# ON THE T-ELASTICITY OF NUMERICAL SEMIGROUPS

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Abstract. In this paper we are going to discuss how extending the definition of length affects the elasticity of different numerical semigroups. The primary focus will be looking at properties that affect the elasticity through both the local and global perspectives. We will fully characterize the global $t$-elasticity for a semigroup with any number of generators, as well as $\tau$, the set of values of $t$ for which the $t$-elasticity is accepted in the two generated case.

## 1. Introduction

In this paper we will be looking at the various properties of numerical semigroups and most specifically the properties regarding the t-elastisity of a numerical semigroup (see [1] for more). We will start with some background information to assist the reader in understanding these concepts.

Throughout this paper, fix $k \in \mathbb{N}$, and let $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$ be a minimal integer vector whose entries are in monotone increasing sequence. A **numerical semigroup**, $S = \langle a_1, \ldots, a_k \rangle$, is a subset of the non-negative integers that is closed under addition and has a finite complement t in the non-negative integers. See [7] or [8] for a more thorough expansion. Namely,

$$S = \{n \in \mathbb{Z}_{\geq 0} : n = a_1 z_1 + \ldots + a_k z_k, z_i \in \mathbb{Z}_{\geq 0}\}$$

Let $S = \langle \mathbf{a} \rangle$ be the semigroup generated by the entries of $\mathbf{a}$. We call $\mathbf{a}$ the *generating vector* of the semigroup. For example, we can look at the numerical semigroup, $S = \langle 4, 7 \rangle$. Written in set notation we have that $S = \{0, 4, 7, 8, 11, 12, 14, ...\}$

A *factorization* $\mathbf{z} \in \mathbb{N}_0^k$ of a semigroup element $n$ is a $k$-tuple which encodes a decomposition of $n$ into atoms (generators of the semigroup). This can be written as $\mathbf{z} = (z_1, \ldots, z_k)$. Keeping with the same semigroup as before, $\langle 4, 7 \rangle$, let's look at the element $n = 53$. While this particular element has only two factorizations, it is important to note that other semigroup elements can have more factorizations. The specific factorizations of 53 are $(8, 3)$ and $(1, 7)$. These can be written out as polynomials which help to visualize why these factorizations work for $n = 53$. Namely,

$$(8, 3) \to 8(4) + 3(7) = 53$$

$$(1, 7) \to 1(4) + 7(7) = 53$$

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In the study of semigroups, the notion of the *length* of a factorization is typically defined as its coordinate sum, or $\ell_1$ norm. We will extend this notion to all $\ell_p$ norms using a real-valued parameter $t \in [0, \infty)$, defining the *t-length* of a factorization in terms of its $\ell_t$ norm. Recall that the $\ell_t$ norm is the map

$$||\cdot||_t : \mathbb{R}^k \to \mathbb{R} \qquad \text{given by} \qquad \begin{cases} \left(\sum_{j=1}^k z_j^t\right)^{1/t} \text{for } 0 \leq t \leq 1 \\ \left(\sum_{j=1}^k z_j^t\right)^{1/t} \text{for} t > 1 \end{cases}$$

As $t \to \infty$, it is known that $||\mathbf{z}||_t \to \max\{z_j\}$, so we define $||\mathbf{z}||_\infty = \max\{z_j\}$ according to that limit. When $\frac{1}{t} + \frac{1}{s} = 1$, $s$ and $t$ are called Hölder conjugates, and $\ell_t$ and $\ell_s$ are dual vector spaces. For any fixed $t$, we let $s(t) = \frac{t}{t-1}$ denote the conjugate parameter value of $t$. Additionally, we will define $q(t) = \frac{t}{t-1}\text{where}\frac{1}{t} + \frac{1}{q} = 1$ as *duals.* We say that 1 and $\infty$ are duals.

Let's continue with the same example and let $t = 2$. Let $\mathbf{z} = (8, 3)\text{and}\mathbf{w} = (1, 7)$. So we have that

$$\|\mathbf{z}\|_2 = (8^2 + 3^2)^{\frac{1}{2}} = \sqrt{73}$$
$$\|\mathbf{w}\|_2 = (1^2 + 7^2)^{\frac{1}{2}} = \sqrt{50}$$

For each semigroup element $n$, we denote its *t-length set*

$$\mathscr{L}_t(n) = \{||\mathbf{z}||_t \mid \mathbf{z} \in \pi^{-1}(n)\}.$$

as well as its minimum and maximum factorization lengths,

$$\ell_t(n) = \min \mathscr{L}_t(n) \qquad \text{and} \qquad L_t(n) = \max \mathscr{L}_t(n),$$

not to be confused with the well-known sequence/function spaces. For $S = \langle 4, 7\rangle, n = 53$ and $t = 2$ we have that

$$\mathscr{L}_2(53) = \{\sqrt{50}, \sqrt{73}\}$$
$$\min \mathscr{L}_2(53) = \sqrt{50}$$
$$\max \mathscr{L}_2(53) = \sqrt{73}$$

For more in depth information on lengths, see [**?**], or [6].

We define the *t-elasticity* $\rho_t$ of a semigroup element as

$$\rho_t(n) = \frac{L_t(n)}{\ell_t(n)},$$

referred to as *local* $t$-elasticity, in contrast to the elasticity of the semigroup, which we define

$$\rho_t(S) = \sup_{n \in S}\{\rho_t(n)\},$$

called the *global* $t$-elasticity.If one is interested in reading more about Elasticity, see [3], and [4], for more.
Finally, we will complete this example by looking at the local and global elasticises for 53 and $S$ respectively. So we have

$$\rho_2(53) = \frac{L_t(n)}{\ell_t(n)} = \frac{\sqrt{73}}{\sqrt{53}} \quad \text{and} \quad \rho_2(S) = \sup_{53\in S}\{\rho_2(53)\} = \sqrt{73}$$

For instructions on how to generate information on numerical semigroups go to [5].

## 2. Global Elasticity

### 2.1. Global Elasticity on $[1, \infty]$.

**Lemma 2.1.** *(Extreme values over $\mathbb{R}^k_{\geq 0}$) Fix $n \geq 1 \in \mathbb{N}_0$ and $t > 1 \in \mathbb{R}$. Define $C : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$ by $C(\mathbf{x}) = \mathbf{a}\cdot\mathbf{x} - n$. Let $\mathbf{e}_i$ denote the $i$-th standard basis vector, and let $\mathbf{V}(C) \subset \mathbb{R}^k_{\geq 0}$ denote the vanishing set of $C$. Finally, let $q(t) = t/(t-1)$ for $t \in (1, \infty)$. For all $k \geq 2$ and $\mathbf{x} \in \mathbb{N}^k_0 \cap \mathbf{V}(C)$,*

$$\min_{\mathbb{R}^k_0\cap\mathbf{V}(C)}(||\mathbf{x}||_t) = \frac{n}{||\mathbf{a}||_q} \qquad \textit{and} \qquad \max_{\mathbb{R}^k_0\cap\mathbf{V}(C)}(||\mathbf{x}||_t) = \frac{n}{a_1},$$

*which occur at the (possibly non-integer) points*

$$n\mathbf{m}(t) = \frac{n}{(||\mathbf{a}||_q)^q}\sum_{i\in[k]} a_i^{\frac{1}{t-1}}\mathbf{e}_i \qquad \textit{and} \qquad n\mathbf{M} = \frac{n\mathbf{e}_1}{a_1},$$

*respectively.*

*Proof.* We are interested in the extreme values of $||\cdot||_t$ over $\mathbb{N}^k_{\geq 0}\cap\mathbf{V}(C)$. We will obtain bounds on the extrema of $||\cdot||_t$ over $\mathbb{N}^k_0$ by solving the optimization explicitly over $\mathbb{R}^k_{\geq 0}$ using the method of Lagrange multipliers.

Since $g(x) = \sqrt[t]{x}$ increases monotonically for $x \geq 0$, $t\rangle 1$, the set of points where $||\cdot||_t$ takes on its extreme values agrees with the set of points at which extrema of $||\cdot||_t^t$ are achieved. As such, we can simplify the problem to locating the points in $\mathbb{R}^k_{\geq 0}\cap\mathbf{V}(C)$ that optimize the function $f(\mathbf{x}) := ||\mathbf{x}||_t^t$.

The method of Lagrange multipliers relies on the fact that the extrema of $f$ over $\mathbf{V}(C)$ must occur at points $\mathbf{x} \in \mathbf{V}(C)$ satisfying at least one of the following criteria:

i) $(\mathbf{x}, \lambda)$ is a solution to the Lagrange system,

$$L(\mathbf{x}, \lambda) = \nabla f(\mathbf{x}) - \lambda\nabla C(\mathbf{x}) = 0$$
$$\hat{C}(\mathbf{x}, \lambda) = C(\mathbf{x}) = 0.$$

ii) $\mathbf{x} \in \partial(\mathbb{R}^k_{\geq 0}\cap\mathbf{V}(C))$ (that is, $\mathbf{x}$ lies on the boundary of $\mathbf{V}(C) \subset \mathbb{R}^k_{\geq 0}$.

Fix $k > 2$ and $t \in (1, \infty)$ We wish to locate points $\mathbf{x} \in \mathbf{V}(C) \subset \mathbb{R}^k_{\geq 0}$ where the extrema of $f(\mathbf{x}) = \sum_{i=1}^k x_i^t$ are achieved. We begin with the Lagrange system, which consists of the $k+1$ equations.

$$L_i(\mathbf{x}, \lambda) = t x_i^{t-1} - \lambda a_i \tag{2.1}$$

$$\hat{C}(\mathbf{x}, \lambda) = C(\mathbf{x}) = -n + \sum_{i \in [k]} a_i x_i = 0. \tag{2.2}$$

Each $L_i$ can be solved for $x_i$ in terms of $\lambda$, obtaining

$$x_i = \left(\frac{\lambda a_i}{t}\right)^{\frac{1}{t-1}} \tag{2.3}$$

We substitute into (2) to get an equation dependent only on $\lambda$, namely

$$0 = -n + \left(\frac{\lambda}{t}\right)^{\frac{1}{t-1}} \sum_{i \in [k]} a_i^{\frac{t}{t-1}}.$$

We use algebra to solve for $\lambda$, which results in

$$\lambda = t n^{t-1} \left(\sum_{i \in [k]} a_i^q\right)^{-t/q} = t n^{t-1} \left|\left|\mathbf{a}\right|\right|_q^{-t}.$$

Back-substitution into (3) gives, for each $i$,

$$x_i = \left(\frac{t n^{t-1}}{\left|\left|\mathbf{a}\right|\right|_q^t} \cdot \frac{a_i}{t}\right)^{1/(t-1)} = \frac{n a_i^{1/(t-1)}}{\left|\left|\mathbf{a}\right|\right|_q^{t/t-1}} = \frac{n a_i^{q/t}}{\left|\left|\mathbf{a}\right|\right|_q^q}.$$

We can write our solution $n\mathbf{m}$, where

$$\mathbf{m} = \sum_{i \in [k]} \frac{a_i^{q/t} \mathbf{e}_i}{\left|\left|\mathbf{a}\right|\right|_q^q}$$

We take the $t$-norm of $n\mathbf{m}$, our unique solution interior to $\mathbf{V}(C)$, which gives us

$$\left|\left|n\mathbf{m}\right|\right|_t = n \left|\left|\mathbf{m}\right|\right|_t = \frac{n}{\left|\left|\mathbf{a}\right|\right|_q^q} \left(\sum_{i \in [k]} a_i^{t/(t-1)}\right)^{1/t} = \frac{n}{\left|\left|\mathbf{a}\right|\right|_q^q} \left|\left|\mathbf{a}\right|\right|_q^{q/t} = n \left|\left|\mathbf{a}\right|\right|_q^{-q(1-1/t)} = n \left|\left|\mathbf{a}\right|\right|_q^{-1}.$$

Now, we must check the points on the boundaries. We claim $n\mathbf{m}(t)$ has minimal $t$-norm and that $\mathbf{M} = \dfrac{n\mathbf{e}_k}{a_k}$ has maximal $t$-norm. We will prove this by induction on $k$, the number of variables.

In the base case, when $k = 2$, $\partial\mathbf{V}(C)$ consists of the endpoints of the line segment $a_1x_1 + a_2x_2 - n = 0$ in the first quadrant, which are $\frac{n\mathbf{e}_1}{a_1}$ and $\frac{n\mathbf{e}_2}{a_2}$, with $t$-norms $\frac{n}{a_1}$ and $\frac{n}{a_2}$, respectively. It is easy to see that

$$\frac{n}{||a||_q} \leq \frac{n}{a_2} \leq \frac{n}{a_1},$$

since $a_1^q \leq a_2^q \leq a_1^q + a_2^q$. This completes our proof of the base case.

Now, we assume that our lemma holds in $k-1$ variables. By definition, $\mathbf{V}(C)$ is a $(k-1)$-dimensional hyperplane in $\mathbb{R}^k_{\geq 0}$, which is bounded by the $k$ hyperplanes $x_i = 0$ for each $i$. The intersection of each bounding hyperplane with $\mathbf{V}(C)$ consists of points satisfying equations

$$\partial C_j(\mathbf{x}) = -n + \sum_{i \neq j \in [k]} a_i x_i = 0$$

for each $j \in [k]$, obtained by substituting $x_j = 0$ into the defining equation for $\mathbf{V}(C)$.

Notice that each $\partial C_j$ is a constraint equation of the desired form in $k-1$ variables. As such, we can apply our inductive hypothesis to conclude that

$$m_j = \min_{\mathbf{V}(\partial C_j)}(||\mathbf{x}||_t) = n\left(\sum_{i \neq j \in [k]} a_i^q\right)^{-\frac{1}{q}} \qquad \text{and} \qquad M_j = \max_{\mathbf{V}(\partial C_j)}(||\mathbf{x}||_t) = \begin{cases} \frac{n}{a_2} & j = 1 \\ \frac{n}{a_1} & \text{otherwise} \end{cases}.$$

for each $j$, which fully characterizes the extrema on the boundary of $\mathbf{V}(C)$.

We can therefore conclude that

$$\min_{\mathbf{V}(C)}(||\mathbf{x}||_t) = \min\left(\left\{\frac{n}{||\mathbf{a}||_q}\right\} \cup \{m_j \mid j \in [k]\}\right) = \frac{n}{||\mathbf{a}||_q}$$

and

$$\max_{\mathbf{V}(C)}(||\mathbf{x}||_t) = \max\left\{\frac{n}{a_1}, \frac{n}{a_2}\right\} = \frac{n}{a_1},$$

as desired.

$\square$

**Lemma 2.2.** *(Global $\infty$-elasticity.)*
*For all $S = \langle a_1, a_2, \ldots, a_k \rangle$ where $a_1 < a_2 < \cdots < a_k$ and $a = (a_1, a_2, \ldots, a_k)$,*

$$\rho_\infty(S) = \frac{\|a\|_1}{a_1},$$

*and the elasticity is accepted.*

*Proof.* Let $S = \langle a_1, a_2, \ldots, a_k \rangle$ and $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{N}_0^k$. Let $n = \sum_{i=1}^{k} a_i x_i \in S$, where $\mathbf{x} \in \mathbb{R}^k_{\geq 0}$. We have that $\mathbf{z} = (\frac{n}{a_1}, 0, \ldots, 0)$ is a factorization of $n$ Any other real factorization can be written as $\mathbf{x} = \mathbf{z} + \boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is a real-valued trade in $\ker \mathbf{a}^T$ with $\alpha_1 < 0$ satisfying

$$-\alpha_1 = \frac{a_2}{a_1}\alpha_2 + \frac{a_3}{a_1}\alpha_3 + \cdots + \frac{a_k}{a_1}\alpha_k > 0 \qquad \text{and} \qquad \alpha_1, \ldots, \alpha_k \in \mathbb{R}.$$

Each entry of the solution must be non-negative, so $\frac{n}{a_1} - \alpha_1 \geq 0$, and $\alpha_2, \ldots, \alpha_k \geq 0$. For any $\alpha_i$ such that $2 \leq i \leq k$, we have $\alpha_i \leq \frac{a_1}{a_i}\alpha_1$, since $\alpha_i = \frac{a_1}{a_i}\alpha_1$ when all other values $\alpha_2, \ldots, \alpha_k = 0$, excluding $\alpha_i$. For each $i$, $\frac{a_1}{a_i}\alpha_1 \geq 0$ and we know that $a_i > a_1$, so we must have that $\alpha_1 > \frac{a_1}{a_i}\alpha_1$. Then, because $\frac{n}{a_1} \geq \alpha_1$, we find that

$$\frac{n}{a_1} \geq \alpha_1 \geq \frac{a_1}{a_i}\alpha_1.$$

Therefore, $\|\mathbf{x}\|_\infty \leq \frac{n}{a_1}$.

Now, consider the rational factorization $\mathbf{z} = (\frac{n}{\|a\|_1}, \frac{n}{\|a\|_1}, \ldots, \frac{n}{\|a\|_1})$ of $n \in S$. Any other factorization may be written as $\mathbf{z} + \boldsymbol{\alpha}$ where $\boldsymbol{\alpha}$ is a non-zero trade in $\ker \mathbf{a}^T$. Since at least one $\alpha_i$ must be positive, $||\mathbf{z}||_\infty \geq z_i + \alpha_i \geq z_i = \frac{n}{||a||_1}$. We have thus shown that

$$\max_{\mathbb{R}^k_{\geq 0}} \mathscr{L}_\infty(n) \leq \frac{n}{a_1} \qquad \text{and} \qquad \min_{\mathbb{R}^k_{\geq 0}} \mathscr{L}_\infty \geq \frac{n}{||\mathbf{a}||_1}$$

Since $\mathbb{N}^k \subseteq \mathbb{R}^k$, these bounds hold for integral factorizations of $n$ as well. As such,

$$\rho_\infty(n) = \frac{\max\{\mathscr{L}_\infty(n)\}}{\min\{\mathscr{L}_\infty(n)\}} \leq \frac{\frac{n}{a_1}}{\frac{n}{\|a\|_1}} = \frac{\|a\|_1}{a_1}$$

.

Let $n = a_1\|a\|_1 \in S$ for $q \in \mathbb{Z}_{\geq 0}$. $\frac{n}{a_1} = \|a\|_1$ and $\frac{n}{\|a\|_1} = a_1$. We have that $(\|a\|_1, 0, \ldots, 0)$ is a factorization of $n$, so $\max\{\mathscr{L}_\infty(n)\} = \|a\|_1 = \frac{n}{a_1}$. Also, $(a_1, a_1, \ldots, a_1)$ is a factorization of $n$, so $\min\{\mathscr{L}_\infty(n)\} = a_1 = \frac{n}{\|a\|_1}$. Thus,

$$\rho_\infty(n) = \frac{\max\{\mathscr{L}_\infty(n)\}}{\min\{\mathscr{L}_\infty(n)\}} = \frac{\|a\|_1}{a_1}.$$

Therefore, $\rho_\infty(S) = \frac{\|a\|_1}{a_1}$ and this elasticity is accepted. $\square$

**Theorem 2.3.** *(Global t-Elasticity Formula for Numerical Semigroups.)*

*Fix* $\mathbf{a} \in \mathbb{N}_0^k$ *with* $a_i < a_j$ *when* $i < j$ *and* $\gcd(\{a_i\}) = 1$. *Define* $q(t) = \begin{cases} \infty & t = 1 \\ t/(t-1) & t \in (1, \infty) \\ 1 & t = \infty \end{cases}$, *so that* $1/t + 1/q = 1$. *Fix* $S = \langle \mathbf{a} \rangle$. *Then*

$$\rho_t(S) = \frac{||\mathbf{a}||_{q(t)}}{a_1}.$$

*Furthermore, if* $t \in \{1, 2, \infty\}$, *then the elasticity is accepted.*

*Proof of Theorem.* The 1-elasticity of any numerical semigroup $S$ is known to be accepted, and is given by $\rho_1(S) = \frac{a_k}{a_1} = \frac{||\mathbf{a}||_\infty}{a_1}$, which agrees with the claimed formula, if we set $q = \infty$ when $t = 1$. Karina's lemma verifies our claim in the $t = \infty$ case, so it remains to be shown for $t \in (1, \infty)$.

Fix $t > 1$. Lemma 1 gives an upper bound on the global elasticity of $S$, namely

$$\rho_t(S) \leq \frac{||\mathbf{M}||_t}{||\mathbf{m}||_t} = \frac{||\mathbf{a}||_q}{a_1}.$$

Recall that the minimal and maximal $t$-norms over the points satisfying $C(\mathbf{x}) = 0$ occur at

$$n\mathbf{m}(t) = \frac{n}{(||\mathbf{a}||_q)^q} \sum_{i \in [k]} a_i^{1/(t-1)} \mathbf{e}_i \qquad \text{and} \qquad n\mathbf{M} = \frac{n}{a_1}\mathbf{e}_1,$$

respectively.

For $t = 2$, notice that when $n = a_1 ||\mathbf{a}||_2^2 \in \mathbb{N}_0$, the points that give extreme values evaluate to

$$a_1 ||\mathbf{a}||_2^2 \mathbf{m}(2) = a_1 \sum_{i \in [k]} a_i \mathbf{e}_i \qquad \text{and} \qquad a_1 ||\mathbf{a}||_2^2 \mathbf{M} = ||\mathbf{a}||_2^2 \mathbf{e}_1,$$

which are both integer points. As such, $\rho_2(n) = \frac{||a||_2}{a_1}$ is maximal, which verifies that in this case, the elasticity is as claimed and that it is accepted.

To verify our elasticity formula for any other fixed $t > 1$, we must ensure the of existence of $n \in S$ with elasticity arbitrarily close to our upper bound. Consider the set of points

$$\mathbf{V} = \{\mathbf{x} \in \mathbb{R}^k \mid \sum_{j \in [k]} a_j x_j = a_1\}.$$

It is easy to see that $a_1\mathbf{M} = \mathbf{e}_1$ is a factorization of $a_1$ in $S$ of maximal t-length. Since $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$, and thus dense in $\mathbf{V} \cap \mathbb{R}^k$, we can ensure the existence of an infinite sequence of rational factorizations of $a_1$, $\{\mathbf{z}_i\} \in \mathbf{V}$ that converges to $a_1\mathbf{m}$, which is irrational for $t \neq 2$. Since $\{\mathbf{z}_i\} \to a_1\mathbf{m}$, it follows that $\{||\mathbf{z}_i||_t\} \to ||a_1\mathbf{m}||_t$. As such, for every $\epsilon > 0$, there exists $N \in \mathbb{N}_0$ so that $\alpha > N \Rightarrow ||\mathbf{z}_\alpha||_t - a_1 ||\mathbf{m}||_t < \epsilon$, which is a positive quantity since $a_1\mathbf{m}$ has minimal t-norm.

Fix $\epsilon > 0$, and choose $\alpha > N(\epsilon)$. We can write the rational factorization $\mathbf{z}_\alpha$ as

$$\mathbf{z}_\alpha = \sum_{j \in [k]} \frac{b_j}{c_j} \mathbf{e}_j,$$

with relatively prime $b_j \in \mathbb{N}_0$, $c_j \in \mathbb{Z}_{>0}$ for each $j$.

Let $C = \operatorname{lcm}(\{c_j\})$. then $C\mathbf{z}_\alpha$ is an integer point, and hence a factorization of $a_1C \in S$, and $a_1C\mathbf{M}$ is as well. As such, we have

$$\rho_t(Ca_1) = \frac{||a_1C\mathbf{M}||_t}{||C\mathbf{z}_\alpha||_t} = \frac{a_1 ||\mathbf{M}||_t}{||\mathbf{z}_\alpha||_t} > \frac{a_1 ||\mathbf{M}||_t}{a_1 ||\mathbf{m}||_t + \epsilon} = \frac{||\mathbf{M}||_t}{||\mathbf{m}||_t + \epsilon/a_1}.$$

Since our choice of $\epsilon$ was arbitrary, this implies that one can obtain an element with elasticity arbitrarily close to $\frac{||\mathbf{M}||_t}{||\mathbf{m}||_t} = \frac{||\mathbf{a}||_q}{a_1}$. More formally, we have shown that

$$\frac{||\mathbf{M}||_t}{||\mathbf{m}||_t + \epsilon/a_1} < \rho_t(S) \leq \frac{||\mathbf{M}||_t}{||\mathbf{m}||_t}$$

which verifies equality of our formula for $\rho_t(S)$ by the squeeze theorem. $\square$

**Theorem 2.4.** *For $t \geq 1$, $\rho_t(S)$ is non-decreasing.*

*Proof.* By Theorem 3.3, if $t \geq 1$, $\rho_t(S) = \frac{\|a\|_{q(t)}}{a_1}$ where $q(t) = \frac{t}{t-1}$. First, we will show that $q(t)$ is decreasing. $q'(t) = \frac{(t-1)-t}{(t-1)^2} = \frac{-1}{(t-1)^2} < 0$. Then, if $t_1 \geq t_2 \geq 1$, $q(t_1) \leq q(t_2)$. We know that this implies $\|a\|_{q(t_1)} \geq \|a\|_{q(t_2)}$ [9]. Thus, $\rho_{t_1}(S) = \frac{\|a\|_{q(t_1)}}{a_1} \geq \frac{\|a\|_{q(t_2)}}{a_1} = \rho_{t_2}(S)$, and $\rho_t(S)$ is non-decreasing. $\square$

**Remark 2.5.** It is possible for elasticity to be accepted for any other rational value of $t > 1$. This was illuminated by the reparametrization and manipulation of our minimal $t$-norm solution using the substitution $u = t - 1$. Then for $u \in (0, \infty)$,

$$\begin{aligned}
\mathbf{m}(u) &= \sum_{j=1}^{k} a_j^{1/u} \left( \sum_{i=1}^{k} a_i^{(u+1)/u} \right)^{-1} \mathbf{e}_j \\
&= \sum_{j=1}^{k} a_j^{1/u} \left( \sum_{i=1}^{k} a_i a_i^{1/u} \right)^{-1} \mathbf{e}_j \\
&= \sum_{j=1}^{k} a_j^{1/u} \left( \left( \frac{a_j}{a_j} \right)^{1/u} \sum_{i=1}^{k} a_i a_i^{1/u} \right)^{-1} \mathbf{e}_j \\
&= \sum_{j=1}^{k} \left( \sum_{i=1}^{k} a_i \left( \frac{a_i}{a_j} \right)^{1/u} \right)^{-1} \mathbf{e}_j.
\end{aligned}$$

**Claim 2.6.** *Fix any rational $t > 1$. There exist infinitely many generating vectors $\mathbf{b} \in \mathbb{R}^k$ such that $\rho_t(\langle \mathbf{b} \rangle)$ is accepted.*

*Proof.* Fix rational $t = \alpha/\beta > 1$ for coprime, non-negative integers with $\alpha > \beta$ and $\beta \neq 0$ and let $p_i$ be the $i$-th positive prime. Then we can write

$$u = t - 1 = \frac{\alpha - \beta}{\beta}.$$

Define $\mathbf{b}(t) \in \mathbb{R}^k$ by $\mathbf{b}(t) = \sum_{i=1}^{k} p_i^{q(\alpha-\beta)} \mathbf{e}_i$, choosing $q \in \mathbb{N}$ such that $p_1^{q(\alpha-\beta)} > k$, to ensure that the multiplicity is at least the number of generators. Consider $S = \langle \mathbf{b}(t) \rangle$, and let $n = p_1^{q(\alpha-\beta)} \sum_{i=1}^k p_i^{q\alpha}$. Then the minimal $t$-length real factorization of $n$ occurs at the point

$$\begin{aligned}
n\mathbf{m}(u) &= p_1^{q(\alpha-\beta)} \left(\sum_{i=1}^k p_i^{q\alpha}\right) \sum_{j=1}^k \left(\sum_{i=1}^k b_i \left(\frac{b_i}{b_j}\right)^{1/u}\right)^{-1} \mathbf{e}_j \\
&= p_1^{q(\alpha-\beta)} \left(\sum_{i=1}^k p_i^{q\alpha}\right) \sum_{j=1}^k \left(\sum_{i=1}^k p_i^{q(\alpha-\beta)} \left(\frac{p_i^{q(\alpha-\beta)}}{p_j^{q(\alpha-\beta)}}\right)^{\beta/(\alpha-\beta)}\right)^{-1} \mathbf{e}_j \\
&= p_1^{q(\alpha-\beta)} \left(\sum_{i=1}^k p_i^{q\alpha}\right) \sum_{j=1}^k \left(\sum_{i=1}^k \frac{p_i^{q(\alpha-\beta)+q\beta}}{p_j^{q\beta}}\right)^{-1} \mathbf{e}_j \\
&= p_1^{q(\alpha-\beta)} \left(\sum_{i=1}^k p_i^{q\alpha}\right) \sum_{j=1}^k \left(\sum_{i=1}^k \frac{p_i^{q\alpha}}{p_j^{q\beta}}\right)^{-1} \mathbf{e}_j \\
&= p_1^{q(\alpha-\beta)} \sum_{j=1}^k p_j^{q\beta} \mathbf{e}_j
\end{aligned}$$

which is an integer point. We can also easily see that $n\mathbf{M} = \frac{p_1^{q(\alpha-\beta)} \sum_{i=1}^k p_i^{q\alpha} \mathbf{e}_1}{p_1^{q(\alpha-\beta)}} = \sum_{i=1}^k p_i^{q\alpha} \mathbf{e}_1$, is an integer factorization as well. Since $n$ achieves the bounds for both minimal and maximal length, the elasticity is accepted. $\square$

**Example 2.7.** Consider $n = 198 \in S = \langle 3, 5\rangle$, which has factorization $(66, 0)$ of maximal length. $(1, 39)$ is also a factorization of 198, and when $t = 1 + \kappa(3, 5)/\kappa(1, 39) = \ln(65)/\ln(39)$, it has minimal $t$-length, so $S$ accepts this elasticity. We claim that $t$ is irrational. BWOC,

$$\frac{\ln(65)}{\ln(39)} = \frac{c}{d}$$

for some integers $c$ and $d$. This is true when there are integer solutions to $(13(5))^d = (3)(13)^c$. Since 3 is not a prime factor of 65, this value cannot be rational. So, $S$ accepts $\rho_t$ for at least one irrational value of $t$.

**Remark 2.8.** Recall the parametrized curve of Lagrange critical points when $n = 1$,

$$\mathbf{m}(t) = \frac{1}{||\mathbf{a}||_{q(t)}^{q(t)}} \sum_{j=1}^k a_j^{1/(t-1)} \mathbf{e}_j.$$

We can rewrite

$$\mathbf{m}(t) = \sum_{j=1}^{k} \left( \frac{a_j^{1/(t-1)}}{\sum_{i=1}^{k} a_i^{q(t)}} \right) \mathbf{e}_j.$$

So that

$$m_j(t) = \frac{a_j^{1/(t-1)}}{\sum_{i=1}^{k} a_i^{t/(t-1)}} = \left( \sum_{i=1}^{k} a_i \left( \frac{a_i}{a_j} \right)^{1/(t-1)} \right)^{-1}.$$

We can reparametrize, letting $u = t - 1$, so that for $u \in (0, \infty)$

$$\mathbf{m}(u) = \sum_{j=1}^{k} \left( \sum_{i=1}^{k} a_i \left( \frac{a_i}{a_j} \right)^{1/u} \right)^{-1} \mathbf{e}_j,$$

for $u \in (0, \infty)$. $\mathbf{m}$ is not defined at $u = 0$ or $\infty$, but we will define it at its endpoints by its limits at those points, which we will see in the following proposition.

**Proposition 2.9.** $\mathbf{m}(u)$ *has limits* $\frac{\mathbf{e}_k}{a_k}$ *and* $\sum_{j=1}^{k} \frac{\mathbf{e}_j}{||\mathbf{a}||_1}$ *as* $u$ *approaches* $0^+$ *and* $\infty$ *respectively.*

*Proof.* Given that $a_i < a_j$ whenever $i < j$, it is easy to see that

$$\lim_{u \to 0^+} \left( \frac{a_i}{a_j} \right)^{1/u} \to \begin{cases} 0 & i < j \\ 1 & i = j \\ \infty & i > j \end{cases}.$$

It follows that when $j < k$,

$$\lim_{u \to 0^+} \frac{1}{m_j(u)} = \sum_{i=1}^{k} a_i \left( \lim_{u \to 0^+} \left( \frac{a_i}{a_j} \right)^{1/u} \right) = 0 + a_j + \sum_{i=j+1}^{k} a_i \lim_{u \to 0^+} \frac{a_i}{a_j}^{1/u}$$

diverges to infinity. Since its reciprocal diverges, this proves that $\lim\limits_{u \to 0^+} m_j(u) = 0$. When $j = k$, we have

$$\lim_{u \to 0^+} \frac{1}{m_k(u)} = \sum_{i=1}^{k} a_i \left( \lim_{u \to 0^+} \left( \frac{a_i}{a_j} \right)^{1/u} \right) = 0 + a_k,$$

which gives that $\lim_{u \to 0^+} \mathbf{m}(u) = \frac{\mathbf{e}_k}{a_k}$, as claimed.

Now, to verify the limit at infinity, we note that for any $i, j$, $\lim\limits_{u \to \infty} \frac{a_i}{a_j}^{1/u} \to 1$, so that

$$\lim_{u \to \infty} \frac{1}{m_j(u)} = \sum_{i=1}^{k} a_i \lim_{u \to \infty} \frac{a_i}{a_j}^{1/u} = \sum_{i=1}^{k} a_i = ||\mathbf{a}||_1, \quad \Rightarrow \quad \lim_{u \to \infty} \mathbf{m}(u) = \sum_{j=1}^{k} \frac{\mathbf{e}_j}{||a||_1}.$$

$\square$

**Definition 2.10.** Define the map $\kappa : \mathbb{R}^2_{>0} \to \mathbb{R}$ by $(x, y) \longmapsto \ln y/x$.

**Proposition 2.11.** *(The solution in $t$ to $\mathbf{m}(t) = (x, t)$)*

*Fix* $\mathbf{a}$, $n \in S$, *and let* $\mathbf{x} = (x, y)$, *with* $x \leq \frac{n}{a_1+a_2}$, *a real factorization of* $n \in S$. *Define* $T(\mathbf{x}) = 1 + \frac{\kappa(\mathbf{a})}{\kappa(\mathbf{x})}$. *Then*

$$n\mathbf{m}(T(\mathbf{x})) = \mathbf{x}.$$

*Proof.* In 2.8, we derived the representation of the lagrange curve, reparametrized in terms of $u = t - 1$, given by

$$\mathbf{m}(1+u) = \sum_{j=1}^{k} \left( \sum_{i=1}^{k} a_i \left( \frac{a_i}{a_j} \right)^{1/u} \right)^{-1} \mathbf{e}_j,$$

for $u \in (0, \infty)$.

When $k = 2$, this gives us the curve

$$\mathbf{m}(1+u) = \left( \frac{1}{a_1 + a_2 \left(\frac{a_2}{a_1}\right)^{1/u}}, \frac{1}{a_1 \left(\frac{a_1}{a_2}\right)^{1/u} + a_2} \right)$$

We refer to the coordinates of as $m_1$ and $m_2$. We evaluate $nm_1$ at $u = T(\mathbf{x}) - 1 = \frac{\kappa(\mathbf{a})}{\kappa(\mathbf{x})}$, and take its reciprocal (to make it look nicer), obtaining

$$\frac{1}{nm_1}\bigg|_{u=\frac{\kappa(\mathbf{a})}{\kappa(\mathbf{x})}} = \frac{1}{n} \left( a_1 + a_2 \left(\frac{a_2}{a_1}\right)^{\frac{\kappa(\mathbf{x})}{\kappa(\mathbf{a})}} \right) = \frac{1}{n} \left( a_1 + a_2 e^{\ln\left(\frac{a_2}{a_1}\right)\frac{\kappa(\mathbf{x})}{\kappa(\mathbf{a})}} \right)$$

We substitute $\kappa(\mathbf{a}) = \ln\left(\frac{a_2}{a_1}\right)$, obtaining

$$\begin{aligned}
\frac{1}{nm_1}\bigg|_{u=\frac{\kappa(\mathbf{a})}{\kappa(\mathbf{x})}} &= \frac{1}{n} \left( a_1 + a_2 e^{\kappa(\mathbf{a})\frac{\kappa(\mathbf{x})}{\kappa(\mathbf{a})}} \right) = \frac{1}{n} \left( a_1 + a_2 e^{\kappa(\mathbf{x})} \right) \\
&= \frac{1}{n} \left( a_1 + a_2 e^{\ln y/x} \right) = \frac{1}{n} \left( a_1 + \tfrac{a_2 y}{x} \right) = \frac{1}{xn}(a_1 x + a_2 y) = 1/x.
\end{aligned}$$

We evaluate $nm_2$ at $u = \frac{\kappa(\mathbf{a})}{\kappa(\mathbf{x})}$ similarly,

$$\frac{1}{nm_2}\bigg|_{u=\frac{\kappa(\mathbf{a})}{\kappa(\mathbf{x})}} = \frac{1}{n} \left( a_1 \left(\frac{a_1}{a_2}\right)^{\frac{\kappa(\mathbf{x})}{\kappa(\mathbf{a})}} + a_2 \right) = \frac{1}{n} \left( a_1 e^{\ln\left(\frac{a_1}{a_2}\right)\frac{\kappa(\mathbf{x})}{\kappa(\mathbf{a})}} + a_2 \right)$$

Notice that $\ln\left(\frac{a_1}{a_2}\right) = \ln\left(\left(\frac{a_2}{a_1}\right)^{-1}\right) = -\ln\left(\frac{a_2}{a_1}\right) = -\kappa(\mathbf{a})$. We use this to substitute

$$\frac{1}{nm_2}\bigg|_{u=\frac{\kappa(\mathbf{a})}{\kappa(\mathbf{x})}} = \frac{1}{n} \left( a_1 e^{-\ln\left(\frac{y}{x}\right)} + a_2 \right) = \frac{1}{n} \left( a_1 e^{\ln\left(\frac{x}{y}\right)} + a_2 \right) = \frac{1}{n} \left( \frac{a_1 x + a_2 y}{y} \right) = 1/y,$$

as desired. $\square$

**Corollary 2.12.** *$T(\mathbf{x})$ is a two-sided inverse of $\mathbf{m}(t)$. In particular, $(T \circ \mathbf{m})(t)=t$.*

*Proof.* We have that $T(\mathbf{m}(t)) = \frac{\ln(\frac{a_2}{a_1})}{\ln\left(\frac{a_1+a_2(\frac{a_2}{a_1})^{\frac{1}{t-1}}}{a_1(\frac{a_1}{a_2})^{\frac{1}{t-1}}+a_2}\right)} + 1$. When looking specifically at the bottom term, we can simplify:

$$\ln\left(\frac{a_1+a_2(\frac{a_2}{a_1})^{\frac{1}{t-1}}}{a_1(\frac{a_1}{a_2})^{\frac{1}{t-1}}+a_2}\right) = \frac{1}{t-1}\ln\left(\frac{(a_1+a_2(\frac{a_2}{a_1})^{\frac{1}{t-1}})^{t-1}}{(a_1(\frac{a_1}{a_2})^{\frac{1}{t-1}}+a_2)^{t-1}}\right)$$

$$= \frac{1}{t-1}\ln\left(\frac{((\frac{a_2}{a_1})^{\frac{1}{t-1}}[(a_1(\frac{a_1}{a_2})^{\frac{1}{t-1}}+a_2)])^{t-1}}{(a_1(\frac{a_1}{a_2})^{\frac{1}{t-1}}+a_2)^{t-1}}\right) = \frac{1}{t-1}\ln\left(\frac{a_2}{a_1}\right).$$

Now, we have that

$$T(\mathbf{m}(t)) = \frac{\ln(\frac{a_2}{a_1})}{\frac{1}{t-1}\ln(\frac{a_2}{a_1})} + 1 = t-1+1 = t.$$

□

**Remark 2.13.** The limits as $\mathbf{m}(t)$ approaches $0^+$ and $\infty$ given by Proposition 2.9 imply that we must define its inverse as follows:

$$T(x,y) = \begin{cases} 1 & x=0 \\ 1+\frac{\ln(\frac{a_2}{a_1})}{\ln(\frac{y}{x})} & 0<x<y \\ \infty & x=y \end{cases}$$

2.2. **Global Elasticity on $[0,1)$.**

**Lemma 2.14.** *Fix a numerical semigroup $S$ having minimal generating vector $\boldsymbol{a} \in \mathbb{Z}^k_{>0}$. Suppose $t \in (0,1)$ and $q$ is the dual of $t$. If $\boldsymbol{x}$ is a factorization of $n$ in $S$, then*

$$\left(\frac{n}{a_k}\right)^t \le \|\boldsymbol{x}\|_t \le \left(\frac{n}{\|\boldsymbol{a}\|_q}\right)^t. \tag{2.4}$$

*Moreover, these lower and upper bounds are achieved by the vectors*

$$n\boldsymbol{m} = \frac{n}{a_k}\boldsymbol{e}_k \qquad \text{and} \qquad n\boldsymbol{M}(t) = \frac{n}{(\|\boldsymbol{a}\|_q)^q}\sum_{i\in[k]} \sqrt[t-1]{a_i}\boldsymbol{e}_i$$

*respectively*

*Proof.* When $n=0$, the only factorization is the zero vector with $k$ components. In this case, (2.4) reduces to $0 \le 0 \le 0$.

We may now assume that $n$ is positive. Consider the functions $C_k\colon \mathbb{R}^k_{\ge 0} \to \mathbb{R}$ by $\boldsymbol{x} \mapsto \boldsymbol{a}\cdot\boldsymbol{x} - n$ and $\|\cdot\|_t$. Any factorization $\boldsymbol{x}$ of $n$ must be an element of $\mathbb{R}^2_{\ge 0}$ with integer components satisfying $C_k(\boldsymbol{x}) = 0$. It follows that the $t$-norm of a factorization of $n$ is bounded by the extreme values of $\|\cdot\|_t$ constrained by $C_k(\boldsymbol{x}) = 0$. We may use the method of Lagrange multipliers to find these extreme values.

The interior of the set of $\boldsymbol{x}$ satisfying $C_k(\boldsymbol{x}) = 0$ contains only vectors with no zero components. Any critical point $\boldsymbol{x} = (x_1, x_2, \ldots, x_k)$ in the interior of this set must satisfy

$$\begin{cases} tx_1^{t-1} = \lambda a_1 \\ tx_2^{t-1} = \lambda a_2 \\ \qquad\vdots \\ tx_k^{t-1} = \lambda a_k \\ \boldsymbol{a} \cdot \boldsymbol{x} = n \end{cases} \tag{2.5}$$

for some real $\lambda$. We have that $\lambda > 0$, since otherwise we could never have $\boldsymbol{a} \cdot \boldsymbol{x} = n$ for the positive integer $n$. Since $tx_i^{t-1}$ is monotone decreasing,

$$x_i = \sqrt[t-1]{\frac{\lambda a_i}{t}}. \tag{2.6}$$

is the only possible value of any component $x_i$. Equation (2.6) and $\boldsymbol{a} \cdot \boldsymbol{x} = n$ implies

$$\begin{aligned} \sum_{i=1}^{k} a_i \sqrt[t-1]{\frac{\lambda a_i}{t}} = n &\Rightarrow \sqrt[t-1]{\frac{\lambda}{t}} \sum_{i=1}^{k} a_i^q = n \\ &\Rightarrow \sqrt[t-1]{\frac{\lambda}{t}} (\|\boldsymbol{a}\|_q)^q = n \\ &\Rightarrow \lambda = t\left(\frac{n}{(\|\boldsymbol{a}\|_q)^q}\right)^{t-1}. \end{aligned}$$

Therefore,

$$x_i = \frac{n \sqrt[t-1]{a_i}}{(\|\boldsymbol{a}\|_q)^q}.$$

We have now identified $\boldsymbol{x} = n\boldsymbol{M}(t)$ as the only critical point in the interior of the set of vectors satisfying $C_k(\boldsymbol{x}) = 0$. This gives

$$\begin{aligned} \|\boldsymbol{x}\|_t &= \sum_{i=1}^{k} \left(\frac{n \sqrt[t-1]{a_i}}{(\|\boldsymbol{a}\|_q)^q}\right)^t \\ &= \left(\frac{n}{(\|\boldsymbol{a}\|_q)^q}\right)^t (\|\boldsymbol{a}\|_q)^q \\ &= \left(\frac{n}{\|\boldsymbol{a}\|_q}\right)^t \end{aligned}$$

as the only possible extreme value of $\|\cdot\|_t$ constrained by $C_k(\boldsymbol{x}) = 0$ not on the boundary of the set of vectors satisfying the constraint. Note that for any vector $\boldsymbol{a}' \in \mathbb{R}^i_{>0}$ and function $C_i$ defined in the same way as $C_k$, the same result holds for the possible extreme value of $\|\cdot\|_t$ restricted by $C_i(\boldsymbol{x}) = 0$.

We now consider possible extreme values on the boundary of the set of $\boldsymbol{x}$ satisfying $C_k(\boldsymbol{x}) = 0$. This boundary is the set containing elements of $\mathbb{R}^k_{\geq 0}$ having at least one zero component. To find the extreme values, we will induct on the number of nonzero components of boundary elements. First, consider the case where $\boldsymbol{x}$ has only one nonzero component. The nonzero component of these vectors take the form $n/a_i$. The possible $t$-norms of such vectors have values of the form $(n/a_i)^t$. Among these values and the possible extreme value we have already identified, the minimum is $(n/a_k)^t$ (achieved at $n\boldsymbol{m}$ and the maximum is $(n/\|\boldsymbol{a}\|_q)^t$ (achieved at $n\boldsymbol{M}(t)$).

Now suppose that among all boundary vectors having either at most $1 \leq j < k$ nonzero components or no nonzero components, the minimum and maximum values of $\|\cdot\|_t$ are $(n/a_k)^t$ and $(n/\|\boldsymbol{a}\|_q)^t$ respectively. Consider the vectors having exactly $j+1$ nonzero components. We may partition these vectors further by which components of the vector are zero. Suppose $D$ is one such partition. Associate a partial sequence $\alpha_1, \ldots, \alpha_{j+1}$ of the indices of the nonzero components of the vectors in $D$. Now let $\boldsymbol{a'} = (a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_{j+1}})$. Then define a new function $C_{j+1} \colon \mathbb{R}^{j+1}_{\geq 0} \to \mathbb{R}$ by $\boldsymbol{x} \mapsto \boldsymbol{a'} \cdot \boldsymbol{x} - n$. We have already shown that the only extreme value of $\|\cdot\|_t$ constrained to $C_{j+1}(\boldsymbol{x}) = 0$ having no nonzero components must be $(n/\|\boldsymbol{a'}\|_q)^t$. Since the image of $D$ under $\|\cdot\|_t$ is equal to the image of vectors in $\mathbb{R}^{j+1}_{>0}$ with $C_{j+1}(\boldsymbol{x}) = 0$, they must have the same extreme values. We have that $\|\boldsymbol{a}\|_q \leq \|\boldsymbol{a'}\|_q \leq a_k$, so the $t$-norm of any vector in $D$ is not less than $(n/a_k)^t$ or greater than $(n/\|\boldsymbol{a}\|_q)^t$. Since this is true regardless of which partition of vectors we used, the inductive hypothesis must hold. It follows that (2.4) holds for all factorizations of $n$. $\square$

**Theorem 2.15.** *Suppose $S$ is a numerical semigroup generated minimally by $\boldsymbol{a} \in \mathbb{Z}^k_{>0}$. If $t \in [0, 1)$, then*

$$\rho_t(S) = \begin{cases} k & \text{if } t = 0 \\ (a_k/\|\boldsymbol{a}\|_q)^t & \text{if } t \in (0,1) \end{cases}$$

*where $q$ is the dual of $t$.*

*Proof.* First we will prove the case when $t = 0$. In this case the length of a factorization is simply the number of nonzero elements in it, so the minimum and maximum factorization possible for any nonzero element is 1 and $k$ respectively. The element $a_1 a_2 \cdots a_k$ has factorizations with both of these lengths, so the elasticity of $S$ is $k$.

Now consider the case when $t \in (0, 1)$. We have from lemma 2.14 that $\rho_t(S) \leq (a_k/\|\boldsymbol{a}\|_q)^t$. Consider the set $V$ of non-negative real factorizations of $a_k$. The factorization $\boldsymbol{e}_k$ of $a_k$ has the smallest $t$-norm of the elements of $V$. There also exists a factorization $\boldsymbol{M}$ of $a_k$ in the interior of $V$ having the maximum $t$-norm. It has previously been shown that $\|\boldsymbol{M}\|_t / \|\boldsymbol{e}_k\|_t = (a_k/\|\boldsymbol{a}\|_q)^t$.

Fix $\epsilon > 0$. Since $\|\cdot\|_t$ is continuous at $\boldsymbol{M}$ and $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$, there exists a rational factorization $\boldsymbol{z} \in \mathbb{Q}^k_{>0}$ of $a_k$ with

$$\left| \|\boldsymbol{z}\|_t - \|\boldsymbol{M}\|_t \right| < \epsilon.$$

Let $C$ be an integer such that $C\boldsymbol{z}$ has integer components. The vector $C\boldsymbol{z}$ is an integer factorization of $Ca_k$ with $\|C\boldsymbol{z}\|_t > \|C\boldsymbol{M}\|_t - C^t\epsilon$ and $C\boldsymbol{e}_k$ is also a factorization of $Ca_k$. Therefore,

$$\begin{aligned}
\rho_t(S) &\geq \frac{\|C\boldsymbol{z}\|_t}{\|C\boldsymbol{e}_k\|_t} \\
&> \frac{\|C\boldsymbol{M}\|_t - C^t\epsilon}{\|C\boldsymbol{e}_k\|_t} \\
&= \left(\frac{a_k}{\|\boldsymbol{a}\|_q}\right)^t - \epsilon
\end{aligned}$$

We now have that $\rho_t(S)$ is less than or equal to $(a_k/\|\boldsymbol{a}\|_q)^t$ and greater than anything less than $(a_t/\|\boldsymbol{a}\|_q)^t$. Therefore $\rho_t(S) = (a_t/\|\boldsymbol{a}\|_q)^t$. □

We now have an explicit formula for the global elasticity of an arbitrary numerical semigroup over the interval $[0, \infty]$. Figure 1 shows a graph of elasticity as a function of $t$. Note that even though the graph is defined piecewise, it is continues on its entire domain.

**Theorem 2.16.** *Let $S = \langle a_1, \ldots, a_k \rangle$ and $\mathbf{a} = (a_1, \ldots, a_k)$. Then, $\rho_0(S) < \rho_\infty(S)$.*

*Proof.* By Theorem 2.15, we know that $\rho_0(S) = k$, and by Lemma 3.2, we know that $\rho_\infty(S) = \frac{\|\mathbf{a}\|_1}{a_1} = \frac{a_1 + \ldots + a_k}{a_1}$. Then,

$$\begin{aligned}
\frac{a_1 + \ldots + a_k}{a_1} - k &= \frac{1}{a_1}(a_1 + \ldots + a_k - a_1k) \\
&= \frac{1}{a_1}((a_1 - a_1) + (a_2 - a_1) + \ldots + (a_k - a_1))\frac{a_1 + \ldots + a_k}{a_1} - k \\
&= \frac{1}{a_1}(a_1 + \ldots + a_k - a_1k) \\
&= \frac{1}{a_1}((a_1 - a_1) + (a_2 - a_1) + \ldots + (a_k - a_1)) \\
&= \frac{1}{a_1}((a_2 - a_1) + \ldots + (a_k - a_1)).
\end{aligned}$$

Since $a_1 < a_2 < \cdots < a_k$, we have that $a_j - a_1 > 0$ for all $2 \leq j \leq k$. Hence, $(a_2-a_1)+\ldots+(a_k-a_1) > 0$ and so $\rho_\infty(S)-\rho_0(S) = \frac{1}{a_1}((a_2-a_1)+\ldots+(a_k-a_1)) > 0$. □

**Remark 2.17.** Computational evidence suggests that the graph of global elasticity from $t = 0$ to $t = 1$ is U-shaped (concave up).

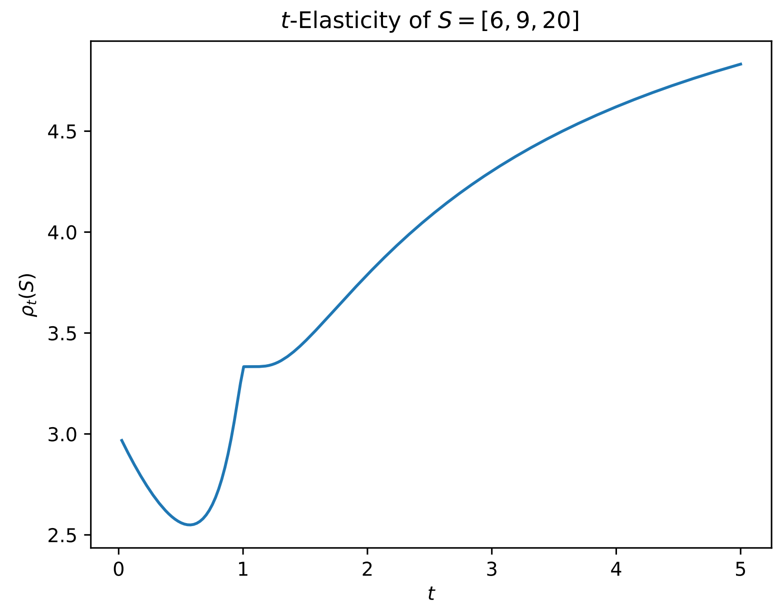



FIGURE 1. Global $t$-elasticity of the semigroup generated by $(6, 9, 20)$ as a function of $t$.

**Conjecture 2.18.** *For* $0 \leq t \leq 1$, $\frac{\partial^2}{\partial t^2}\rho_t(S) > 0$.

**Remark 2.19.** Based on Theorem 2.4, Theorem 2.16, and Conjecture 2.18, it would naturally follow that a semigroup's $\infty$-elasticity is the maximum of all of its other $t$-elasticities.

**Conjecture 2.20.** *Let* $\rho_*(S) = \{\rho_t(S) \mid t \in [0, \infty]\}$. *Then,* $\max(\rho_*(S)) = \rho_\infty(S)$.

## 3. 2-Generated Elasticity Curves

We will now look deeper into two generated semigroups. There is a lot of information that can be drawn upon about the elasticity curves of two generated semigroups, and most of that information comes from that factorizations of a given element. Before we dive into results we will look at a few figures that will motivate our results.

First, in Figure 2, we will look at a graph of the line $n = a_1x_1 + a_2x_2$ where $\mathbf{x} = (x_1, x_2)$ is an arbitrary factorization of $n$. Every point on this line is a real factorization of $n$ and all of the points labeled with red dots are the integer factorizations of $n$.

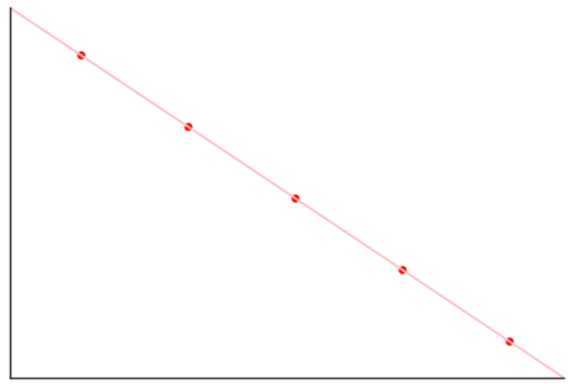
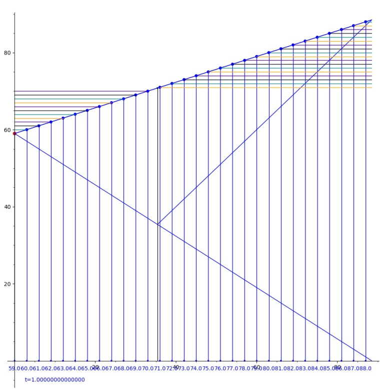
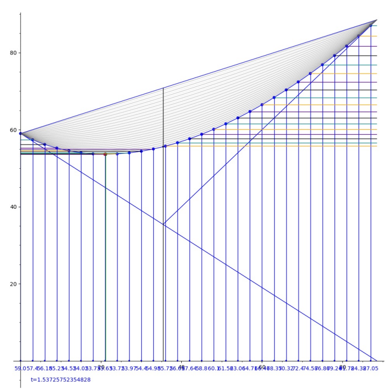


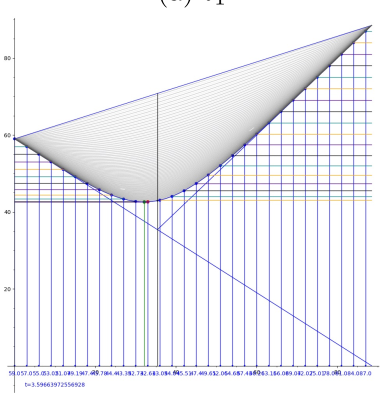
FIGURE 2. Line of Factorizations



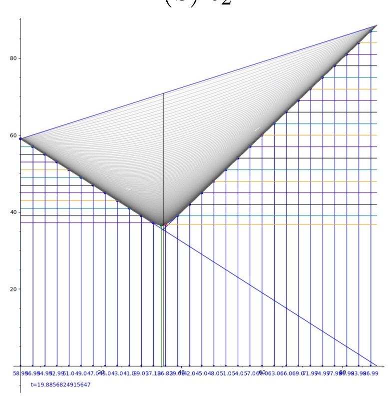
(a) $t_1$



(b) $t_2$



(c) $t_3$



(d) $t_4$

FIGURE 3. Relating factorizations of $n \in S$ to their $t$-norm

Now we will expand on this by looking at a graph that related the factorizations of an element $n$ to their respective $t$-norms. Above in Figure 3 is a graphic showing, for a particular element, $n \in S$, showing the factorizations of that element in relation to their $t$-norm. As $t$ increases we see the $t$-norm change as the line becomes concave up as $t$ approaches its $t = \infty$-norm. The blue dots show the integer factorizations of $n$ and the lines show all of the real numbered factorizations. The green dot is the real factorization that has the minimum $t$-length and the red dot is the integer factorization that has the minimum $t$-length. Figure 3 shows four instances of this figure as $t$ increases; $t_1 < t_2 < t_3 < t_4$. However, think of this an more of an animation than a figure where these curves will smoothly change over time.

When $t$ is in the interval $[0, 1]$ we can create the same graph that is shown in Figure 3. Figure 4 shows various versions of this graph for the element 30 in the semigroup $\langle 5, 7\rangle$ as $t$ varies.

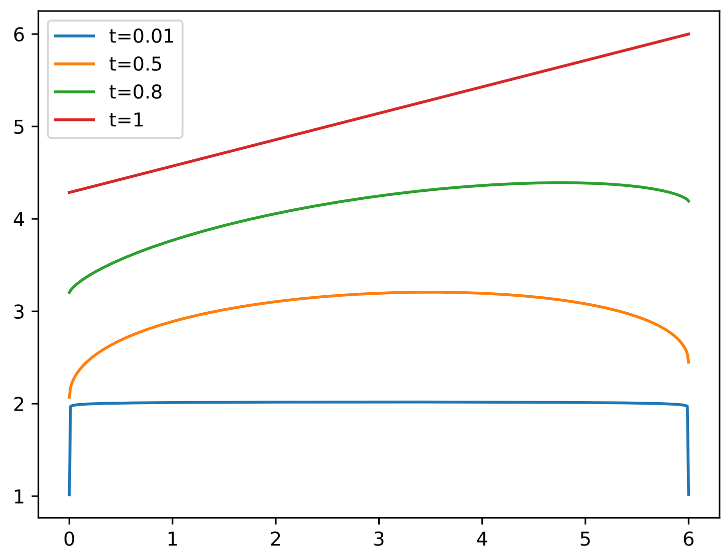



FIGURE 4. Factorization $t$-length for 30 in the semigroup $\langle 5, 7\rangle$ as a function of the first component of the factorization.

**Proposition 3.1.** *Suppose $S$ is a numerical semigroup generated minimally by $\boldsymbol{a} = (a_1, a_2)$ and $r$ is a positive real number. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are unique real-valued factorizations of $r$, then $\|\boldsymbol{x}\|_t = \|\boldsymbol{y}\|_t$ at most twice on $(0, \infty)$.*

*Proof.* First note that for all $t$ in $(0, \infty)$, $\|\boldsymbol{x}\|_t = \|\boldsymbol{y}\|_t$ if and only if $\|\boldsymbol{x}/r\|_t = \|\boldsymbol{y}/r\|_t$. Since both $\boldsymbol{x}/r$ and $\boldsymbol{y}/r$ are real-valued factorizations of 1, we only need to prove the statement of Lemma 1 for the case $r = 1$. We may assume without loss of generality that $\boldsymbol{x}$ and $\boldsymbol{y}$ are factorizations of 1. Suppose that $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$. We may further simplify what we are looking to prove by noting that

$$\|\boldsymbol{x}\|_t = \|\boldsymbol{y}\|_t \iff x_1^t + x_2^t = y_1^t + y_2^t. \tag{3.1}$$

We will prove some edge cases by focusing on the right side of (3.1). If $\boldsymbol{x}$ and $\boldsymbol{y}$ both have only one nonzero component $c_{\boldsymbol{x}}$ and $c_{\boldsymbol{y}}$ respectively, then the right side of (3.1) is equivalent to $(c_{\boldsymbol{x}}/c_{\boldsymbol{y}})^t = 1$. Since $c_{\boldsymbol{x}} \neq c_{\boldsymbol{y}}$ ($\boldsymbol{x}$ and $\boldsymbol{y}$ are unique), there is no $t$ in $(0, \infty)$ satisfying this equation. Now suppose that one of the factorizations has two nonzero components and the other has only one. We can assume without loss of generality that $\boldsymbol{x}$ has two nonzero components and $\boldsymbol{y}$ has only $c_{\boldsymbol{y}}$ as a nonzero component. We can get what we want by showing that $(x_1/c_{\boldsymbol{y}})^t + (x_2/c_{\boldsymbol{y}})^t = 1$ is satisfied by at most a single positive $t$. Since $\boldsymbol{x}$ and $\boldsymbol{y}$ are factorizations of 1, we know that $1/c_{\boldsymbol{y}} \leq a_2$ and $x_2 < 1/a_2$, so $x_2/c_{\boldsymbol{y}} < 1$. If $x_1/c_{\boldsymbol{y}} \geq 1$, then $(x_1/c_{\boldsymbol{y}})^t + (x_2/c_{\boldsymbol{y}})^t > 1$ for all positive $t$. If $x_1/c_{\boldsymbol{y}} < 1$, then $(x_1/c_{\boldsymbol{y}})^t + (x_2/c_{\boldsymbol{y}})^t$ is strictly decreasing and can only equal 1 for a single positive $t$.

The only case left to prove is when both $\boldsymbol{x}$ and $\boldsymbol{y}$ have no nonzero components. For the rest of the proof we will assume that this is the case. We may assume without loss of generality that the first component of $\boldsymbol{x}$ is greater than the first component of $\boldsymbol{y}$, so that

$$x_1 < y_1 \qquad \text{and} \qquad y_2 < x_2. \tag{3.2}$$

We know the second of these inequalities holds because $x_2 = (1 - a_1x_1)/a_2$ and $y_2 = (1 - a_1y_1)/a_2$. For positive $t$, we get from (3.1) that

$$\|\boldsymbol{x}\|_t = \|\boldsymbol{y}\|_t \iff \ln\big(x_2^t - y_2^t\big) - \ln\big(y_1^t - x_1^t\big) = 0. \tag{3.3}$$

Let $f$ be the difference of logarithms on the right side of (3.3) defined for positive $t$.

We can show that $f(t) = 0$ at most twice over its domain by proving that $f'$ has at most a single zero. Before differentiating $f$, we will rewrite it in a form that will be easier to work with.

$$\begin{aligned}
f(t) &= \ln\big(x_2^t - y_2^t\big) - \ln\big(y_1^t - x_1^t\big) \\
&= \ln\big(y_2^t\big((x_2/y_2)^t - 1\big)\big) - \ln\big(x_1^t\big((y_1/x_1)^t - 1\big)\big) \\
&= t\ln(y_2) + \ln\big((x_2/y_2)^t - 1\big) - t\ln(x_1) - \ln\big((y_1/x_1)^t - 1\big) \\
&= t\ln(y_2/x_1) + \ln\big((x_2/y_2)^t - 1\big) - \ln\big((y_1/x_1)^t - 1\big)
\end{aligned}$$

For simplicity, let $A = x_2/y_2$ and $B = y_1/x_1$ so that

$$f(t) = t\ln(y_2/x_1) + \ln\big(A^t - 1\big) - \ln\big(B^t - 1\big).$$

Note that (3.2) guarantees that both $A$ and $B$ are greater than 1. Now we differentiate $f$ to get

$$f'(t) = \ln(y_2/x_1) + \ln(A)\frac{A^t}{A^t - 1} - \ln(B)\frac{B^t}{B^t - 1}.$$

We wish to show that $f'$ has at most a single zero. Before proceeding to prove this generally, we will pick out the special case of when $A = B$. When this is true, $f'(t) = \ln(y_2/x_1)$. As long as $y_2 \neq x_1$ we have that $f'$ has no zeros, so $f(t)$ for at most one positive $t$. Suppose $y_2 = x_1$ and remember that if $A = B$, then $x_2/y_2 = y_1/x_1$. Therefore, we get $x_2 = y_1$, so $\boldsymbol{y} = (x_2, x_1)$. Finally, using the fact that both $\boldsymbol{x}$ and $\boldsymbol{y}$ are factorizations of 1 we get

$$\begin{aligned} a_1x_1 + a_2x_2 &= a_1y_1 + a_2y_2 \\ a_1x_1 + a_2x_2 &= a_1x_2 + a_2x_1 \\ x_2(a_2 - a_1) &= x_1(a_2 - a_1) \\ x_2 &= x_1. \end{aligned}$$

But if this is true, then $\boldsymbol{x} = \boldsymbol{y}$. Since we are given that $\boldsymbol{x} \neq \boldsymbol{y}$, we must have that $y_2 \neq x_1$. This implies $f'$ has no zeros when $A = B$, so $f$ can only have at most one zero in $(0, \infty)$.

We can now assume that $A \neq B$ and look to show that $f'$ has only one zero in $(0, \infty)$. To prove this, we will show that $f''(t)$ is either always negative or always positive, so that $f'$ is strictly monotone. Note before taking the derivative of $f'$ that for any constant $C$ greater than 1,

$$\begin{aligned} \frac{d}{dt}\left(\frac{C^t}{C^t - 1}\right) &= \frac{\ln(C)C^t(C^t - 1) - C^t\ln(C)C^t}{(C^t - 1)^2} \\ &= \ln(C)\frac{C^t((C^t - 1) - C^t)}{(C^t - 1)^2} \\ &= -\ln(C)\frac{C^t}{(C^t - 1)^2}. \end{aligned}$$

Therefore,

$$f''(t) = \ln(B)^2\frac{B^t}{(B^t - 1)^2} - \ln(A)\frac{A^t}{(A^t - 1)^2}.$$

We want to show that $f''$ is either always positive or always negative regardless of $t$. Fix an arbitrary $t_0$ in $(0, \infty)$ and let

$$g(z) = \ln(z)^2\frac{z^{t_0}}{(z^{t_0} - 1)^2}$$

for $z \in (1, \infty)$. This means $f''(t_0) = g(B) - g(A)$. If we show that $g$ is strictly decreasing, then we will have that $f''$ is either always negative or always positive (depending on which of $A$ or $B$ is greater) regardless of $t$. To do this, we will show that $g'$ is always negative on its domain. First note that

$$\begin{aligned}\frac{d}{dz}\left(\frac{z^{t_0}}{(z^{t_0}-1)^2}\right) &= \frac{t_0 z^{t_0-1}(z^{t_0}-1)^2 - z^{t_0}(2(z^{t_0}-1)t_0 z^{t_0-1})}{(z^{t_0}-1)^4}\\ &= \frac{t_0 z^{t_0-1}(z^{t_0}-1)((z^{t_0}-1)-2z^{t_0})}{(z^{t_0}-1)^4}\\ &= \frac{-t_0 z^{t_0-1}(z^{t_0}+1)}{(z^{t_0}-1)^3}.\end{aligned}$$

Then we can use the product rule to differentiate $g$:

$$\begin{aligned}g'(t) &= \frac{2\ln(z)z^{t_0}}{z(z^{t_0}-1)^2} + \frac{\ln(z)^2(-t_0 z^{t_0-1}(z^{t_0}+1))}{(z^{t_0}-1)^3}\\ &= \frac{2\ln(z)z^{t_0-1}}{(z^{t_0}-1)^2} + \frac{\ln(z)^2(-t_0 z^{t_0-1}(z^{t_0}+1))}{(z^{t_0}-1)^3}\\ &= \frac{\ln(z)z^{t_0-1}}{(z^{t_0}-1)^3}\big(2(z^{t_0}-1) - t_0\ln(z)(z^{t_0}+1)\big).\end{aligned}$$

Since everything outside the parentheses is positive, $g'$ can only be negative if everything inside the parentheses is negative for all $z$ in $(1,\infty)$. Let $h(z)$ be this quantity. This means that we have proven what we want if we show that $h(z)$ is always negative on its domain. We can see that $\lim_{z\to 1^+} h(z) = 0$, so if $h$ is strictly decreasing, then we have shown what we are looking for. Once again, we will take a derivative and then attempt to show that it is negative:

$$\begin{aligned}h'(z) &= 2t_0 z^{t_0-1} - t_0\left(\frac{z^{t_0}+1}{z} + t_0\ln(z)z^{t_0-1}\right)\\ &= \frac{t_0}{z}(2z^{t_0} - ((z^{t_0}+1) + t_0\ln(z)z^{t_0}))\\ &= \frac{t_0}{z}(z^{t_0} - 1 - t_0\ln(z)z^{t_0})\\ &= \frac{t_0}{z}(z^{t_0}(1-t_0\ln(z)) - 1).\end{aligned}$$

The fraction outside the parentheses is positive. This means we want the quantity in the parentheses to be negative. We can see that

$$\lim_{z\to 1^+} z^{t_0}(1-t_0\ln(z)) - 1 = 0.$$

So if this quantity is decreasing as $z$ increases, then we have shown what is needed. We complete this proof by taking one final derivative and showing it is negative:

$$\begin{aligned}\frac{d}{dz}\big(z^{t_0}(1-t_0\ln(z))-1\big) &= t_0z^{t_0-1}(1-t_0\ln(z))-t_0z^{t_0-1}\\ &= t_0z^{t_0-1}((1-t_0\ln(z))-1)\\ &= -t_0z^{t_0-1}\ln(z).\end{aligned}$$

Because $z>1$, this is certainly negative. Working backwards, we have just shown that $g$ is strictly decreasing. This means that $f''(t_0)=g(B)-g(A)$ is either always negative or always positive (depending on the values of $A$ and $B$, which have already been determined). Since $t_0$ is arbitrary, $f'$ is strictly monotone. From that, we get that $f$ has at most two zeros, so $\|\boldsymbol{x}\|_t=\|\boldsymbol{y}\|_t$ for at most two positive $t$. □

The proof of Proposition 3.1 shows that for all cases except where both factorizations have no nonzero components and $A\neq B$, there can only be at most one crossing in $(0,\infty)$. What we did show however, is that the function $f'$ is strictly monotone in that case. Data shows that whenever $f'(t)=0$ for some positive $t$, $f(t)$ is always negative. Proving that this is the case would be enough to show that two factorizations of an element of a two-generated numerical semigroup can only be equal at most once over $(0,\infty)$.

**Conjecture 3.2.** *Suppose $S$ is a numerical semigroup generated minimally by $a_1$ and $a_2$ with $a_1<a_2$ and $r$ is a positive real number. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are unique real-valued factorizations of $r$, then $\|\boldsymbol{x}\|_t=\|\boldsymbol{y}\|_t$ at most once on $(0,\infty)$.*

**Proposition 3.3.** *Suppose $S$ is minimally generated by $(a_1,a_2)$ and $N$ is a positive integer. There exists an $n>N$ in $S$ and $t_0\in(0,1)$ such that the factorization of $n$ with the smallest $t_0$-norm is $(n/a_1,0)$.*

*Proof.* Let $k$ be an integer greater than $N/a_1$ that is not divisible by $a_2$. The integer $n=a_1k$ is greater than $N$ and $(k,0)$ is the only factorization of $n$ having a zero component. Suppose $\boldsymbol{x}$ is a different factorization of $n$. Since

$$\lim_{t\to0^+}\|(k,0)\|_t=1 \qquad \text{and} \qquad \lim_{t\to0^+}\|\boldsymbol{x}\|_t=2$$

and there are only a finite number of factorization of $n$, there exist a $t_0\in(0,1)$ such that $(k,0)$ has the minimum $t_0$-norm of all factorizations of $n$. □

When $t=0$, the factorization length graph shown in Figure 4 is no longer continuous. Moreover, it discontinuous only where it reaches its minimum, which is at the endpoints of the domain. This property is what allows us to always pick out a large element of the semigroup having an minimum-length integer factorization on the far right side of the domain. However, when we only consider $t\in(\epsilon,1)$ where $\epsilon\in(0,1)$ then we can not always find large elements having a minimum-length integer factorization on the

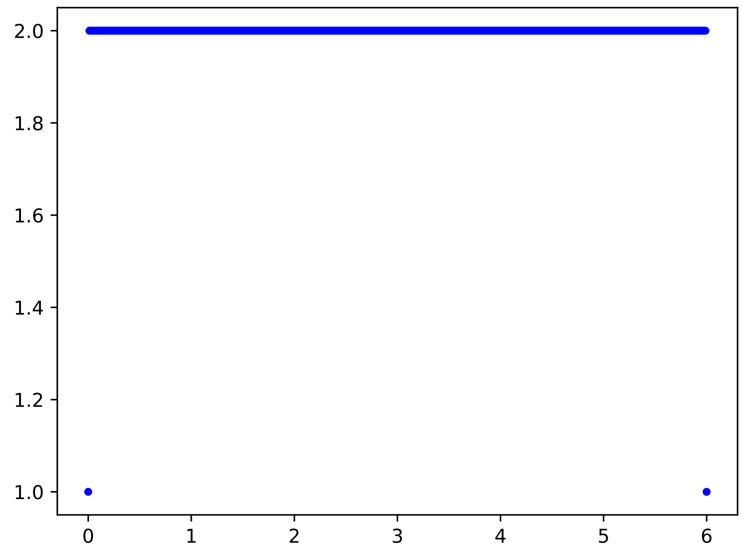



FIGURE 5. Factorization 0-length for 30 in $\langle 5,7\rangle$ as a function of the first component of the factorization.

right side of the domain. In fact, we can for very large elements of the semigroup, the minimum-length integer factorization is garanteed to be on the left side of the graph.

**Proposition 3.4.** *Suppose $S$ is generated by $(a_1, a_2)$ and $\epsilon \in (0,1)$. There exists an integer $N$ such that for all $n > N$ in $S$ and $t \in (\epsilon, 1)$, the factorization of $n$ with the smallest $t$-norm is the factorization having the smallest first component.*

*Proof.* Let $f$ be the bijection from $[0, 1/a_1]$ to the real-valued factorizations of 1 by

$$x \to \left(x, \frac{1 - a_1 x}{a_2}\right).$$

Fix any $t$ in $(0,1)$ and consider the function

$$\|f(x)\|_t = x^t + \left(\frac{1 - a_1 x}{a_2}\right)^t.$$

In the interior of the domain, the first and second derivatives of this function are

$$(\|f(x)\|_t)' = t\left(x^{t-1} - \frac{a_1}{a_2}\left(\frac{1 - a_1 x}{a_2}\right)^{t-1}\right)$$

and

$$(\|f(x)\|_t)'' = t(t-1)\left(x^{t-2} + \left(\frac{a_1}{a_2}\right)^2 \left(\frac{1-a_1x}{a_2}\right)^{t-2}\right).$$

Since $t \in (0,1)$, $(\|f(x)\|_t)''$ is negative for all $x$ in $(0, 1/a_1)$, so is concave down. Looking at $(\|f\|_t)'$, we find that

$$\lim_{x\to 0^+} (\|f\|_t)' = \infty \qquad \text{and} \qquad \lim_{x\to(1/a_1)^-} (\|f\|_t)' = -\infty.$$

Therefore, $(\|f\|_t)'$ has at least one zero. Since the derivative of this function is negative, we also know that this is the only zero. This implies $\|f\|_t$ has a maximum that is achieved at a single point in $(0, 1/a_1)$. We may also confirm that $\|f(0)\|_t < \|f(1/a_1)\|_t$.

Now focus specifically on $\|f\|_\epsilon$. As $x$ increases from 0 to the point in $(0, 1/a_1)$ that achieves the maximum, continuity of $\|f\|_\epsilon$ guarantees there exists an $x$ with $\|f(x)\|_\epsilon = \|f(1/a_1)\|_\epsilon$. Call this point $c$ and consider the function

$$\frac{\|f(c)\|_t}{\|f(1/a_1)\|_t} = (a_1c)^t + \left(\frac{a_1 - a_1^2c}{a_2}\right)^t$$

of $t$ on $(0,1)$. Since $c < 1/a_1$ and $a_1 < a_2$, both terms being raised to the $t^{\text{th}}$ power are positive and less than 1 $(a_1 - a_1^2c < a_1)$. Therefore, this function is strictly decreasing. This implies $\|f(c)\|_t < \|f(1/a_1)\|_t$ when $t \in (\epsilon, 1)$.

Suppose that $x \in [0, c)$, $y \in (x, 1/a_1]$, and $t \in (\epsilon, 1)$. We wish to show that $\|f(x)\|_t < \|f(y)\|_t$. Because $\|f\|_t$ is concave down and $\|f(c)\|_t < \|f(1/a_1)\|_t$, the function must be increasing on $[0, c)$. If $y \in (x, c]$, then $\|f(x)\|_t < \|f(y)\|_t$. If $y \in (c, 1/a_1]$, then $\|f(c)\|_t < \|f(y)\|_t$, so $\|f(x)\|_t < \|f(y)\|_t$.

Let $N$ be an integer satisfying $Nc > a_2$ and suppose $n$ in $S$ is greater than $N$. Let $\boldsymbol{x}$ be the factorization of $n$ with the smallest first component. For any factorization $\boldsymbol{w}$, the vector $\boldsymbol{w} + (-a_2, a_1)$ is also a factorization, provided the first component is not negative. Since $nc > a_2$, the first component of $\boldsymbol{x}$ must be less than $nc$. This means that $\boldsymbol{x}$ takes the form $nf(x)$ for some $x \in [0, c)$. Suppose $\boldsymbol{y}$ is a different factorization of $n$. It must take the form $nf(y)$ for some $y$ in $[0, 1/a_1]$. If $y < x$, then $\boldsymbol{y}$ must have a smaller first component than $\boldsymbol{x}$. Since we assume this to be false, $y \in (x, 1/a_2]$. It follows that for all $t$ in $(\epsilon, 1)$, $\|f(x)\|_t < \|f(y)\|_t$. Therefore, $\|\boldsymbol{x}\|_t < \|\boldsymbol{y}\|_t$. $\square$

**Proposition 3.5.** *Let $t > 1$, and let $a, b \in \mathbb{R}$ satisfy $a, b > 0$. Then $(a+b)^t > a^t + b^t$.*

*Proof.* Without loss of generality, assume that $a \geq b$. We fix $a, t$ and study $f(b) = (a+b)^t - a^t - b^t$ for $b \in [0, a]$. We calculate $f'(b) = t(a+b)^{t-1} - tb^{t-1}$. If this were zero, $a + b = b$, so $a = 0$, which is impossible. Hence $f'(b)$ does not change sign. Since $f'(0) = ta^{t-1} > 0$, the function $f(b)$ is strictly increasing for all $b \in [0, a]$. Since $f(0) = a^t - a^t = 0$, the function $f(b)$ is strictly positive for all $b \in (0, a]$. $\square$

Note that we can apply this repeatedly, e.g. $(a+b+c)^t > (a+b)^t + c^t > a^t + b^t + c^t$. Also, if $a > b > 0$, we have $(a-b+b)^t > (a-b)^t + b^t$, so $(a-b)^t < a^t - b^t$.

**Proposition 3.6.** *Let $t>1$ and let $x_1,x_2,a_1,a_2\in\mathbb{N}_0$ with $a_2>a_1\geq 1$ and $x_1+x_2\neq 0$. Consider the function $f(u)=(x_1+a_2u)^t+(x_2-a_1u)^t$ for $u\in[-\frac{x_1}{a_2},\frac{x_2}{a_1}]$. This function is decreasing from the left endpoint down to a single minimum, then is increasing to the right endpoint, which is the maximum on the whole interval. Further, if $x_1a_1+x_2a_2>\frac{a_1-1}{a_2-a_1}a_2^2$, then $f(\lfloor\frac{x_2}{a_1}\rfloor)$ gives the maximum value of $f(u)$ on all integers in that interval.*

*Proof.* We calculate $f'(u)=ta_2(x_1+a_2u)^{t-1}-ta_1(x_2-a_1u)^{t-1}$, and set to zero. We get $\left(\frac{a_2}{a_1}\right)^{1/(t-1)}(x_1+a_2u)=x_2-a_1u$. This equation, linear in $u$, has some single solution $u^\star$, which might (for now) not be in our interval for $u$. Next, we calculate $f(-\frac{x_1}{a_2})=(x_2+\frac{a_1}{a_2}x_1)^t=a_2^{-t}(a_2x_2+a_1x_1)^t$ and $f(\frac{x_2}{a_1})=(x_1+\frac{a_2}{a_1}x_2)^t=a_1^{-t}(a_1x_1+a_2x_2)^t$. Comparing these, we see that $f(\frac{x_2}{a_1})>f(-\frac{x_1}{a_2})$. Next we calculate $f'(-\frac{x_1}{a_2})=-ta_1(x_2+\frac{a_1}{a_2}x_1)^{t-1}<0$ and $f'(\frac{x_2}{a_1})=ta_2(x_1+\frac{a_2}{a_1}x_2)^{t-1}>0$. Because $f'(u)$ enjoyed a sign change, in fact $u^\star\in[-\frac{x_1}{a_2},\frac{x_2}{a_1}]$.

If we only wanted to maximize $f(u)$, we would be done; however, we seek the largest value of $f(u)$ for integer $u$. We know that $f(u)$ is maximal at the right endpoint $\frac{x_2}{a_1}$, so if $f(\lfloor\frac{x_2}{a_1}\rfloor)$ is larger than its value at the left endpoint[1], $f(-\frac{x_1}{a_2})$, we have found that largest integer $u$.

We begin with our assumption $x_1a_1+x_2a_2>\frac{a_1-1}{a_2-a_1}a_2^2$. Setting $X=x_1a_1+x_2a_2$ for convenience, we have $\left(\frac{a_2-a_1}{a_2}\right)X>(a_1-1)a_2$, which rearranges to $X-(a_1-1)a_2>\frac{a_1}{a_2}X$, and hence $(X-(a_1-1)a_2)^t>\left(\frac{a_1}{a_2}\right)^tX^t$.

If $\frac{x_2}{x_1}\in\mathbb{Z}$ we are done; otherwise, choose $\alpha\in(0,1)$ with $\frac{x_2}{a_1}-\alpha\in\mathbb{Z}$. By considering $x_2$ modulo $a_1$, we see that in fact $\alpha\leq\frac{a_1-1}{a_1}$, and consequently $-a_1a_2\alpha\geq -a_2(a_1-1)$.

Now, we calculate $f(\frac{x_2}{a_1}-\alpha)-f(-\frac{x_1}{a_2})=(x_1+\frac{a_2}{a_1}x_2-a_2\alpha)^t+(a_1\alpha)^t-a_2^{-t}(a_2x_2+a_1x_1)^t>a_1^{-t}(X-a_1a_2\alpha)^t+(a_1\alpha)^t-a_1^{-t}\left(\frac{a_1}{a_2}\right)^tX^t>a_1^{-t}(X-(a_1-1)a_2)^t+(a_1\alpha)^t-a_1^{-t}\left(\frac{a_1}{a_2}\right)^tX^t>a_1^{-t}\left(\frac{a_1}{a_2}\right)^tX^t+(a_1\alpha)^t-a_1^{-t}\left(\frac{a_1}{a_2}\right)^tX^t=(a_1\alpha)^t>0$. □

**Proposition 3.7.** *If $x=(x_1,y_1)$ is a factorization of $n$ with $x_1\leq\frac{n}{a_1+a_2}$, then $x_1\leq y_1$.*

*Proof.* By definition, we have $a_1x_1+a_2y_1-n=0$. When $x_1\leq\frac{n}{a_1+a_2}$,

$$0=a_1x_1+a_2y_1-n\leq\frac{a_1n}{a_1+a_2}+a_2y_1-n=n\left(\frac{a_1}{a_1+a_2}-1\right)+a_2y_1=-\frac{a_2n}{a_1+a_2}+a_2y_1.$$

We continue the chain, rearrange, and rewrite

$$0\leq a_2y_1-\frac{a_2n}{a_1+a_2}=a_2\left(y_1-\frac{n}{a_1+a_2}\right),$$

[1]Strictly speaking, we just need $f(\lfloor\frac{x_2}{a_1}\rfloor)>f(\lceil-\frac{x_1}{a_2}\rceil)$, but this is harder to calculate.

since the rightmost quantity above is non-negative it follows that $y_1 \geq \frac{n}{a_1+a_2} \geq x_1$, as desired. $\square$

Let $S = \langle a_1, a_2 \rangle$, $n \in S$. If $\mathbf{x} = (x_1, x_2)$ is a real factorization of $n$ then $y$ can be expressed in terms of $x_1$, $x_2(x_1) = \frac{n-a_1x_1}{a_2}$. Define the function $f(x_1, t) = x_1^t + x_2^t = ||(x_1, x_2)||_t^t$.

**Lemma 3.8.** *(Baton is passed once in each interval.)*

*Fix $x_1$ so that $x_1 + a_2 \leq \frac{n}{a_1+a_2}$. Let $T(x_1) = 1 + \frac{\ln(a_2/a_1)}{\ln(x_2(x_1)/x_1)}$, so that $\mathbf{m}(T(x_1)) = (x_1, x_2)$. That is, $T(x_1)$ gives the value of $t$ such that $(x_1, x_2)$ is the real factorization of $n$ with minimal $t$-norm.*

*The function*

$$F(t) = f(x_1, t) - f(x_1 + a_2, t),$$

*has exactly one zero for $t \in [T(x_1), T(x_1 + a_2)]$.*

*Proof.* First, we notice that $F(T(x_1)) < 0$ and $F(T(x_1 + a_2)) > 0$, since $\mathbf{m}(T(x_1)) = (x_1, x_2(x_1))$ and $\mathbf{m}(T(x_1+a_2)) = (x_1+a_2, x_2(x_1+a_2))$ have minimal $T(x_1)$ and $T(x_1+a_2)$ norms respectively, at the endpoints of our interval. By the IVT, $F$ has at least one zero in this interval. It remains to show that $F$ has at most one zero in this interval. Notice that $x_2(x_1 + a_2) = \frac{n-a_1(x_1+a_2}{a_2} = \frac{n-a_1x_1}{a_2} - a_1 = x_2(x_1) - a_1$. We use this and the definition of $f$, to rewrite $F$ as

$$F(t) = f(x_1, t) - f(x_1 + a_2, t) = x_1^t + x_2^t - \left[(x_1 + a_2)^t + (x_2 - a_1)^t\right].$$

We use algebra to decompose $F$ into a difference of two positive quantities, namely

$$F(t) = \underbrace{(x_2^t - (x_2 - a_1)^t)}_{F_2(t)} - \underbrace{((x_1 + a_2)^t - x_1^t)}_{F_1(t)}.$$

It is easy to see that $F_1$ and $F_2$ are strictly positive. When $F_1 = F_2$, clearly $\ln(F_1) = \ln(F_2)$. Let $X_1(t) = \ln(F_1(t))$ and $X_2(t) = \ln(F_2(t))$. We see that the zeroes of $F$ must agree with the zeroes of the difference

$$\Delta(t) = x_2(t) - X_1(t).$$

To prove our claim, we will show that $\Delta'(t) > 0$ for $t \in [T(x_1), T(x_1 + a_2)]$. We can use algebra to rewrite $X_1$ and $X_2$ in the forms

$$Y(t) = \ln(x_2^t - (x_2 - a_1)^t) = \ln\left(x_2^t\left(1 - [\frac{x_2 - a_1}{x_2}]^t\right)\right) = t\ln x_2 + \ln\left(1 - [\frac{x_2 - a_1}{x_2}]^t\right).$$

and

$$X_1(t) = \ln((x_1 + a_2)^t - x_1^t) = \ln\left((x_1 + a_2)^t\left(1 - [\frac{x_1}{x_1 + a_2}]^t\right)\right) = t\ln(x_1 + a_2) + \ln\left(1 - [\frac{x_1}{x_1 + a_2}]^t\right)$$

For brevity, let $\xi(w,t) = \ln(1-w^t)$, $A = \frac{x_2-a_1}{x_2}$ and $B = \frac{x_1}{x_1+a_2}$ Notice that both $A$ and $B$ are non-negative and strictly less than 1. We can write

$$X_2(t) = t\ln x_2 + \xi(A,t) \qquad\qquad X_1(t) = t\ln(x_1+a_2) + \xi(B,t).$$

We differentiate $X_2$ and $X_1$ with respect to $t$, obtaining

$$X_2'(t) = \ln x_2 + \frac{\partial\xi(A,t)}{\partial t} \qquad \text{and} \qquad X_1'(t) = \ln(x_1+a_2) + \frac{\partial\xi(B,t)}{\partial t}.$$

The difference of these derivatives gives us

$$\Delta'(t) = X_2' - X_1' = \ln x_2 + \frac{\partial\xi(A,t)}{\partial t} - \ln(x_1+a_2) - \frac{\partial\xi(B,t)}{\partial t} = \ln\left(\frac{x_2}{x_1+a_2}\right) + \left(\frac{\partial\xi(A,t)}{\partial t} - \frac{\partial\xi(B,t)}{\partial t}\right).$$

By assumption, $x_1 + a_2 \le \frac{n}{a_1+a_2}$, which, by Proposition 3.7, implies that $x_1 \le x_1 + a_2 \le x_2 - a_1 \le x_2$. As such, $\frac{x_2}{x_1+a_2} \ge 1$ and therefore $\ln\left(\frac{x_2}{x_1+a_2}\right) \ge 0$. Our problem has been reduced to showing that $\left(\frac{\partial\xi(A,t)}{\partial t} - \frac{\partial\xi(B,t)}{\partial t}\right)$ is non-negative.

$A$ and $B$ can be rewritten

$$A = \frac{x_2-a_1}{x_2} = 1 - \frac{a_1}{x_2} \qquad\qquad B = \frac{x_1}{x_1+a_2} = \frac{x_1+a_2-a_2}{x_1+a_2} = 1 - \frac{a_2}{x_1+a_2}.$$

Now, since $x_2 \ge x_1 + a_2$, it follows that

$$\frac{1}{x_2} \le \frac{1}{x_1+a_2} \Rightarrow \frac{a_1}{x_2} \le \frac{a_2}{x_1+a_2} \Rightarrow -\frac{a_1}{x_2} \ge -\frac{a_2}{x_1+a_2} \Rightarrow 1 - \frac{a_1}{x_2} \ge 1 - \frac{a_2}{x_1+a_2},$$

or equivalently, $1 \ge A \ge B$. If $\frac{\partial^2\xi(w,t)}{\partial w\partial t} \ge 0$ i.e. $\frac{\partial\xi(w,t)}{\partial t}$ is non-decreasing with respect to $w \in (0,1)$, we are done. Recall that $\xi(w,t) = \ln(1-w^t)$. We take the partial with respect to $t$, which gives

$$\frac{\partial\xi(w,t)}{\partial t} = \frac{-w^t\ln w}{1-w^t}.$$

We differentiate once more, this time with respect to $w$ after rewriting

$$\frac{\partial}{\partial w}\left(\frac{\partial\xi(w,t)}{\partial t}\right) = \frac{\partial}{\partial w}\left(-\frac{w^t\ln w}{1-w^t}\right) = -\frac{\partial}{\partial w}\left(\frac{w^t\ln w}{1-w^t}\right)$$

We will use the quotient rule, first calculating the derivative of the numerator and denominator seperately. Let $u = w^t\ln w$ and $v = 1-w^t$. Then $u_w = w^{t-1} + tw^{t-1}\ln w = w^{t-1}(1 + t\ln w)$ and $v_w = -tw^{t-1}$. We are ready to take the partial with respect to $w$, giving

$$\begin{aligned}
\frac{\partial}{\partial w}\left(\frac{\partial \xi(w,t)}{\partial t}\right) &= -\frac{\partial}{\partial w}\left(\frac{u}{v}\right) = -\left(\frac{vu_w - uv_w}{v^2}\right) \\
&= -\left(\frac{(1-w^t)w^{t-1}(1+t\ln w) + tw^{2t-1}\ln w}{(1-w^t)^2}\right) \\
&= -\left(\left((1-w^t)(1+t\ln w) + tw^t \ln w\right)\frac{w^{t-1}}{(1-w^t)^2}\right) \\
&= -\left(\left(1+t\ln w - w^t - tw^t\ln w + tw^t\ln w\right)\frac{w^{t-1}}{(1-w^t)^2}\right) \\
&= -\left(-w^t + t\ln w + 1\right)\frac{w^{t-1}}{(1-w^t)^2} \\
&= \left(w^t - t\ln w - 1\right)\frac{w^{t-1}}{(1-w^t)^2}
\end{aligned}$$

The fraction $\frac{w^{t-1}}{(1-w^t)^2}$ is positive, so we need only show that the quantity $W = w^t - t\ln w - 1$ is non-negative in $(0,1)$. $W$ can be rewritten $W = w^t - \ln(w^t) - 1$. Letting $Q(z) = z - \ln(z) - 1$, we notice that $W = Q(w^t)$. We can finally verify our claim by showing that $Q(z)$ is non-negative for all $z \in (0,1)$. We evaluate $Q(1) = 1 - \ln(1) - 1 = 0$, and differentiate to see that

$$Q'(z) = 1 = 1 - 1/z.$$

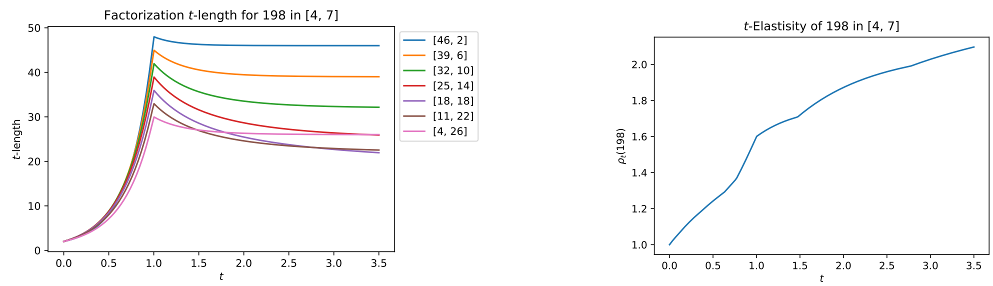
Since $z < 1$, it follows that $1 < 1/z$, and hence $Q'(z) = 1 - 1/z < 0$ for all $z \in (0,1)$. Since $Q$ has negative derivative for all $z \in (0,1)$ and is non-negative at its right endpoint, it must take on non-negative values everywhere in this interval. This completes the proof. $\square$

**Theorem 3.9.** *Fix $\mathbf{a} \in \mathbb{N}_0^2$ and let $S = \langle a_1, a_2\rangle$. Fix $n \geq \frac{a_1-1}{a_2-a_1}a_2^2 \in S$. For $t \in [1,\infty)$, let $q(t) = \frac{t}{t-1}$, and let $\mathbf{z}_m(t)$ and $z_M(t)$ denote the factorizations of $n$ with minimal (resp. maximal) $t$-norm as $t$ varies. Let $\boldsymbol{\alpha} = (a_2, -a_1)$, the unique minimal trade in $S$.*

*There exists a partition $\{s_0, s_1, \ldots, s_c\}$ of $[1,\infty)$ where $s_0 = 1$, and for $1 \leq j \leq c-1$, $s_j$ lies in the open interval $\Big(T(\mathbf{z}_m(1) + \boldsymbol{\alpha}(j-1)),\, T(\mathbf{z}_m(1) + j\boldsymbol{\alpha})\Big)$, $\rho_t(n) = \dfrac{||\mathbf{z}_M||_t}{||\mathbf{z}_m(t)_t||}$, with $z_M$ fixed as $t$ varies, and $z_m(t)$ defined piece-wise as*

$$\mathbf{z}_m(t) = \mathbf{z}_m(1) + \begin{cases} 0 & t \in [1, s_1) \\ j\boldsymbol{\alpha} & t \in (s_j, s_{j+1}) \\ c\boldsymbol{\alpha} & t \in (s_c, \infty). \end{cases}$$

*Proof.* By the Baton Lemma, each factorization, $\mathbf{x} = (x_1, x_2)$ of $n$ with $x_1 \leq x_2$ is the minimal $t$-length factorization for all $t$ in an interval containing $T(x_1, x_2)$ in order of increasing $x_1$, these factorizations can be ordered based on how many applications of the minimal trade are applied to the factorization with minimal $t$-length. Furthermore, we have that $z_m$ is fixed for all $t$ by 3.6. $\square$




(a) $t$-lengths for all factorizations of element 198 in semigroup $\langle 4, 7\rangle$

(b) $t$-elasticity of element 198 in semigroup $\langle 4, 7\rangle$

FIGURE 6

## 3.1. **Optimistic, Pesemistic and Crepusuculum Elements.**

**Definition 3.10.** Given a numerical semi group, $S = \langle a_1, a_2, ..., a_k\rangle$, an element, $n \in S$, is called *optimistic* if for $t$ large enough, the $t$-elasticity of $n$, $\rho_t(n)$, is always increasing.

**Definition 3.11.** Given a numerical semi group, $S = \langle a_1, a_2, ..., a_k\rangle$, an element, $n \in S$, is called *pessimistic* if for $t$ large enough, the $t$-elasticity of $n$, $\rho_t(n)$, is always decreasing.

**Definition 3.12.** Given $S = \langle a_1, a_2\rangle$ where $a_1 < a_2$ and some $n \in S$ for $1 \leq t$ has $\max \mathscr{L}_t(n) = a_1x_1 + a_2x_2$ and $\min \mathscr{L}_t(n) = a_1w_1 + a_2w_2$, if $x_2 > w_1$, then $n$ is called a **crepusculum** element.

**Theorem 3.13.** *Given $S = \langle a_1, a_2\rangle$, $n \in S$, and $t > 1$. Let $\mathbf{x} = (x_1, x_2)$ be the factorization of $n$ with maximum length and $\mathbf{w} = (w_1, w_2)$ be the factorization of $n$ with the minimum length where $x_1 > w_1 \geq w_2 > x_2$. Then for $t$ large enough, $n$ is an optimistic element of $S$.*

*Proof.* we are given that $S = \langle a_1, a_2\rangle$ where $n \in S$ and $t > 1$. Since $(x_1, x_2)$ is the factorization of $n$ with the maximum length and $(w_2, w_1)$ is the factorization of $n$ with the minimum length then the elasticity of $n$ is given by

$$\rho_t(n) = \left(\frac{x_1^t + x_2^t}{w_1^t + w_2^t}\right)^{\frac{1}{t}}$$

For simplicity, let $f(t) = \frac{x_1^t + x_2^t}{w_1^t + w_2^t}$ so $\rho_t(n) = (f(t))^{\frac{1}{t}}$. Taking natural log of $\rho_t(n)$ and then differentiating we get

$$\begin{aligned}
\ln(\rho_t(n)) &= \ln(f(t))^{\frac{1}{t}} \\
\rho_t'(n) &= \frac{\rho_t(n)}{t^2}\left(\frac{f''(t)}{f(t)}t - \ln(f(t))\right)
\end{aligned}$$

Notice that $\frac{\rho_t(n)}{t^2} > 0$ so we can ignore that term and focus on showing that $\left(\frac{f''(t)}{f(t)}t - \ln(f(t))\right) > 0$. For simplicity, let $r(t) = \left(\frac{f''(t)}{f(t)}t - \ln(f(t))\right)$. It is difficult to show that $r(t) > 0$ just from this equation. Instead, we will show that $\lim_{t\to\infty} r(t) = 0$ and that $r(t)$ is decreasing. This combination of facts will show that $r(t) > 0$ which is what we want.

First, let us define some functions that will help us throughout the proof. Let $g(t) = x_1^t + x_2^t$ and $h(t) = w_1^t + w_2^t$, so then we have that

$$\begin{aligned}
f(t) &= \frac{g(t)}{h(t)} \\
f'(t) &= \frac{g'(t)h(t) - h'(t)g'(t)}{(h(t))^2} \\
f''(t) &= \frac{g''(t)(h(t))^2 - g(t)h''(t)h(t) - 2g'(t)h'(t)h(t) + 2g(t)(h'(t))^2}{(h(t))^3}
\end{aligned}$$

Additionally,

$$\begin{aligned}
g(t) &= x_1^t + x_2^t & h(t) &= w_2^t + w_1^t \\
g'(t) &= x_1^t ln x_1 + x_2^t ln x_2 & h'(t) &= w_2^t ln w_2 + w_1^t ln w_1 \\
g''(t) &= x_1^t (ln x_1)^2 + x_2^t (ln x_2)^2 & h''(t) &= w_2^t (ln w_2)^2 + w_1^t (ln w_1)^2
\end{aligned}$$

Now we are ready to show that $\lim_{t\to\infty} r(t) = 0$ Expanding $r(t)$ we get

$$\begin{aligned}
r(t) &= \frac{f''(t)}{f(t)}t - \ln(f(t)) \\
&= \frac{g'(t)h(t) - h'(t)g(t)}{(h(t))^2} * \frac{h(t)}{g(t)}t - \ln(f(t)) \\
&= \frac{g'(t)}{g(t)}t - \frac{h'(t)}{h(t)}t - \ln(f(t))
\end{aligned}$$

Recall that $x_1 > w_1 \geq w_2 > x_2$. will will now substitute in for $g(t), g'(t), h(t), h'(t)$ and takes the limit of $r(t)$ as $t \to \infty$. So we have

$$\begin{aligned}
\lim_{t\to\infty} r(t) &= \lim_{t\to\infty}\left(\frac{x_1^t \ln x_1 + x_2^t \ln x_2}{x_1^t + x_2^t}\right) t - \lim_{t\to\infty}\left(\frac{w_2^t \ln w_2 + w_1^t \ln w_1}{w_2^t + w_1^t}\right) t - \lim_{t\to\infty} \ln \frac{x_1^t + x_2^t}{w_2^t + w_1^t} \\
&= \lim_{t\to\infty}\left(\frac{\ln x_1 + (\frac{x_2}{x_1})^t \ln x_2}{1 + (\frac{x_2}{x_1})^t}\right) \lim_{t\to\infty} t - \lim_{t\to\infty}\left(\frac{(\frac{w_2}{w_1})^t \ln w_2 + \ln w_1}{(\frac{w_2}{w_1})^t + 1}\right) \lim_{t\to\infty} t - \ln\left(\lim_{t\to\infty} \frac{(\frac{x_1}{w_1})^t + (\frac{x_2}{w_1})^t}{(\frac{w_2}{w_1})^t + 1}\right) \\
&= (\ln x_1) \lim_{t\to\infty} t - (\ln w_1) \lim_{t\to\infty} t - \ln\left(\frac{\lim_{t\to\infty} (\frac{x_1}{w_1})^t + \lim_{t\to\infty} (\frac{x_2}{w_1})^t}{\lim_{t\to\infty}\left((\frac{w_2}{w_1})^t + 1\right)}\right) \\
&= \lim_{t\to\infty} (t \ln x_1) - \lim_{t\to\infty} (t \ln w_1) - \lim_{t\to\infty} \ln\left(\frac{x_1}{w_1}\right)^t \\
&= \lim_{t\to\infty}\left(\ln\left(\frac{x_1}{w_1}\right)^t - \ln\left(\frac{x_1}{w_1}\right)^t\right) = 0
\end{aligned}$$

Since we know that $r(t)$ is tending towards zero, we will now show that $r'(t) < 0$ in order to show that $r(t)$ is increasing. Taking the derivative of $r(t)$ we get

$$\begin{aligned}
r'(t) &= f''(t)f(t) - (f'(t))^2 \\
r'(t) &= \frac{(g(t))^2(h'(t))^2 - (g(t))^2 h''(t)h(t) + g''(t)(h(t))^2 g(t) - (g'(t))^2(h(t))^2}{(h(t))^4}
\end{aligned}$$

Since $(h(t))^4 > 0$ we only need to show that the numerator of $r'(t)$ is less than zero, since a negative number divided by a positive number is still negative. Denote the numerator as $\mathrm{num}(r'(t))$. we want to expand $\mathrm{num}(r'(t))$ and take its limit as $t \to \infty$, since we are looking at large $t$.

First, we will expand $\mathrm{num}(r'(t))$. we will do this by first expanding each of the four terms in it, and then adding those together where we will see many simplifications. Our term order will be correspond to the fact that $x_1 > w_1 \geq w_2 > x_2$. The coefficients of the variables will not affect the term order. Here are the expansions of the four different terms:

$$\begin{aligned}
(g(t))^2(h'(t))^2 &= \left(x_1^{2t}+2(x_1x_2)^t+x_2^{2t}\right)\left(w_2^{2t}(\ln w_2)^2+2w_2^tw_1^t\ln w_2\ln w_1+w_1^{2t}(\ln w_1)^2\right)\\
&= (x_1w_1)^{2t}(\ln w_1)^2+2x_1^{2t}(w_2w_1)^t\ln w_2\ln w_1+(x_1w_2)^{2t}(\ln w_2)^2+2(x_1x_2)^tw_2^{2t}(\ln w_2)^2\\
&\quad +2(x_1x_2)^tw_1^{2t}(\ln w_1)^2+4(x_1x_2w_2w_1)^t\ln w_2\ln w_1+(x_2w_1)^{2t}(\ln w_2)^2\\
&\quad +2x_2^{2t}\ln w_2\ln w_1+(x_2w_2)^{2t}(\ln w_2)^2
\end{aligned}$$

$$\begin{aligned}
-(g(t))^2h''(t)h(t) &= -\left(x_1^{2t}+2(x_1x_2)^t+x_2^{2t}\right)\left(w_2^t(\ln w_2)^2+w_1^t(\ln w_1)^2\right)\left(w_1^t+w_2^t\right)\\
&= -(x_1w_1)^{2t}(\ln w_1)^2-x_1^{2t}(w_1w_2)^t(\ln w_1)^2-x_1^{2t}(w_1w_2)^t(\ln w_2)^2-(x_1w_2)^{2t}(\ln w_2)^2\\
&\quad -2(x_1x_2)^tw_1^{2t}(\ln w_1)^2-2(x_1x_2)^tw_2^{2t}(\ln w_2)^2-2(x_1x_2w_2w_1)^t(\ln w_2)^2-2(x_1x_2w_2w_1)\\
&\quad -(x_2w_1)^{2t}(\ln w_1)^2-x_2^{2t}(w_1w_2)^t(\ln w_2)^2-x_2^{2t}(w_1w_2)^t(\ln w_1)^2-(x_2w_2)^{2t}(\ln w_2)^2
\end{aligned}$$

$$\begin{aligned}
g''(t)(h(t))^2g(t) &= \left(x_1^t(\ln x_1)^2+x_2^t(\ln x_2)^2\right)\left(w_2^t+w_1^t\right)^2\left(x_1^t+x_2^t\right)\\
&= (x_1w_1)^{2t}(\ln x_1)^2+2x_1^{2t}(w_1w_2)^t(\ln x_1)^2+(x_1w_2)^{2t}(\ln x_1)^2+(x_1x_2)^tw_1^{2t}(\ln x_1)^2\\
&\quad +(x_1x_2)^tw_1^{2t}(\ln x_2)^2+(x_1x_2)^tw_2^{2t}(\ln x_1)^2+(x_1x_2)^tw_2^{2t}(\ln x_2)^2\\
&\quad +2(x_1x_2w_2w_1)^t(\ln x_1)^2+2(x_1x_2w_2w_1)^t(\ln x_2)^2+(x_2w_1)^{2t}(\ln x_2)^2\\
&\quad +2x_2^{2t}(w_1w_2)^t(\ln x_2)^2+(x_2w_2)^{2t}(\ln x_2)^2
\end{aligned}$$

$$\begin{aligned}
-(g'(t))^2(h(t))^2 &= -\left(x_1^{2t}(\ln x_1)^2+2(x_1x_2)^t\ln x_1\ln x_2+x_2^{2t}(\ln x_2)^2\right)\left(w_2^{2t}+2(w_2w_1)^t+w_1^2t\right)\\
&= -(x_1w_1)^{2t}(\ln x_1)^2-2x_1^{2t}(w_2w_1)^t(\ln x_1)^2-(x_1w_2)^{2t}(\ln x_1)^2-2(x_1x_2)^tw_1^{2t}\ln x_1\ln x_2\\
&\quad -2(x_1x_2)^tw_2^{2t}\ln x_1\ln x_2-4(x_1x_2w_2w_1)^t\ln x_1\ln x_2-(x_2w_1)^{2t}(\ln x_2)^2\\
&\quad -2x_2^{2t}(w_2w_1)^t(\ln x_2)^2-(x_2w_2)^{2t}(\ln x_2)^2
\end{aligned}$$

Now, summing all of those terms together, and keeping with the above term order convention, we get

$$\begin{aligned}
\operatorname{num}(r(t)) &= 2x_1^{2t}(w_2w_1)^t\ln w_1\ln w_2-x_1^{2t}(w_1w_2)^t(\ln w_1)^2-x_1^{2t}(w_1w_2)^t(\ln w_2)^2+\dots\\
&\quad -2x_2^{2t}(w_1w_2)^t\ln w_1\ln w_2-x_2^{2t}(w_1w_2)^t(\ln w_1)^2-x_2^{2t}(w_1w_2)^t(\ln w_2)^2
\end{aligned}$$

The limit and sign of $\operatorname{num}(r'(t))$ will be determined by its leading terms. Define those leading terms as $\mathrm{L}(\operatorname{num}(t)) = 2x_1^{2t}(w_2w_1)^t\ln w_1\ln w_2 - x_1^{2t}(w_1w_2)^t(\ln w_1)^2 - x_1^{2t}(w_1w_2)^t(\ln w_2)^2$. We can re-write this as

$$\begin{aligned}
\mathrm{L}(\operatorname{num}(t)) &= 2x_1^{2t}(w_2w_1)^t\ln w_1\ln w_2-x_1^{2t}(w_1w_2)^t(\ln w_1)^2-x_1^{2t}(w_1w_2)^t(\ln w_2)^2\\
&= x_1^{2t}(w_2w_1)^t\left(2\ln w_1\ln w_2-(\ln w_1)^2-(\ln w_2)^2\right)\\
&= -x_1^{2t}(w_2w_1)^t(\ln w_1-\ln w_2)^2
\end{aligned}$$

Notice that $x_1^{2t}(w_2w_1)^t(\ln w_1 - \ln w_2)^2 > 0$, but $-1 < 0$. Since $\mathrm{L}(\mathrm{num}(t))$ is going to determine the sign of the limit we can conclude that $\lim_{t\to\infty} \mathrm{num}(r'(t)) < 0$. This implies that $\lim_{t\to\infty}(r'(t)) < 0$. Since $\lim_{t\to\infty}(r(t)) = 0$ and $r(t)$ is decreasing, then we know that $r(t)$ is positive. Therefore, we know that $\rho_t'(n) > 0$ so $n$ is optimistic.

□

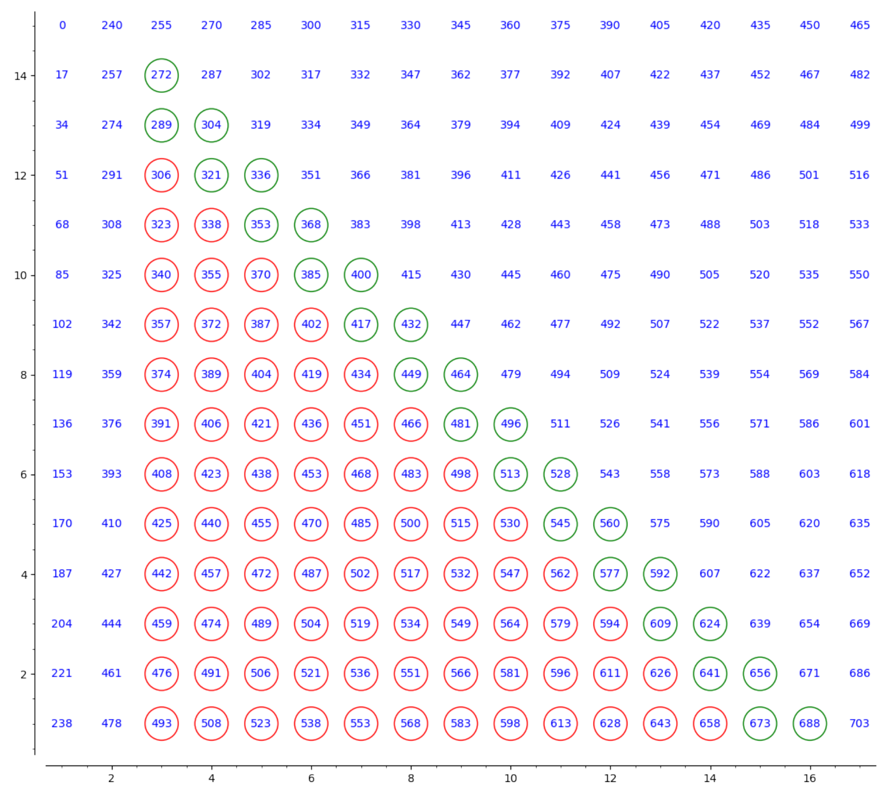



FIGURE 7. Apery Table of $S = \langle 15, 17\rangle$ where green=pessimistic and red=crepusculum

**Theorem 3.14.** *In $S = \langle a_1, a_2\rangle$ where $a_1 < a_2$ $\exists$ some element for $n \le a_2^2 + (a_1 - 2)(a_1 + a_2)$ where $n$ is pessimistic.*

*Proof.* Consider the Apery set of $S$ with respect to $a_1$. Then the smallest non-zero element of $Ap(S) = a_2$.

Consider an element $n \in S$ such that, $n = a_2 + (a_1)(a_2)$. Then the factorizations of $n$ are $(0, a_1 + 1)$ or $(a_2, 1)$ since the only trade for a 2-generated semigroup is $(a_2, 0) \sim (0, a_1)$.

Then the max $\mathscr{L}_t(c) = (a_2^t + 1^t)^{1/t}$ and the min $\mathscr{L}_t(c)) = ((a_1+1)^t + 0^t)^{1/t}$ for $1 \le t$. Then $\rho_t(n) = \frac{(a_2^t+1^t)^{\frac{1}{t}}}{((a_1+1)^t+0^t)^{\frac{1}{t}}} = \frac{(a_2^t+1)^{\frac{1}{t}}}{a_1+1}$ To see if the elasticity decreases, meaning $n$ is a pessimistic element, we must show that for $\epsilon > 0$, $\rho_{t+\epsilon}(n) < \rho_t(n)$, or $\frac{\rho_{t+\epsilon}(n)}{\rho_t(n)} < 1$. Then,

$$\frac{\frac{(a_2^{t+\epsilon}+1)^{\frac{1}{t+\epsilon}}}{a_1+1}}{\frac{(a_2^t+\epsilon)^{\frac{1}{t}}}{a_1+1}} = (\frac{(a_2^{t+\epsilon}+1)^{\frac{1}{t+\epsilon}}}{a_1+1})(\frac{a_1+1}{(a_2^t+1)^{\frac{1}{t}}}) = \frac{(a_2^{t+\epsilon}+1)^{\frac{1}{t+\epsilon}}}{(a_2^t+1)^{\frac{1}{t}}}.$$

We must show that $(a_2^{t+\epsilon}+1)^{\frac{1}{t+\epsilon}} < (a_2^t+1)^{\frac{1}{t}}$

By Hölders inequality (see [9]), we know that for some $1 \le s < t$ where $y \in \mathbb{R}^n$, $\|y\|_t \le \|y\|_s$. Therefore, since $1 \le t < t+\epsilon$, $(a_2^{t+\epsilon}+1)^{\frac{1}{t+\epsilon}}) < (a_2^t+1)^{\frac{1}{t}}$. So as $t$ increases, $\rho_t(n)$ decreases or is asymptotic.

**Theorem 3.15.** *There exists $a_2 - a_1$ pessimistic elements in $Ap_{a_2}(S)$ column for $a_2 - a_1 < a_1$ If $a_2 - a_1 + 1 < a_1$, then there exists crepusculum elements in this numerical semigroup.*

*Proof.* By the theorem above we know that $n \in S$ where $n = a_2 + a_2(a_1)$ is pessimistic. Consider this for the next element $n + a_2$, or $2a_2 + a_2(a_1)$. Since this only includes one more copy of $a_2$, we can use the same method we used to show $n$ is pessimistic. We can continue this pattern until we get to the element $(a_2 - a_1 + 1)a_2 + (a_2)a_1$ which we will call $r$. Then $r$ can be factored into $(a_2, a_2 - a_1 + 1)$ and $(0, a_2 + 1)$. Then max $\mathscr{L}_t(r) = ((a_2+1)^t + (0)^t)^{\frac{1}{t}}$ and the min $\mathscr{L}_t(r) = ((a_2)^t + (a_2 - a_1 + 1)^t)^{\frac{1}{t}}$ for $t$ large enough, meaning this is a crepusculum element. Then $\rho_t((r) = \frac{((a_2+1)^t+0^t)^{\frac{1}{t}}}{(a_2^t+(a_2-a_1+1)^t)^{\frac{1}{t}}} = \frac{a_2+1}{(a_2^t+(a_2-a_1+1)^t)^{\frac{1}{t}}}$ We must show that $\rho_{t+\epsilon}(r) > \rho_t(r)$, or $\frac{\rho_t(r)}{\rho_{t+\epsilon}(r)} < 1$. Consider

$$\frac{\frac{a_2+1}{(a_2^t+(a_2-a_1+1)^t)^{\frac{1}{t}}}}{\frac{a_2+1}{(a_2^{t+\epsilon}+(a_2-a_1+1)^{t+\epsilon})^{\frac{1}{t+\epsilon}}}} = (\frac{a_2+1}{(a_2^t+(a_2-a_1+1)^t)^{\frac{1}{t}}})(\frac{(a_2^{t+\epsilon}+(a_2-a_1+1)^{t+\epsilon})^{\frac{1}{t+\epsilon}}}{a_2+1})$$

$$= \frac{(a_2^{t+\epsilon}+(a_2-a_1+1)^{t+\epsilon})^{\frac{1}{t+\epsilon}}}{(a_2^t+(a_2-a_1+1)^t)^{\frac{1}{t}}} < \frac{(a_2^{t+\epsilon}+(a_2-a_1+1)^{t+\epsilon})^{\frac{1}{t+\epsilon}}}{(a_2^{t+\epsilon}+(a_2-a_1+1)^{t+\epsilon})^{\frac{1}{t}}}$$

$= (a_2^{t+\epsilon} + (a_2 - a_1 + 1)^{t+\epsilon})^{\frac{1}{t+\epsilon}-\frac{1}{t}} = (a_2^{t+\epsilon} + (a_2 - a_1 + 1)^{t+\epsilon})^{\frac{-\epsilon}{t(t+\epsilon)}} < 1$. Therefore, $\rho_t(r)$ is not decreasing, so $\exists$ only $a_2 - a_1$ pessimistic elements in the column created by $Ap(S) + a_2$. □

There are a lot of questions that about 2-generated elasticity curves that we have yet to answer, but have some thoughts about. here are some conjectures regarding 2-generated elasticity curves. □

**Conjecture 3.16.** *Given a numerical semigroup, $S = \langle a_1, a_2, ..., a_n \rangle$. Then there is some $n \in S$ such that for all $n_i > n$ each $n_i$ is optimistic.*

**Conjecture 3.17.** *In $S = \langle a_1, a_2 \rangle$ where $a_1 < a_2$, if $a_2 - a_1 < a_1$, then $\exists$ $a_2 - a_1$ at most pessimistic elements from the element $a_2 + a_2(a_1)$ to $(a_1 - 1)a_2 + (a_2 + a_1 - 1)a_1$ in a triangular form organized in the Apery Set.*

**Conjecture 3.18.** *If $a_2 - a_1 > a_1$, then there exists $a_1 - 1$ pessimistic elements in the column $Ap(S) + (a_2)a_1$ ,and no crepusculum elements.*

## 4. Elasticity Curves

One very interesting question that I did not ask in the initial problem list[2] was whether the elasticity curve of a numerical semigroup (i.e. $\rho_t(S)$ for all $t$) uniquely determines the semigroup. However we can solve it now, using a theorem we found[3] and a theorem we proved.

**Theorem 4.1.** *Let $T$ be an infinite subset of $\mathbb{R}^{\geq 1}$ containing a limit point. Let $k \in \mathbb{N}$. Let $a, b \in \mathbb{R}^k$. Assume that $a_1 \leq a_2 \leq \cdots \leq a_k$ and $b_1 \leq b_2 \leq \cdots \leq b_k$. Now, if $\|a\|_t = \|b\|_t$ for all $t \in T$, then $a = b$.*

**Theorem 4.2.** *Let $k \in \mathbb{N}$. Let $a \in \mathbb{N}_0^k$. Assume that $a_1 \leq a_2 \leq \cdots \leq a_k$, and that $A = \langle a \rangle$ is a numerical semigroup. Let $t \in [1, \infty)$. Then $\rho_t(A) = \frac{\|a\|_q}{a_1}$, where $q = t/(t-1)$.*

**Theorem 4.3.** *Let $T$ be an infinite subset of $\mathbb{R}^{\geq 1}$ containing a limit point. Let $k \in \mathbb{N}$. Let $a, b \in \mathbb{N}_0^k$. Assume that $a_1 \leq a_2 \leq \cdots \leq a_k$ and $b_1 \leq b_2 \leq \cdots \leq b_k$, and that $A = \langle \mathbf{a} \rangle, B = \langle \mathbf{b} \rangle$ form numerical semigroups. Now, if $\rho_t(A) = \rho_t(B)$ for all $t \in T$, then $A = B$.*

*Proof.* By Theorem 4.2, for all $t \in T$ we have $\rho_t(A) = \rho_t(B) = \frac{\|a\|_q}{a_1} = \frac{\|b\|_q}{b_1}$. Now, set $Q = \{t/(t-1) : t \in T\}$. $Q$ is itself a subset of $\mathbb{R}^{\geq 1}$, and must have a limit point since $T$ does (if $t, t'$ are close, then $t/(t-1)$, $t'/(t'-1)$ are close). Consider now the vectors $\overline{a} = \frac{a}{a_1} = (1, \frac{a_2}{a_1}, \ldots, \frac{a_k}{a_1}), \overline{b} = \frac{b}{b_1}$. Because the $q$-norm is homogeneous, we have $\|\overline{a}\|_q = \frac{1}{a_1}\|a\|_q = \frac{1}{b_1}\|b\|_q = \|\overline{b}\|_q$, for all $q \in Q$. By Theorem 4.1, we must have $\overline{a} = \overline{b}$. Hence, $a = \frac{a_1}{b_1}b$. Consider $\frac{a_1}{b_1} = \frac{u}{v}$, in simplified form. Since $a, b \in \mathbb{Z}^k$, we must have $v | \gcd(b)$, so $v = 1$ since $B$ is a numerical semigroup. But now $u | \gcd(a)$, so $u = 1$ since $A$ is a numerical semigroup. Hence $a_1 = b_1$, and thus $a = b$. $\square$

NOTE: For $t \in (0, 1)$, all bets are off! We do not have a version of Theorem 1, we do not yet have a version of Theorem 2, and the homogeneity of the $q$-norm for negative $q$ has not been established.

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[2] I thought it might be too hard.

[3] Not fully proved in the form given here, but I for one believe it's true.

## 5. Accepted Elasticity

**Lemma 5.1.** *Let $n = a_1 m \in S = \langle a_1, a_2 \rangle$ with factorization $(z_1, z_2)$. Then, $a_1 \mid z_2$.*

*Proof.* We have that $n = a_1 m = a_1 z_1 + a_2 z_2$. Then, $a_1 m = a_1(z_1 + \frac{a_2 z_2}{a_1})$. Since $\gcd(a_1, a_2) = 1, a_1 \nmid a_2$. Thus, $a_1 \mid z_2$. $\square$

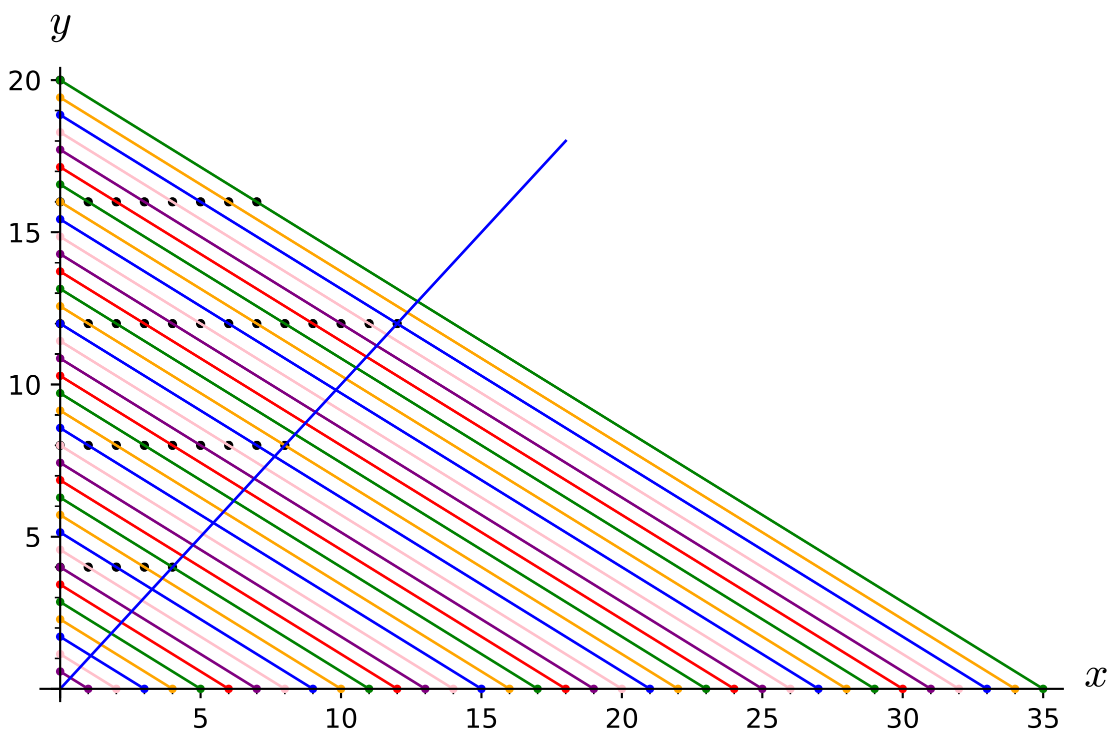


Figure 8. Elements $n = 4m$ in the semigroup $\langle 4, 7 \rangle$ and their integer factorization points $(x, y)$ for which $x \leq y$

**Theorem 5.2.** *Let $S = \langle a_1, a_2 \rangle$ and $\tau(S) = \{t \in (1, \infty) \mid \exists n \in S$ satisfying $\rho_t(n) = \rho_t(S)\}$. For each fixed $k \in \mathbb{Z}_{>0}$, define*

$$\tau_k(S) = \left\{ t = \frac{\ln(\frac{a_2 k}{j})}{\ln(\frac{a_1 k}{j})} \middle| j \in [1, a_1 k - 1] \right\}.$$

*Then,*

$$\tau(S) = \bigcup_{k=1}^{\infty} \tau_k(S).$$

*Proof.* For $1 < t < \infty$, $\rho_t(S) = \frac{\|M\|_t}{\|\mu\|_t}$ where $M$ is the real maximum factorization and $\mu$ is the real minimum factorization of an element in the semigroup. In order for $\rho_t(n) = \rho_t(S)$, the real factorization of $n$ with maximal length, $M = (\frac{n}{a_1}, 0)$ must be an integer point. It follows that $\rho_t(n) = \rho_t(S)$ implies that $n = a_1 m$ for some positive integer $m$.

Let $\mu(t) = (x, y)$. When $t = 1$, we know that $x = 0$ and when $t = \infty$, we know that $x = y$ by Proposition 2.9. So, in order for a factorization to have minimal $t$-length for the values of $t \in (1, \infty)$ we must have that $0 < x < y$. If $y \in \mathbb{Z}$, then, by Lemma 5.1, we know that $a_1 \mid y$. So, we can write $y = a_1 k$ for $k \in \mathbb{Z}_{>0}$. Now, we have that all possible integer factorizations of $a_1 m$ with minimal $t$-length can be written in the form $(j, a_1 k)$, where $1 \leq j \leq a_1 k - 1$. By Proposition 2.11 and Corollary 2.12, we know that $\|(j, a_1 k)\|_t = \|\mu\|_t$ only when

$$t = T((j, a_1 k)) = \frac{\ln\left((\frac{a_2}{a_1})(\frac{a_1 k}{j})\right)}{\ln(\frac{a_1 k}{j})} = \frac{\ln(\frac{a_2 k}{j})}{\ln(\frac{a_1 k}{j})}.$$

Thus, any $t$ for which $\rho_t(S)$ is accepted must be of the aforementioned form. In other words, $\bigcup_{k=1}^{\infty} \tau_k(S) \subseteq \tau(S)$.

Now, we must also ensure that $\tau(S) \setminus \bigcup_{k=1}^{\infty} \tau_k(S) = \emptyset$. Let $1 < q < \infty$ such that $q \notin \tau_k(S)$, so $q \neq \frac{\ln(\frac{a_2 k}{j})}{\ln(\frac{a_1 k}{j})}$. Assume, by way of contradiction, that $q \in \tau(S)$, so $\rho_q(n) = \rho_q(S) = \frac{\|M\|_t}{\|\mu\|_t}$. By the above reasoning, we must have that $n = a_1 m$ for $m \in \mathbb{Z}_{>0}$ and that all $(x, y) = \mu$ where $x, y \in \mathbb{Z}_{>0}$ are of the form $(j, a_1 k)$ for $j, k \in \mathbb{Z}_{>0}$. Then, $\|(j, a_1 k)\|_q = \|\mu\|_q \iff q = T((j, a_1 k))$. However, since $q \neq \frac{\ln(\frac{a_2 k}{j})}{\ln(\frac{a_1 k}{j})} = T((j, a_1 k))$, $\|(j, a_1 k)\|_q \neq \|\mu\|_q$. And thus, $\rho_q(n) \neq \rho_q(S)$. Hence, we cannot have an element of $\tau(S)$ that is not contained in any $\tau_k(S)$. Therefore,

$$\tau(S) = \bigcup_{k=1}^{\infty} \tau_k(S).$$

$\square$

**Proposition 5.3.** *Let $t \in \tau(S)$. Then, $\{n \mid \rho_t(n) = \rho_t(S)\} = \{n \in \mathbb{Z} \mid \exists k \geq 1$ for which $n = a_1 k((\frac{a_1^t}{a_2})^{\frac{1}{t-1}} + a_2)\}$.*

*Proof.* If $t \in \tau(S)$, then there exists a $k \geq 1$ such that $t \in \tau_k(S)$. So, $t = \frac{\ln(\frac{a_2 k}{j})}{\ln(\frac{a_1 k}{j})}$. Then,

$$t \ln\left(\frac{a_1 k}{j}\right) = ln\left(\frac{a_2 k}{j}\right) \Rightarrow \left(\frac{a_1 k}{j}\right)^t = \frac{a_2 k}{j} \Rightarrow a_1^t k^{t-1} = a_2 j^{t-1} \Rightarrow \left(\frac{j}{k}\right)^{t-1} = \frac{a_1^t}{a_2} \Rightarrow \frac{j}{k} = \left(\frac{a_1^t}{a_2}\right)^{\frac{1}{t-1}}.$$

Since $t \in \tau(S)$, we know that $\rho_t(n) = \rho_t(S)$ only when $(j, a_1k)$ is a factorization of $n$. So, we can write $n = a_1j + a_2(a_1k) = a_1k\left(\frac{j}{k} + a_2\right) = a_1k((\frac{a_1^t}{a_2})^{\frac{1}{t-1}} + a_2)$.

$\square$

**Proposition 5.4.** *Fix* $t = \frac{\ln(\frac{a_2k}{j})}{\ln(\frac{a_1k}{j})}$. *Let* $P = a_1(\frac{a_2k+j}{q})$ *where* $q = \gcd(k,j)$. *Then,* $\rho_t(n) = \rho_t(S) \iff P \mid n$.

*Proof.* First, we will prove that $P \mid n \Rightarrow \rho_t(n) = \rho_t(S)$. If $P \mid n$, then $n = \alpha P$ for some $\alpha \in \mathbb{Z}_{>0}$. Then, $n = \frac{\alpha j}{q}a_1 + \frac{\alpha a_1 k}{q}a_2$. So, we can write the factorizations $(x, y)$ of $n$ for which $x < y$ in the form $(\alpha\frac{j}{q}, \alpha\frac{a_1k}{q})$. We know that $\rho_d(n) = \rho_d(S)$ when

$$d = T((\alpha\tfrac{j}{q}, \alpha\tfrac{a_1k}{q})) = \frac{\ln(\frac{a_2(\alpha\frac{a_1k}{q})}{a_1(\alpha\frac{j}{q})})}{\ln(\frac{\alpha\frac{a_1k}{q}}{\alpha\frac{j}{q}})} = \frac{\ln(\frac{a_2k}{j})}{\ln(\frac{a_1k}{j})} = t.$$

Next, we must show that $\rho_t(n) = \rho_t(S) \Rightarrow P \mid n$. By Proposition 2.11, we can use $n\mathbf{m}(t)$ to return the factorization of $n$ whose $t$-length is equal to the real minimal $t$-length. Note, by reasoning from Theorem 5.2, that among these factorizations is $(j, a_1k)$, and consequently $n = a_1j + a_2a_1k$. So, we must evaluate $n\mathbf{m}\left(\frac{\ln(\frac{a_2k}{j})}{\ln(\frac{a_1k}{j})}\right)$. Let $u = t - 1 = \frac{\ln(\frac{a_2}{a_1})}{\ln(\frac{a_1k}{j})}$. We know that $n\mathbf{m}(1+u) = \left(\frac{n}{a_1+a_2\left(\frac{a_2}{a_1}\right)^{1/u}}, \frac{n}{a_1\left(\frac{a_1}{a_2}\right)^{1/u}+a_2}\right)$. Let us call the coordinates $m_1$ and $m_2$. First, we can evaluate $nm_1$ at $u$, and take its reciprocal.

$$\left.\frac{1}{nm_1}\right|_u = \frac{1}{n}\left(a_1 + a_2(\frac{a_2}{a_1})^{\frac{\ln(\frac{a_1k}{j})}{\ln(\frac{a_2}{a_1})}}\right) = \frac{1}{n}\left(a_1 + a_2e^{\ln(\frac{a_2}{a_1})\frac{\ln(\frac{a_1k}{j})}{\ln(\frac{a_2}{a_1})}}\right) = \frac{1}{n}(a_1 + a_2e^{\ln(\frac{a_1k}{j})})$$

$$= \frac{1}{n}(a_1 + a_2(\frac{a_1k}{j})) = \frac{1}{n}(\frac{a_1j + a_2a_1k}{j}) = \frac{1}{j}.$$

Now, we will evaluate the reciprocal of $nm_2$ at $u$.

$$\left.\frac{1}{nm_2}\right|_u = \frac{1}{n}\left(a_1(\frac{a_1}{a_2})^{\frac{\ln(\frac{a_1k}{j})}{\ln(\frac{a_2}{a_1})}} + a_2\right) = \frac{1}{n}\left(a_1e^{\ln(\frac{a_1}{a_2})\frac{\ln(\frac{a_1k}{j})}{\ln(\frac{a_2}{a_1})}} + a_2\right)$$

Since $\ln(\frac{a_1}{a_2}) = -\ln(\frac{a_2}{a_1})$,

$$\left.\frac{1}{nm_2}\right|_u = \frac{1}{n}(a_1e^{-\ln(\frac{a_1k}{j})}+a_2) = \frac{1}{n}(a_1e^{\ln(\frac{j}{a_1k})}+a_2) = \frac{1}{n}(a_1(\frac{j}{a_1k})+a_2) = \frac{1}{n}(\frac{a_1k + a_2a_1k}{a_1k}) = \frac{1}{a_1k}.$$

Therefore, $n\mathbf{m}(t) = (j, a_1k)$ Then, we can write $n = a_1j + a_2a_1k = a_1(j + a_2k)$. Let $\frac{P}{a_1} = \frac{j+a_2k}{q} = r$. Then, $j + a_2k = rq$, so $r \mid j + a_2k = \frac{n}{a_1}$. So, we have that $\frac{P}{a_1} \mid \frac{n}{a_1}$. Therefore, $P \mid n$. $\square$

**Proposition 5.5.** *Let us define* $K = \mathbb{Z} \cap (\frac{n}{a_1(a_1+a_2)}, \frac{n}{a_1a_2})$ *and* $\tau(n) = \{t \mid \rho_t(n) = \rho_t(S)\}$. *Then, given* $n \in S$, $\tau(n) = \left\{ \frac{\ln(\frac{a_2k}{\frac{n}{a_1}-a_2k})}{\ln(\frac{a_1k}{\frac{n}{a_1}-a_2k})} \mid k \in K \right\}$.

*Proof.* Since we have $n$ such that $\rho_t(n) = \rho_t(S)$, $(j, a_1k)$ must be a factorization of $n$ where $j < a_1k$. If $a_1k \geq \frac{n}{a_2}$, then $n \geq a_1j + a_2(\frac{n}{a+2}) = a_1j + n$, which is untrue. Thus, we must have that $a_1k < \frac{n}{a_2}$, and so $k < \frac{n}{a_1a_2}$. If $a_1k \leq \frac{n}{a_1+a_2}$, then $j = \frac{n-a_2a_1k}{a_1} \geq \frac{n-a_2\frac{n}{a_1+a_2}}{a_1} = \frac{a_1n+a_2n-a_2n}{a_1(a_1+a_2)} = \frac{n}{a_1+a_2}$. This implies that $j \geq \frac{n}{a_1+a_2} \geq a_1k$, which is also untrue since $j < a_1k$. Thus, we must have that $a_1k > \frac{n}{a_1+a_2}$, and so $k > \frac{n}{a_1(a_1+a_2)}$. Therefore, we can only examine factorizations with $k$-values within the set $K = \mathbb{Z} \cap (\frac{n}{a_1(a_1+a_2)}, \frac{n}{a_1a_2})$. To find the $t$-values at which these factorizations accept the minimal real $t$-length, we can use the function $T((j, a_1k)) = \frac{\ln(\frac{a_2k}{j})}{\ln(\frac{a_1}{j})}$. Since $n = a_1j + a_2a_1k \Rightarrow j = \frac{n-a_2a_1k}{a_1}$, we can write $T((j, a_1k)) = T((\frac{n}{a_1} - a_2k, a_1k)) = \frac{\ln(\frac{a_2k}{\frac{n}{a_1}-a_2k})}{\ln(\frac{a_1k}{\frac{n}{a_1}-a_2k})}$. Therefore, $\tau(n) = \left\{ \frac{\ln(\frac{a_2k}{\frac{n}{a_1}-a_2k})}{\ln(\frac{a_1k}{\frac{n}{a_1}-a_2k})} \mid k \in K \right\}$. □

**Proposition 5.6.** $|\tau(n)| = |K|$.

*Proof.* Fix $t_1, t_2 \in \tau(n)$. Then, $t_1 = T((\frac{n}{a_1} - a_2k_1, a_1k_1))$ for some $k_1 \in K$. Similarly, $t_2 = T((\frac{n}{a_1} - a_2k_2, a_1k_2))$ for some $k_2 \in K$. Let $t_1 = t_2$. So, $T((\frac{n}{a_1} - a_2k_1, a_1k_1)) = T((\frac{n}{a_1} - a_2k_2, a_1k_2))$. By Corollary 2.12, $T((\frac{n}{a_1} - a_2k_1, a_1k_1)) = 1 + \frac{\ln(\frac{a_2}{a_1})}{\ln(\frac{a_1k_1}{\frac{n}{a_1}-a_2k_1})} = 1 + \frac{\ln(\frac{a_2}{a_1})}{\ln(\frac{a_1k_2}{\frac{n}{a_1}-a_2k_2})} = T((\frac{n}{a_2} - a_2k_2, a_1k_2))$. It follows that this implies

$$\frac{a_1k_1}{\frac{n}{a_1} - a_2k_1} = \frac{a_1k_2}{\frac{n}{a_1} - a_2k_2} \Rightarrow a_1k_1(\frac{n}{a_1} - a_2k_2) = a_1k_2(\frac{n}{a_1} - a_2k_1) \Rightarrow nk_1 - a_1a_2k_1k_2 = nk_2 - a_1a_2k_1k_2$$

$$\Rightarrow nk_1 = nk_2 \Rightarrow k_1 = k_2.$$

Since $t_1 = t_2 \Rightarrow k_1 = k_2$, we know that each distinct $t \in \tau(n)$ corresponds to a unique $k \in K$. Hence, $|\tau(n)| = |K|$. □

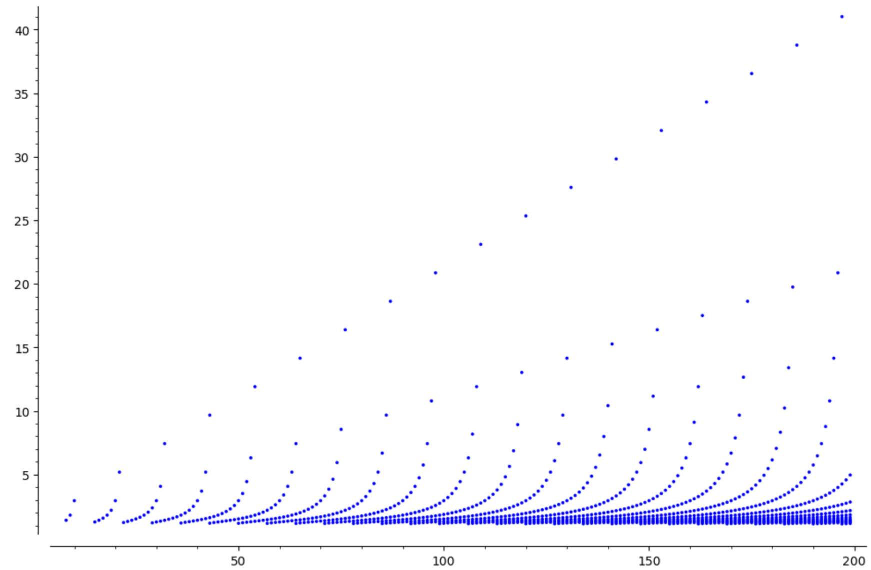


FIGURE 9. Accepted t-elasticities for elements, n, of S where n=4x

5.1. **Accumulation of $\tau$.** Notice that when plotting elements of $\tau(S)$ or, accepted t-elasticities, there is a large grouping around $t = 1$. We conjectured that 1 would be the only accumulation point of $\tau(S)$, but with further research we saw that was not the case. First we found that the limits of $\tau(S)$ as $k \liminf$ for different $j$ values did not always approach 1.

**Lemma 5.7.** *For $t(k, j)$ being an element of $\tau(S)$, where $j = 1$, then* $\lim_{k\to\infty} t(k, 1) = 1$.

*Proof.* Consider $t(k,1) = \frac{\ln(\frac{a_2 k}{1})}{\ln(\frac{a_1 k}{1})}$. Then

$$\lim_{k\to\infty} t(k,1) = \lim_{k\to\infty} \frac{\ln(\frac{a_2 k}{1})}{\ln(\frac{a_1 k}{1})} = \lim_{k\to\infty} \frac{\ln a_2 + \ln k}{\ln a_1 + \ln k} = \lim_{k\to\infty} \frac{\ln a_2}{\ln a_1 + \ln k} + \lim_{k\to\infty} \frac{\ln k}{\ln a_1 + \ln k}$$

$$= 0 + \lim_{k\to\infty} \frac{1}{\frac{\ln a_1}{\ln k} + 1} = \frac{1}{0+1} = 1$$

$\square$

**Lemma 5.8.** *For $t(k, j)$ being an element of $\tau(S)$, where $j = a_1 k - 1$, then* $\lim_{k\to\infty} t(k, a_1 k - 1) = \infty$.

*Proof.* Consider, $t(k, a_1 k - 1) = \frac{\ln(\frac{a_2 k}{a_1 k - 1})}{\ln(\frac{a_1 k}{a_1 k - 1})}$. Then,

$$\lim_{k\to\infty} t(k, a_1k-1) = \lim_{k\to\infty} \frac{\ln(\frac{a_2k}{a_1k-1})}{\ln(\frac{a_1k}{a_1k-1})} = \lim_{k\to\infty} \frac{\ln a_2 + \ln k - \ln a_1 - \ln(k - \frac{1}{a_1})}{\ln a_1 + \ln k - \ln a_1 - \ln(k - \frac{1}{a_1})}$$

$$= \lim_{k\to\infty} \frac{\ln a_2}{\ln k - \ln(k - \frac{1}{a_1})} + \lim_{k\to\infty} \frac{\ln k}{\ln k - \ln(k - \frac{1}{a_1})} + \lim_{k\to\infty} \frac{-\ln a_1}{\ln k - \ln(k - \frac{1}{a_1})} + \lim_{k\to\infty} \frac{-\ln(k - \frac{1}{a_1})}{\ln k - \ln(k - \frac{1}{a_1})}$$

$$= \lim_{k\to\infty} \frac{\ln \frac{a_2}{a_1}}{\ln k - \ln(k - \frac{1}{a_1})} + \lim_{k\to\infty} \frac{\ln k - \ln(k - \frac{1}{a_1})}{\ln k - \ln(k - \frac{1}{a_1})} = \lim_{k\to\infty} \frac{\ln \frac{a_2}{a_1}}{\ln k - \ln(k - \frac{1}{a_1})} + 1$$

Since $\lim_{k\to\infty} \ln k - \ln(k - \frac{1}{a_1}) = \lim_{k\to\infty} \ln k - \ln(k(1 - \frac{1}{a_1k})) = \lim_{k\to\infty} \ln k - \ln k - \ln(1 - \frac{1}{a_1k}) = \lim_{k\to\infty} -\ln(1 - \frac{1}{a_1k})$ approaches 0 , and $\ln \frac{a_2}{a_1}$ is a positive constant,

$$\lim_{k\to\infty} \frac{\ln \frac{a_2}{a_1}}{\ln k - \ln(k - \frac{1}{a_1})} + 1 = \infty$$

□

This led us to believe that every element in $\mathbb{R}_{\geq 1}$ would be an accumulation point of $\tau(S)$.

**Theorem 5.9.** *Every $\alpha \in \mathbb{R}_{\geq 1}$ is an accumulation point of $\tau(S)$.*

*Proof.* Consider the elements of $\tau(S)$; each $t = \frac{\ln(\frac{a_2k}{j})}{\ln(\frac{a_1k}{j})}$. For simplicity, call $\frac{k}{j} = r$ and we want to find values of $t$ that get infinitely close to any $\alpha$.
Consider the dual of $\alpha$ as a function, where $q(\alpha) = \frac{\alpha}{\alpha-1} = \frac{\ln(a_2r)}{\frac{\ln a_2}{\ln a_1}}$. Then,

$$q(\alpha)(\frac{\ln a_2}{\ln a_1}) = \ln(a_2r) \longrightarrow a_2r = e^{q(\alpha)(\frac{\ln a_2}{\ln a_1})}$$

So, $r = \frac{e^{q(\alpha)(\frac{\ln a_2}{\ln a_1})}}{a_2}$ meaning $r$ is either rational, or irrational. Either way, $r \in \mathbb{R}^+$. We know that every real number is an accumulation point of the rationals, so $r$ is an accumulation point of $\mathbb{Q}$. Then, $\exists\ S_k$, a sequence of real numbers that converge to $r$. Pick $s_k \in S_k$ where $s_k = \frac{a_2^{q(t_k)-1}}{a_1^{q(t_k)}}$, where $t \in \tau(S)$. Then

$$|r - s_k| = |q(r) - q(s_k)| < \epsilon$$

which we know to be true since $q(r)$ is a continuous function. From this we can derive that:

$$|\frac{\ln(a_2r)}{\ln(\frac{a_2}{a_1})} - \frac{\ln(a_2s_k)}{\ln(\frac{a_2}{a_1})}| < \epsilon \longrightarrow |\frac{\ln(a_2r) - \ln(a_2s_k)}{\ln(\frac{a_2}{a_1})}| < \epsilon \longrightarrow |\ln(\frac{r}{s_k})| < \epsilon(\ln(\frac{a_2}{a_1}))$$

Next we want to see if $|q(r)-q(s_k)| = |q(q(\alpha)) - q(q(t_k))| < \delta$. Notice that $q(t)$ is its own inverse, so for $|\alpha - t_k| < \delta$, pick $\delta > |\frac{ln(\frac{a_2}{a_1})}{\ln(a_1 r)\ln(a_1 s_k)}||\epsilon \ln(\frac{a_2}{a_1})|$.

This means that there is a sequence of $t_k$-values that approaches $\alpha$ by using the sequences of rationals, $S_k$ that approach $r$. This means that we can find values of $t_k$ that approach $\alpha$ for every $\alpha \in \mathbb{R}_{\geq 1}$. Therefore, every $\alpha \in \mathbb{R}_{\geq 1}$ is an accumulation point of $\tau(S)$.

□

5.2. $\tau$ **as a Unique Semigroup.**

**Proposition 5.10** (MORE STRONGERER). *Fix $S$ generated by $\mathbf{a} \in \mathbb{N}_0^2$, and let $\gamma$ denote the greatest common divisor of all positive exponents appearing in the prime factorization of $a_1 a_2$. Then*

$$\tau(S) \cap \mathbb{Q} = 1 + \left(\{0\} \cup \{\tfrac{c-d}{d} : (c-d) \mid \gamma\}\right).$$

*Proof.* We will prove this claim by verifying containment in both directions. Suppose $t = \frac{c}{d} \in \tau(S) \cap \mathbb{Q}$. If $\rho_t(S) = \rho_t(n)$, for some $n$, then $n$ must have a factorization achieving the minimum real $t$-length. This can occur if and only if $\mathbf{m}(t)$ falls on a rational point or, equivalently, the reciprocal of each of its entries is rational. When $\mathbf{m}$ has all rational entries, $\mathbf{m}(t)$ lifts to an integer point upon scaling by the least common denominator of its entries, and can be scaled further by $a_1$ to ensure that it has a factorization that achieves the maximal length, so this is sufficient. Consider

$$\frac{1}{m_1(t)} = a_1 + a_2\left(\tfrac{a_2}{a_1}\right)^{1/(t-1)} \qquad \Rightarrow \qquad \frac{1}{a_2}\left(\frac{1}{m_1(t)} - a_1\right) = \left(\tfrac{a_2}{a_1}\right)^{1/(t-1)}.$$

As such, $1/m_1(t)$ and $\left(\frac{a_2}{a_1}\right)^{\frac{1}{t-1}}$ lie in the same coset of $\mathbb{Q}$, so $1/m_1 \in \mathbb{Q} \iff \left(\frac{a_2}{a_1}\right)^{\frac{1}{t-1}} = \frac{u}{v}$ for some co-prime integers $u$ and $v$. Rewrite $t-1 = \frac{c-d}{d}$, which we substitute to obtain a new expression, $\star : \left(\frac{a_2}{a_1}\right)^{\frac{d}{c-d}} = \frac{u}{v}$. By assumption, $\gcd(a_1,a_2) = 1$ which means we can write

$$a_1 a_2 = \prod_{\substack{p_i \mid a_1 \\ p_i \text{ prime}}} p_i^{\gamma\alpha_i} \prod_{\substack{q_i \mid a_2 \\ q_i \text{ prime}}} q_i^{\gamma\beta_i}$$

where each exponent $\alpha_i$ and $\beta_i$ is positive and $\gcd(\{\alpha_i, \beta_i\}) = 1$. We substitute these products into $\star$ to see that

$$\left(\frac{a_2}{a_1}\right)^{d/(c-d)} = \left(\frac{\prod_{p_i \mid a_1} p_i^{\gamma\alpha_i}}{\prod_{q_i \mid a_2} q_i^{\gamma\beta_i}}\right)^{d/(c-d)} = \frac{\prod_{p_i \mid a_1} p_i^{d\gamma\alpha_i/(c-d)}}{\prod_{q_i \mid a_2} q_i^{d\gamma\beta_i/(c-d)}}$$

Each prime factor in the numerator and denominator of the ratio above must have a positive integer exponent in order for our desired quantity to be rational, which requires that any cancellation be independent of $\alpha_i$ and $\beta_i$. If $(c-d) \mid \gamma$, then $t \in \tau(S) \cap \mathbb{Q}$, which verifies the containment

$$1 + \left(\{0\} \cup \{\tfrac{c-d}{d} : (c-d) \mid \gamma\}\right) \subseteq \tau(S) \cap \mathbb{Q}.$$

We prove the reverse containment by way of contradiction. Suppose $t = 1 + \frac{c-d}{d}$ with $\gcd(d, c-d) = 1$, so that its rational part is written in lowest terms. If $(c-d) \nmid \gamma$ and $d\gamma/(c-d) \in \mathbb{Z}^+$, it must be true that $(c-d) \mid d\gamma$. Let $g = \gcd(\gamma, c-d)$, $b' = \frac{c-d}{g}$, and write $\gamma = g\gamma'$ and $c-d = gb'$ with $b' > 1$ and $\gcd(b', \gamma') = 1$. By assumption, we must have $gb' \mid gd\gamma'$, which implies $b' \mid d$. It follows that $\gcd(d, c-d) \geq b' > 1$, which is a contradiction.

A symmetric result holds for the reciprocal $1/m_2(t)$, since it lies in the same $\mathbb{Q}$-coset as $\frac{a_1}{a_2}^{1/(t-1)}$, which also lies in the same $\mathbb{Q}$-coset as its reciprocal. $\square$

**Remark 5.11.** Resuming notation from the preceding proposition, let $D(\gamma)$ denote the set of positive integer divisors of $\gamma$. We may notice that the preceding result induces a natural decomposition of $\tau(S) \cap \mathbb{Q}$, into (not-necessarily disjoint) subsets indexed by positive integer denominators $d \geq 1$. For brevity, write $r = c - d$ and for each such $d$, define $\tau_S(d) = 1 + \{\frac{r}{d} : r \in D(\gamma)\}$. We can say that

$$(\tau(S) \setminus \{1\}) \cap \mathbb{Q} = \bigcup_{d=1}^{\infty} \tau_S(d).$$

**Theorem 5.12** (THE STRONGEREST?!)**.**

$$\tau(S) \cap \mathbb{Q} = 1 + \left(\{0\} \cup \{\tfrac{\gamma}{n} : n \in \mathbb{Z}^+\}\right).$$

*Proof.* Since each rational $t \in \tau(S)$ lies in some $\tau_S(d)$ (or is equal to 1), it can be written $t = 1 + \frac{r}{d}$ where $r \mid \gamma$. Let $rz = \gamma$. Then we can write

$$t = 1 + \frac{\gamma/z}{d} = \frac{\gamma}{zd}.$$

To see that every $n$ value appears as a denominator in $\tau(S) \cap \mathbb{Q}$, fix an arbitrary positive integer $n$. If $\gcd(\gamma, n) = 1$, we simply take $1 + \gamma/n \in \tau_S(n)$. Otherwise, write $g = \gcd(\gamma, n) > 1$. Write $n = gn'$ and $\gamma = g\gamma'$. We can find $1 + \frac{\gamma}{n} = 1 + \frac{\gamma'}{n'}$ in $\tau_S(n')$ since $\gamma'$ is, by definition, a positive divisor of $\gamma$. $\square$

**Theorem 5.13.** *The set*

$$q(\tau) = \frac{\tau}{\tau - 1} = \{\tfrac{t}{t-1} : t \in \tau\} = \{\tfrac{\log(a_2 k/j)}{\log(a_2/a_1)} : k, j \in \mathbb{Z}_{>1}, k/j > 1/a_1\}$$

*is an additive semigroup. Moreover,* $(q(\tau), +) \cong (\mathbb{Q}_{\geq \frac{a_2}{a_1}}, \cdot)$.

*Proof.* We will first show that every $q \in q(\tau)$ has the claimed form. We have seen that every $t \in \tau$ can be written

$$t = \frac{\log(a_2 k/j)}{\log(a_1 k/j)} \qquad \Rightarrow \qquad t - 1 = \frac{\log(a_2/a_1)}{\log(a_1 k/j)}.$$

Any $t \in \tau$ corresponds uniquely to its Hölder conjugate in $q(\tau)$, which we can write

$$q(t) = \frac{t}{t-1} = \frac{\log(a_2 k/j)}{\log(a_1 k/j)} \cdot \frac{\log(a_1 k/j)}{\log(a_2/a_1)} = \frac{\log(a_2 k/j)}{\log(a_2/a_1)},$$

as claimed. Associativity is clear since $\frac{\tau}{\tau-1} \subset \mathbb{R}$, where addition is associative. To show closure under addition, take two arbitrary elements, writing $r_1, r_2 > 1/a_1$ for their rational parts. Using algebra, we see that

$$t_1 + t_2 = \frac{\log(a_2 r_1)}{\log(a_2/a_1)} + \frac{\log(a_2 r_2)}{\log(a_2/a_1)} = \frac{\log(a_2(a_2 r_1 r_2))}{\log(a_2/a_1)}.$$

We need only verify that $a_2 r_1 r_2 > 1/a_1$. Since $a_2 > a_1$ and $r_1 > 1/a_1$, we have that $a_2 r_1 > 1$. It follows that $a_2 r_1 r_2 > r_2 > 1/a_1$, as desired.

By considering the sequence of maps

$$\varphi : (q(\tau), +) \xrightarrow[\cdot \log(a_2/a_1)]{\psi_1} (\{\log(a_2 r) : r \geq 1/a_1\}, +) \xrightarrow[\exp(\log(a_2 r))]{\psi_2} (\mathbb{Q}_{\geq \frac{a_2}{a_1}}, \cdot).$$

the semigroup isomorphism is easily verified. $\square$

**Corollary 5.14.** *$q(\tau)$ is atomic, and its set of atoms is precisely*

$$\mathcal{A}(q(\tau)) = \varphi^{-1}\left(\mathbb{Q} \cap [\tfrac{a_2}{a_1}, \tfrac{a_2}{a_1}^2)\right) = \left\{ \frac{\log(a_2 r)}{\log(a_2/a_1)} : r \in [\tfrac{a_2}{a_1}, \tfrac{a_2^2}{a_1^2}) \right\}.$$

*Proof.* For ease of notation, let $r$ denote $\frac{a_2}{a_1}$. We will prove that $\mathcal{A}(\mathbb{Q}_{\geq r}, \cdot) = [r, r^2)$. Any rational $s \in [r, r^2)$ is clearly an atom, since $s \geq r > 1$, it cannot factor into a product of smaller atoms in the semigroup. To see that every $s$ outside this interval is not an atom, take $s \geq r^2$. Then $s = r \cdot s/r$, with $s/r \geq r$, which completes the proof. $\square$

**Proposition 5.15.** *Fix $q = \frac{\log(a_2 r)}{\log(a_2/a_1)} \in q(\tau)$ for some $r \in \mathbb{Q}_{\geq \frac{1}{a_1}}$. Then*

$$q - 1 = \frac{\log(a_1 r)}{\log(a_2/a_1)}.$$

*Further, $q - 1 \in q(\tau) \iff r \geq \frac{a_2}{a_1^2}$*

*Proof.* Using algebra,

$$q - 1 = \frac{\log(a_2 r)}{\log(a_2/a_1)} - \frac{\log(a_2/a_1)}{\log(a_2/a_1)} = \frac{\log(a_1 r)}{\log(a_2/a_1)},$$

as claimed. If $q-1 \in \tau$, by definition, we must have $a_1 r \geq \frac{a_2}{a_1}$, hence $r \geq \frac{a_2}{a_1^2}$. Conversely, $r \geq a_2/a_1^2$, then $a_1 r \geq a_2/a_1$, which means that $q-1 \in Q(\tau)$, as desired. ☐

**Remark 5.16.** The bijection $q : \tau \to q(\tau)$ induces a commutative semigroup structure $(\tau, \oplus)$ under the binary operation $\oplus$, given by

$$s \oplus t = q^{-1}\left(q(s) + q(t)\right).$$

We derive the formula for this operation as follows.

$$\begin{aligned} s \oplus t = q^{-1}(q(s) + q(t)) &= q^{-1}\left(\frac{s}{s-1} + \frac{t}{t-1}\right) \\ &= q^{-1}\left(\frac{s(t-1)+t(s-1)}{(s-1)(t-1)}\right) \\ &= \left(\frac{s(t-1)+t(s-1)}{(s-1)(t-1)}\right) \cdot \left(\frac{(s-1)(t-1)}{s(t-1)+t(s-1)-(s-1)(t-1)}\right) \end{aligned}$$

The denominator of the rightmost fraction above simplifies

$$s(t-1)+t(s-1)-(t-1)(s-1) = s(t-1)+t(s-1)-t(s-1)+(s-1) = st-1.$$

Which we substitute, cross-cancelling $(s-1)(t-1)$ to obtain

$$s \oplus t = \left(\frac{s(t-1)+t(s-1)}{st-1}\right) = \frac{2st-(s+t)}{st-1}.$$

Notice that

$$t \oplus t = \frac{2t^2-2t}{t^2-1} = \frac{2t(t-1)}{t^2-1} = \frac{2t}{t+1}.$$

**Remark 5.17.** We will use the notation $k \otimes t$ to denote the result of $t \oplus t \ldots \oplus t$, where the semigroup operation $\oplus$ is applied $k-1$ times,and $k$ is the number of $t$'s in the expression.

**Proposition 5.18.** *For all $k \geq 2$,*

$$k \otimes t = \frac{kt}{(k-1)t+1}.$$

*Proof.* We proceed by induction on $k$. We first verify the base case. Using algebra, we see that

$$2 \otimes t = t \oplus t = \frac{2t^2-2t}{t^2-1} = \frac{2t(t-1)}{t^2-1} = \frac{2t}{t+1},$$

which completes the base case. Now, we assume $k \otimes t$ agrees with the claimed expression, and we wish to show that $(k+1) \otimes t = \frac{(k+1)t}{kt+1}$.

First, we rewrite

$$(k+1) \otimes t = t \oplus (k \otimes t) = t \oplus \left(\frac{kt}{(k-1)t+1}\right) = \frac{N_k(t)}{D_k(t)},$$

where numerator and denominator are given by the rational functions

$$N_k(t) = \frac{2t(kt)}{(k-1)t+1} - \left(t + \frac{kt}{(k-1)t+1}\right) \qquad \text{and} \qquad D_k(t) = \frac{t(kt)}{(k-1)t+1} - 1.$$

We will use algebra to simplify $N_k$ and $D_k$ seperately. For $N_k$, we have

$$\begin{aligned} N_k(t) &= \frac{2kt^2 - (kt + t\,[(k-1)t+1])}{(k-1)t+1} \\ &= \frac{2kt^2 - (k-1)t^2 - (k+1)t}{(k-1)t+1} \\ &= \frac{(2k-(k-1))t^2 - (k+1)t}{(k-1)t+1} \\ &= \frac{(k+1)t^2 - (k+1)t}{(k-1)t+1} \\ &= \frac{(k+1)t(t-1)}{(k-1)t+1} \end{aligned}$$

Similarly,

$$\begin{aligned} D_k(t) &= \frac{t(kt)}{(k-1)t+1} - 1 = \frac{kt^2 - (k-1)t - 1}{(k-1)t+1} \\ &= \frac{kt^2 + t - kt - 1}{(k-1)t+1} \\ &= \frac{t(kt+1) - 1(kt+1)}{(k-1)t+1} \\ &= \frac{(t-1)(kt+1)}{(k-1)t+1}. \end{aligned}$$

After combining these two, we see that

$$\frac{N_k(t)}{D_k(t)} = \frac{(k+1)t(t-1)}{(k-1)t+1} \cdot \frac{(k-1)t+1}{(t-1)(kt+1)} = \frac{(k+1)t}{(kt+1)},$$

as claimed. □

**Proposition 5.19.** *Fix a non-integer rational $r > 1$. If $\varphi : \mathbb{Q}_{\geq r} \to \mathbb{R}$ satisfies*

*i)* $\varphi(xy) = \varphi(x)\varphi(y)$,
*ii)* $x < y \Rightarrow \varphi(x) < \varphi(y)$,
*iii)* $\varphi(r) = s \in \mathbb{Q}_{>1}$,

*then* $\varphi(x) = x^{\log_r s}$.

*Proof.* For any $x \in \mathbb{R}_{\geq r} \setminus \mathbb{Q}_{\geq r}$, there exists a monotone increasing sequence of rational numbers $\{y_k\} \to z$. We can uniquely extend the domain of $\varphi$ to $\mathbb{R}_{\geq r}$ by defining the sequence of functions

$$\varphi_k(x) = \begin{cases} \varphi(x) & x \in \mathbb{Q}_{\geq r} \\ \varphi(y_k) & \{y_k\} \to x \in \mathbb{R}_{\geq r} \setminus \mathbb{Q}_{\geq r} \end{cases}.$$

Define $\tilde{\varphi}(x) = \lim_{k\to\infty} \varphi_k(x)$. By ii), the sequences $\{\varphi(y_k)\}$ are monotone increasing and bounded by $\varphi(s)$ for any rational $s > x$, which ensures that $\tilde{\varphi}(x)$ and that $\{\varphi_k(x)\}$ is uniformly continuous and that $\{\varphi_k(x)\}$ converges uniformly to $\tilde{\varphi}(x)$ for irrational $x$. Continuity then follows immediately. Further, for all $x_1, x_2 \in \mathbb{R}_{\geq r}$, let $\{y_k\} \to x_1$ and $\{z_k\} \to x_2$. We can write

$$\tilde{\varphi}(x_1 x_2) = \lim_{k\to\infty} \varphi(y_k z_k) = \lim_{k\to\infty} \varphi(y_k)\varphi(z_k) = \tilde{\varphi}(x_1)\tilde{\varphi}(x_2),$$

so $\tilde{\varphi}$ also satisfies i) and ii) and iii). We will now write $\varphi$ to mean the unique continuous extension of our original map that satisfies these three properties.

For $x \geq 1$, define $\psi(x) = \log(\varphi(r^x))$. Since $\varphi$ is continuous, so too is $\psi$. Then for $x, y \geq 1$,

$$\psi(x+y) = \log\left(\varphi(r^{x+y})\right) = \log(\varphi(r^x)\varphi(r^y)) = \log(\varphi(r^x)) + \log(\varphi(r^y)) = \psi(x) + \psi(y).$$

Which means that $\psi$ satisfies Cauchy's functional equation [4]. As such, we have that $\psi(x) = cx$ for some real value $x$. Using algebra, it follows that $e^{cy} = \varphi(r^y)$. When $y = 1$, we have $e^c = \varphi(r) = s$, which implies that $c = \log s$, and hence $\varphi(r^y) = e^{x \log s}$. Given $x$ in the domain, we can write $x = r^{\log_r x}$, where the exponent is larger than 1, which means that

$$\varphi(x) = \varphi(r^{\log_r x}) = e^{\frac{\log x \log s}{\log r}} = x^{\log_r s},$$

as desired. □

## 6. Acknowledgements

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