|_\infty + \epsilon] \cap \mathscr{L}_t(x)$. Now, $l' - l \in [|f'|_\infty - |f|_\infty - \epsilon, |f'|_\infty - |f|_\infty]$. It only remains to show that l', l are consecutive in $\mathscr{L}_t(x)$.

Suppose $f'' \in Z(x)$ such that $|f''|_t \in (l, l')$. If f'' witnesses $\mathrm{Ma_1chuP1cchu}(x)$, $f'' \in [|f''|_\infty, |f''|_\infty + \epsilon]$, meaning $|f''|_\infty \in (|f|_\infty, |f|_\infty)$, violating the assumption that $|f|_\infty, |f'|_\infty$ are consecutive in $\mathrm{Ma_1chuP1cchu}(x)$. Else, $|f''| \leq L_\infty(x) - B_1 \leq |f|_\infty \leq |f|_t$, another contradiction.

So l', l are consecutive in $\mathscr{L}_t$, so $l' - l \in \Delta_t(x)$.

QEΔ

We find one more accumulation point of $\Delta_t(S)$ on the lower end.

Remark 12. In the following theorems, we will consider length sets as multisets, allowing repeated elements where multiple factorizations have the same length. Using this convention, we say that $0 \in \Delta_t(x)$ if there exists an element in the length multiset with multiplicity greater than 1; that is, if there is some common length achieved by two different factorizations of x.

Proposition 5. Let S be a numerical semigroup. $0 \in \Delta_t(S)$ for $t = 0, \infty$.

Proof. For $t = 0$, consider elements of the form $x_k = ka_1a_2 \in S,\ k > 2$. Then $(a_2, (k-1)a_1, \ldots), ((k-1)a_2, a_1, \ldots) \in Z(x_k)$, both with a 0-length of 2, so $0 \in \Delta_0(x_k)$. Since there are countably infinite x_k, $0 \in \Delta_0(x)$ for infinitely many x.

For $t = \infty$, consider elements of the form $x_k = ka_1a_2a_2 \in S,\ k \geq 1$. Then $(0, ka_1a_2, \ldots), (ka_1a_2, ka_1(a_2 - a_1), \ldots) \in Z(x_k)$, both with an ∞-length of ka_1a_2, so $0 \in \Delta_\infty(x_k)$. Since there are countably infinite x_k, $0 \in \Delta_\infty(x)$ for infinitely many x. QE∆

We first need another approximation lemma.

Lemma 4.2.3. Let $t \in (0, 1)$ and $\varepsilon, c > 0$. There exists N such that if $x > N$, $(x+c)^t - x^t \in (0, \varepsilon)$.

Proof. Let $\varepsilon, c > 0$, and let $N \geq (\frac{ct}{\varepsilon})^{\frac{1}{1-t}}$.

The first and second derivatives of x^t with respect to x are $t(x)^{t-1}$, $t(t-1)x^{t-2}$. For positive x, the first is always positive, while the second is always negative.

This means that:

1. $(x + c)^t > x^t$
2. The function x^t is concave down, so the linear approximation of $(x + c)^t$ around the point x exceeds the actual value of $(x + c)^t$. In other words. $x^t + ctx^{t-1} > (x + c)^t$.

So $(x + c)^t - x^t \in (0, ctx^{t-1})$. Then since $t - 1 < 0$ and $x > N \geq (\frac{ct}{\varepsilon})^{\frac{1}{1-t}}$,

$$ctx^{t-1} < ct\left(\left(\frac{ct}{\varepsilon}\right)^{\frac{1}{1-t}}\right)^{t-1} = ct\left(\frac{ct}{\varepsilon}\right)^{-1} = \varepsilon$$

. Therefore $(x + c)^t - x^t \in (0, \varepsilon)$. QE∆

The following proves that 0 is an accumulation point of $\Delta_t(S)$ where $t \in (0, 1)$:

Proposition 6. For all Numerical Semigroups $\langle a_1, ..., a_k\rangle$ and norm parameters $t \in (0, 1)$, 0 is an accumulation point of $\Delta_t(S)$ or appears in the Δ_t set of infinitely many x.

Proof. Let $S = \langle a_1, a_2, \ldots, a_k\rangle$ and let $t \in (0, 1) \cup (1, \infty)$. We will be considering elements in S which can be factored using just a_1 and a_2. It is sufficient to show that there exist factorizations of elements of this types that are an arbitrarily small distance apart, since any factorizations with lengths in between will necessarily also be a small distance apart.

From **[ateam]** (Lemma 2.14), we know that for $t < 1$, the maximum length real factorization of x using just these two generators occurs at

$$(w_1, w_2) = \left(\frac{a_1^{q/t}}{a_1^q + a_2^q}x, \frac{a_2^{q/t}}{a_1^q + a_2^q}x\right), \text{ where } q = \frac{t}{t-1} \text{ is the dual of } t.$$

Now let (w'_1, w'_2) be the integer factorization of x with the smallest w'_1 such that $w'_1 \geq w_1$, and let $(w^\star_1, w^\star_2)$ be the integer factorization with the smallest $w^\star_2$ such that $w^\star_2 > w_2$. (Note that if

$w_1' = w_1$, $w_2^\star = w_2$, so (w_1', w_2') and $(w_1^\star, w_2^\star)$ are necessarily distinct). Since a trade of the form $(a_2, 0, \ldots) \sim (0, a_1, \ldots)$ exists between the first two generators, clearly

$$w_1 \leq w_1' < w_1 + a_2, \quad w_2 - a_1 < w_2' \leq w_2 \quad \text{and} \quad w_1 - a_2 \leq w_1^\star < w_1, \quad w_2 < w_2^\star \leq w_2 + a_1 \tag{43}$$

Then, since the t-length of a factorization monotonically decreases from its maximum as either coordinate of (w_1, w_2) increases, $l' = |(w_1', w_2')|_t \leq |(w_1, w_2)|_t$ and $l^\star = |(w_1^\star, w_2^\star)|_t \geq |(w_1 - a_2, w_2 + a_1)|_t$, so

$$\begin{aligned} 0 \leq l' - l^\star &\leq |(w_1, w_2)|_t - |(w_1 - a_2, w_2 + a_1)|_t \\ &= (w_1^t + w_2^t) - ((w_1 - a_2)^t + (w_2 + a_1)^t) \\ &= \left(\left(\frac{a_1^{q/t}}{a_1^q + a_2^q} x\right)^t + \left(\frac{a_2^{q/t}}{a_1^q + a_2^q} x\right)^t\right) - \left(\left(\frac{a_1^{q/t}}{a_1^q + a_2^q} x - a_2\right)^t + \left(\frac{a_2^{q/t}}{a_1^q + a_2^q} x + a_1\right)^t\right) \end{aligned} \tag{44}$$

Since $t \in (0, 1)$, we can apply Lemma 4.2.3 to get:

$$\begin{aligned} &\lim_{x \to \infty} \Big((w_1^t + w_2^t) - ((w_1 - a_2)^t + (w_2 + a_1)^t) \Big) \\ &\quad = \lim_{x \to \infty} \Big(w_1^t - (w_1 - a_1)^t \Big) + \lim_{x \to \infty} \Big(-((w_2 + a_2)^t - w_2^t) \Big) = 0 - 0 = 0 \end{aligned} \tag{45}$$

Therefore as x increases, the upper bound on the distance between l' and $l^\star$ approaches 0. So, either $l' - l^\star$ is equal to 0 for an infinite number of x, or, if $l' - l^\star$ is equal to zero for a finite number of x, their difference be made arbitrarily small under this bound.

QEΔ

The result for $t \in (1, \infty)$ requires a famous result from analytic number theory that allows for efficient approximation of real numbers.

Theorem 4.2.4. For all Numerical Semigroups $\langle a_1, ..., a_k \rangle$ and norm parameters $t \in (1, \infty)$, 0 is an accumulation point of $\Delta_t(S)$ or appears in the Δ_t set of infinitely many x.

Proof. Let $\epsilon^2 > 0$. Consider the function $\mu : [0, 1] \to \mathbb{R}$; $\mu(r) = |\frac{1-r}{a_1}, \frac{r}{a_2}|_t$. We can think of this function as a curve describing the t-norms of the real factorizations of 1. From **[ateam]** (Proposition 3.6), this curve is U-shaped, with $\mu(1)$ not an extremum of the function μ. By the Intermediate Value Theorem, there exists $r \neq 1$ such that $\mu(r) = \mu(1) = \frac{1}{a_2}$. If r is rational, let $\frac{M}{N}$ be any ratio equal to r. Else, by Dirichlet's approximation theorem [38] and the Archimedean property of the reals, there exists $\frac{M}{N}$ such that

$$\left|\frac{M}{N} - r\right| < \frac{1}{N^2} < \left(\frac{\epsilon}{|(a_2, a_1)|_t}\right)^2 \tag{46}$$

.

Let $x := N a_1 a_2$. Then $(0, N a_1); ((N - M) a_2, M a_1) \in Z(x)$, meaning $N a_1 \in \mathscr{L}_t(x)$ and $|((N - M) a_2, M a_1)|_t \in \mathscr{L}_t(x)$. We will show that the second length is within ϵ of the first.

Pulling x out of the second length gives $x|(\frac{1-\frac{M}{N}}{a_1}, \frac{M}{Na_2})|_t$. If r is rational, this equals $x\mu(1) = Na_1$. Else, by equation 46, $(\frac{1-\frac{M}{N}}{a_1}, \frac{M}{Na_2})$ is in the region $(\frac{1-r \pm N^{-2}}{a_1}, \frac{r \pm N^{-2}}{a_2})$. A-team proved that μ is monotonic in a neighborhood around r, so if ϵ is small enough, μ preserves the boundaries of that region. So the second length is within $x|(\frac{1-r \pm N^{-2}}{a_1}, \frac{r \pm N^{-2}}{a_2})|_t$. By the triangle inequality, this is within $x(|(\frac{1-r}{a_1}, \frac{r}{a_2})|_t \pm (\frac{N^{-2}}{a_1}, \frac{N^{-2}}{a_2})_t)$. By definition of r, this equals $x\mu(1) \pm x(\frac{N^{-2}}{a_1}, \frac{N^{-2}}{a_2})_t = Na_1 \pm N^{-1}|(a_2, a_1)|_t$. Since $N^{-1} < \frac{\epsilon}{|(a_2,a_1)|_t}$, this is within $Na_1 \pm \epsilon$.

So the difference between lengths is 0 if r is rational and within $\pm\epsilon$ if r is irrational.

If r is rational, there exist infinitely many ratios $\frac{M}{N}$ equal to r. Each leads to an element of S with 0 in its Δ_t set, so 0 appears in the Δ_t set of infinitely many elements of S.

If r is irrational, $\frac{M}{N} \neq r$, so the difference between lengths cannot be exactly zero. So for all $\epsilon^2 > 0$, there exists $d \in \Delta_t(S)$ such that $d \in (0, \epsilon)$. So 0 is an accumulation point of $\Delta_t(S)$.

QEΔ

One might think that a similar approach could work for factorizations with less extreme disparity in coordinate sizes. However, the following result illustrates that said factorizations have t-norms that diverge increasingly from their ∞-norms. This implies that we cannot, in general, show that elements of $\Delta_\infty(S)$ are accumulation points of $\Delta_t(S)$ by directly approximating the t-norm of a factorization with the ∞-norm of the same factorization.

Proposition 7. Let $S : \langle a_1, ..., a_k \rangle$ be a numerical semigroup. Fix $t > 1; \epsilon > 0$. There exists infinitely many n with $f \in Z(n)$ such that $|f|_t - |f|_\infty > \epsilon$.

Proof. Let $n := x(\sum_{i=1}^{k} a_i)$, where $x > \frac{\epsilon}{k^{1/t}-1}$. Then $f := (x, ..., x) \in Z(n)$, with $|f|_\infty = x$ and $|f|_t = xk^{1/t}$. So $|f|_t - |f|_\infty = x(k^{1/t} - 1) > \epsilon$.

QEΔ

5 Miscellaneous

5.1 Minimal Presentation for Generalized Arithmetic Progression

Remark 13. The following expands the results of Omidali and Rahmati in [34] to provide an explicit minimal presentation for semigroups with generators in generalized arithmetic progression.

Let $S = \langle a,\ ha + d,\ ha + 2d, \ldots,\ ha + xd \rangle = \langle n_0, n_1, n_2, \ldots, n_x \rangle$ with $h \geq 1, x \leq a - 1$, and $\gcd(a, d) = 1$. Furthermore, set $r = a - 1 \mod x$ and $p = \frac{a-1-r}{x}$.

Omidali and Rahmati prove that the cardinality of the minimal presentation of S is $\frac{x(x-1)}{2} + x - r$. In particular, their proof specifies that $\frac{x(x-1)}{2}$ trades occur at Betti elements of the form $n = n_i + n_j,\ 1 \leq i, j \leq x - 1$ (call these Type I) and $x - r$ trades occur at Betti elements of the form $n = pn_x + n_k,\ r + 1 \leq k \leq x$ (call these Type II). Then, we can explicitly describe the full set of trades in the minimal presentation as follows:

Type I trades:

$n = 2n_1$	$(0, 2, 0, \ldots) \sim (h, 0, 1, \ldots)$
$n = n_1 + n_2$	$(0, 1, 1, 0, \ldots) \sim (h, 0, 0, 1, \ldots)$
$n = n_1 + n_3$	$(0, 1, 0, 1, 0, \ldots) \sim (h, 0, 0, 0, 1, \ldots)$
$\vdots$	$\vdots$
$n = n_1 + n_{x-1}$	$(0, 1, \ldots, 1, 0) \sim (h, 0, \ldots, 0, 1)$
$n = 2n_2$	$(0, 0, 2, 0, 0, \ldots) \sim (0, 1, 0, 1, 0, \ldots)$
$n = n_2 + n_3$	$(0, 0, 1, 1, 0, \ldots) \sim (0, 1, 0, 0, 1, \ldots)$
$\vdots$	$\vdots$
$n = n_2 + n_{x-1}$	$(0, 0, 1, \ldots, 1, 0) \sim (0, 1, 0, \ldots, 0, 1)$
$\vdots$	$\vdots$
$n = 2n_{x-2}$	$(\ldots, 0, 2, 0, 0) \sim (\ldots, 1, 0, 1, 0)$
$n = n_{x-2} + n_{x-1}$	$(\ldots, 0, 1, 1, 0) \sim (\ldots, 1, 0, 0, 1)$
$n = 2n_{x-1}$	$(\ldots, 0, 2, 0) \sim (\ldots, 1, 0, 1)$

Type II trades:

$n = pn_x + n_{r+1}$	$(0, \ldots, 1, 0, 0, \ldots, p) \sim ((p+1)h + d, 0, 0, \ldots, 0)$
$n = pn_x + n_{r+2}$	$(0, \ldots, 0, 1, 0, \ldots, p) \sim (ph + d, 1, 0, \ldots, 0)$
$n = pn_x + n_{r+3}$	$(0, \ldots, 0, 0, 1, \ldots, p) \sim (ph + d, 0, 1, \ldots, 0)$
$\vdots$	$\vdots$
$n = (p+1)n_x$	$(0, \ldots, p+1) \sim (ph + d, \ldots, 1, \ldots, 0)$

The first of the Type II trades follows from the fact that

$$pn_x + n_{r+1} = p(ha + xd) + ha + (r+1)d = (p+1)ha + (px + r + 1)d = ((p+1)h + d)a \tag{47}$$

since

$$px + r + 1 = \frac{a - 1 - r}{x}x + ((a-1) \mod x) + 1 = a - 1 - ((a-1) \mod x) + ((a-1) \mod x) + 1 = a. \tag{48}$$

Subsequent Type II trades increase n by d, which can be expressed as

$$pn_x + n_{r+k} = pn_x + n_{r+1} + (k-1)d = ((p+1)h + d)a + (k-1)d = (ph + d)a + ha + (k-1)d, \quad 1 < k \leq x - r \tag{49}$$

Since none of the trades of Type I or Type II variety share a common support, we know that they are all distinct. And, since they total exactly $\frac{x(x-1)}{2} + x - r$ trades, using the Betti elements prescribed by Omidali and Rahmati, we know that they form a minimal presentation for S.

6 Algorithms

This section describes, proves correctness, and analyses asymptotic runtime of dynamic algorithms for computing length and delta sets for the 0- and ∞- norm. Computing those invariants from factorization sets quickly proves intractable, as factorization sets themselves require prohibitive memory and computational resources to compute.

We begin with the 0 norm.

6.1 0-Norm Algorithms

The 0-norm algorithms are based on the following relationship between the support sets of the factorization sets of elements, derived from the relationship between factorization sets described in Lemma 3.1 of [4]

Lemma 6.1.1. Let $S := \langle a_1, ..., a_1 \rangle$ be a numerical semigroup. For each nonzero $x \in S$, let $\text{supp}(x) := \{\text{sgn}(f) : f \in Z(x)\}$. Then

$$\text{supp}(x) = \bigcup_{i=1}^{k} \{\text{sgn}(a + e_i) : a \in \text{supp}(x - a_i)\}$$

.

Proof. Take sgn of all vectors in the sets on either side of the first equality in Lemma 3.1 of [4]. QEΔ

This motivates the following algorithm for computing support sets.

Algorithm 1 Given $n \geq 0$, $S = \langle a_1, ..., a_k \rangle$, computes $\text{supp}(x)$ for all $x \in [0, n]$.

$V[0] \leftarrow \{\hat{0}\}$
for $x \in [1, n]$ **do**
  $v \leftarrow \{\}$
  for $i \in [1, k]$ with $x - a_i \in S$ **do**
    $v \leftarrow v \cup \{\text{sgn}(s + e_i) : s \in V[x - a_i]\}$
  end for
  $V[x] \leftarrow v$
end for
return V

Computing lengths from V is straightforward:

Algorithm 2 Given $n \geq 0$, $S = \langle a_1, ..., a_k \rangle$, computes $\mathscr{L}_0(x)$ for all $x \in [0, n]$.

$V[x] \leftarrow \text{supp}(x)$ for all $x \in [0, n]$.
$L[x] \leftarrow \{|s|_0 : s \in V[x]\}$ for all $x \in [0, n]$. **return** L

After which Δ sets for all elements may be computed in a similar fashion. As in [4], a ring buffer can decrease memory requirements in the Δ set algorithm.

6.2 ∞-Norm Algorithms

The algorithms for ∞-norm are based on the following relationship between $\mathscr{L}_\infty(x, i)$ sets (as defined in Section **??**) of elements of S.

Lemma 6.2.1. Let $S := \langle a_1, ..., a_k \rangle$ be a numerical semigroup, and let $i \in [1, k]$. Let $\mathcal{P}_i := \{s : i \in s; s \in \mathcal{P}([1, k])\}$. For all $x \in S$, $\mathscr{L}_\infty(x, i) = \bigcup_{s \in P_i} \{1 + l : l \in \mathscr{L}_\infty(x - \sum\limits_{j \in s} a_j, i)\}$.

Proof. Suppose $l \in \mathscr{L}_\infty(x, i)$, with the factorization $f = (f_1, ..., f_k)$ witnessing. Let $c := \{j : f_j = \max\{f_1, ..., f_k\}\}$ be the set of indices of maximal coordinates of f. Then $f - \sum_{j \in c} e_j = (f'_1, ..., f'_k)$ is a factorization of $x - \sum\limits_{j \in c} a_j$ with $c \subseteq \{j : f'_j = \max\{f'_1, ..., f'_k\}\}$; in particular, $f'_i = |f - \sum_{j \in c} e_j|_\infty$. Moreover, $f'_i = f_i - 1 = l - 1$. So $l - 1 \in \mathscr{L}_\infty(x - \sum j \in c a_j, i) \subseteq \bigcup_{s \in P_i} \mathscr{L}_\infty(x - \sum\limits_{j \in s} a_j, i)$. This proves one direction of containment.

Now suppose $l \in \mathscr{L}_\infty(x - \sum\limits_{j \in c} a_j, i)$ for some $c \subseteq \mathcal{P}_i$, with the factorization $f = (f_1, ..., f_k)$ witnessing, i.e., $f_i = \max\{f_1, ..., f_k\}$. Let $f' := f + \sum\limits_{j \in c} e_j$. For all $j \in [1, k]$, $f'_j \leq f_j + 1$, meaning $\max\{f'_1, ..., f'_k\} \leq \max\{f_1, ..., f_k\} + 1 \leq f_i + 1$. Since $i \in c$, $f'_i = f_i + 1$. So $f_i + 1 = \max\{f'_1, ..., f'_k\} = |f'|_\infty$. So f' is a factorization of x with maximal coordinate i and length $l + 1$. So $l + 1 \in \mathscr{L}_\infty(x, i)$, proving the reverse containment.

QEΔ

This motivates the following algorithm for computing $\mathscr{L}_\infty(\cdot, i)$ and $\mathscr{L}_\infty$ sets, from which we compute delta sets in the expected fashion.

Algorithm 3 Given $n \geq 0$, $S := \langle a_1, ..., a_k \rangle$, computes $\mathscr{L}_\infty(x, i)$ for all $x \in [0, n]$, $i \in [1, k]$

.

$L[0][i] \leftarrow \{0\}$ for all $i \in [1, k]$
for $x \in [1, n]$ **do**
  $L[x][i] \leftarrow \{\}$ for all $i \in [1, k]$
  for $s \in \mathcal{P}([1, k])$ **do**
    for $i \in s$ **do**
      $L[x][i] \leftarrow L[x][i] \cup \{l + 1 : l \in L[x - \sum\limits_{j \in s} a_j][i]\}$
    end for
  end for
end for
for $x \in [1, n]$ **do**
  $\mathscr{L}[x] \leftarrow \bigcup_{i=1}^{k} L[x][i]$
end forreturn $L, \mathscr{L}$

6.3 Asymptotic and Experimental Runtime

The outer loop of the support set algorithm runs n times. During each iteration, the algorithm performs a single assignment operation on v, then performs k set unions between v and an earlier support set. Since the size of a support set is at most $|\mathcal{P}(\{a_1, ..., a_k\})| = 2^k$, a constant, each set union takes constant time to perform, resulting in constant time taken for each iteration. The overall asymptotic runtime for the support set algorithm is then $O(n)$.

The 0-Length set algorithm performs the support set algorithm, followed by an $O(n)$ operation on its output, requiring $O(n)$ runtime. The 0-Delta set performs the Length set algorithm, followed by an $O(n)$ operation on its output, also requiring $O(n)$ runtime.

The ∞ length set algorithm initializes in constant time before executing its outer loop n times. It then loops over every set in $\mathcal{P}([1, k])$, which has constant cardinality 2^k, and, at each iteration, loops over every element in the selected set, which has cardinality at most k. The innermost line, a set union, then executes at most $k^3 \in O(1)$ times per iteration of the outer loop. Since those sets are bounded above by $\frac{x}{a_i} \in O(n)$, each set union then takes $O(n)$ time, meaning each iteration of the outer loop takes $O(n)$ time. So the outer loop requires $O(n^2)$ time. The final loop executes n times. Its inner loop performs a constant number (k) of set unions, each of which involves sets that are, as mentioned, size $O(n)$. So the final loop requires $O(n^2)$ time as well, for $O(n^2)$ time overall.

Below is a table of experimental runtimes for both algorithms on eight randomly selected numerical semigroups:

Semigroup	n	Δ_0 time	Δ_∞ time
$\langle 10, 17, 19, 25, 31\rangle$	10000	2389 ms	80651 ms
$\langle 7, 15, 17, 18, 20\rangle$	10000	2201 ms	102140
$\langle 7, 19, 20, 25, 29\rangle$	10000	2180 ms	84167
$\langle 11, 53, 73, 87\rangle$	10000	861 ms	18961
$\langle 31, 73, 77, 87, 91\rangle$	50000	3111 ms	546166
$\langle 51, 53, 55, 117\rangle$	50000	4990 ms	253269 ms
$\langle 100, 121, 142, 163\rangle$	10^5	21935 ms	375586 ms
$\langle 1001, 1211, 1421, 1631, 2841$	10^6	327559 ms	–

Note that the machine running these procedures was unable to compute the factorization sets up to the assigned threshold for any of these numerical semigroups. The algorithms in this section made those computations possible.

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