REU Summer 2022 Final Report

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Abstract

There has been recent interest in the relationships between semigroups and a geometric object called the Kunz cone. In this paper we will explore a specific type of numerical semigroup called a symmetric numerical semigroup. We uncover and identify the characteristics of the faces of the Kunz cone where symmetric numerical semigroups lie, named Ripley faces. We look to characterize Ripley faces and discover the relationships between them.

Contents

1 Introduction

A *numerical semigroup* is a set $S \subset \mathbb{Z}_{\geq 0}$ that is cofinite and closed under addition. Every numerical semigroup S has a finite list of generators, $g_0 < g_1 < g_2 < \ldots < g_k$, and can be expressed in the following set notation:

$$S = \langle g_0, \ldots, g_k \rangle = \{a_0 g_0 + \cdots + a_k g_k \mid a_i \in \mathbb{Z}_{\geq 0}\}.$$

In [1], the authors studied semigroups that lied on the faces of different polyhedra. In this paper, we focus on a specific type of numerical semigroup called a symmetric numerical semigroup and determine which faces of the polyhedra contain these semigroups. We do this by studying the Apéry sets of different symmetric numerical semigroups and using partially ordered sets to represent their Apéry sets.

This paper begins with the necessary background information needed to understand semigroups and their geometric representations. We will then introduce symmetric numerical semigroups. To gather a better understanding of this specific type of semigroup, we will look at their geometric properties by studying their Kunz posets. After introducing symmetric numerical semigroups, we will shift our focus towards identifying relationships between different semigroups. We will determine when two different symmetric numerical semigroups are combinatorially isomorphic and create functions connecting them up to bijectivity. Finally, the last section of the paper explores the characteristics of the faces and facets that follow from these symmetric numerical semigroups.

2 Background

Suppose we have the semigroup $S = \langle g_0, \ldots, g_k \rangle$. The smallest generator, g_0, is called the *multiplicity*, denoted m. The *Apéry set* of S with respect to the multiplicity is a subset of S with the form

$$\mathrm{Ap}(S; m) = \{n \in S : n - m \notin S\}.$$

For example, take $S = \langle 4, 6, 7 \rangle$. The resulting Apéry set of S with respect to the multiplicity can be written as $\mathrm{Ap}(S; 4) = \{0, 13, 6, 7\}$. Notice that each element of the Apéry set is the smallest number in the semigroup that represents a mod class with respect to the multiplicity (i.e. $0 = 0 \mod 4$, $13 = 1 \mod 4$, $6 = 2 \mod 4$, etc.). Because S has a finite complement, there exists an element in S in each equivalence class modulo m, and by construction of $\mathrm{Ap}(S; m)$ and the minimality of each $n \in \mathrm{Ap}(S; m)$, there are exactly m elements in $\mathrm{Ap}(S; m)$.

One way in which we can graphically represent Apéry sets is by using partially ordered sets, also called posets. For example, suppose we have the semigroup $S = \langle 4, 6, 7 \rangle$ and $\mathrm{Ap}(S; 4) = \{0, 13, 6, 7\}$. This Apéry set can be represented by the poset in Figure 1a.

These Apéry posets allow us to establish a hierarchy between the elements of the $\mathrm{Ap}(S; m)$. Given that each equivalence class modulo m is represented once in the $\mathrm{Ap}(S; m)$, we can replace elements of the Apéry poset with their corresponding value modulo m, resulting in a particular type of poset: a *Kunz poset*. Looking back at our Apéry set in Figure 1a, the Kunz poset associated with that Apéry set is shown in Figure 1b.

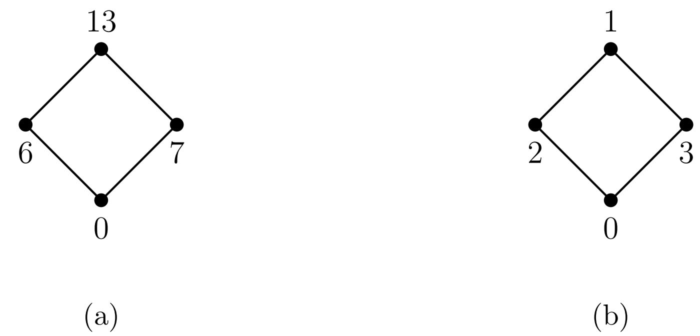


Figure 1: The poset (left) and the Kunz poset (right) associated with Apéry set $\{0, 13, 6, 7\}$

Kunz posets make it easier to work with the Apéry sets in comparison to Apéry posets when dealing with Apéry sets that have large orders.

Throughout this paper, one important geometric object that appears is a *polyhedron.* A polyhedron is the intersection of finitely many inequalities. When some subset of those original inequalities become equalities, we obtain a *face* of the polyhedron. A face may be any dimension from 0 to the dimension of the polyhedron itself (in fact, the entire polyhedron is itself a face). We use the term *codimension* to mean "dimensions less than that of the polyhedron"; If a polyhedron is n-dimensional, a face with codimension 3 is $(n-3)$-dimensional. We use the term *facet* to describe a face with codimension 1.

In this paper, we focus on a particular polyhedron called the *Kunz cone*, denoted C_m where m is the multiplicity (in [2] this is denoted $C(\mathbb{Z}_m)$). The Kunz cone is defined by all of the points $(x_1, \ldots, x_{m-1}) \in \mathbb{R}^m$ that satisfy

$$x_i + x_j \geq x_{i+j} \quad \text{for all} \quad 1 \leq i \leq j \leq m-1.$$

In [1], the authors found a natural correspondence between faces of the Kunz Cone C_m and tuples $(H, \preceq)$ where H is a subgroup and $\preceq$ is a partial ordering on $\mathbb{Z}_m/H$.

Theorem 2.0.1. *There is an injective function*

$$F \mapsto (H, \preceq)$$

sending each face F of $\mathcal{C}(G)$ to a pair $(H, \preceq)$, where

$$H = \{0\} \cup \{h \in G : x_h = 0 \textit{ for all } x \in F\} \subset G$$

is a subgroup of G and $\preceq$ is the Kunz-balanced poset on G/H with minimal element $\overline{0}$ and the property that $x_a + x_b = x_{a+b}$ is a facet equation for F if and only if $\overline{a} \preceq \overline{a} + \overline{b}$.

Given an Apéry set of a semigroup, we can construct a point in $\mathbb{R}^{m-1}$. By construction, the point will satisfy certain inequalities, and some may be satisfied with strict equality. The inequalities that are satisfied with strict equality will indicate what face the semigroup lies on. Even though not every face has a numerical semigroup, every numerical semigroup corresponds to a point in the polyhedron.

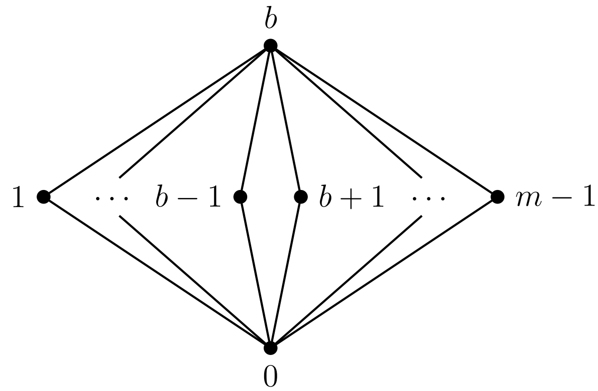
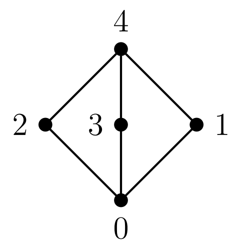


Figure 2: Kunz poset representing $Q_{m,b}$

For a poset to have a maximal element, there has to exist an element that does not precede any of the other elements. If the Kunz poset of a numerical semigroup has a single maximal element, then we classify that numerical semigroup as a *symmetric numerical semigroup*. The faces of the Kunz cone can sometimes be made up of symmetric numerical semigroups. If the Kunz cone is made up of a symmetric numerical semigroup, then we denote these faces as $Q_{m,b}$ where m is the multiplicity and b is the maximal element of the Kunz poset. So for example, suppose we have $Q_{5,4}$. This can be represented by the following poset.



3 Ripley faces

Just as all numerical semigroups are represented as points that live in the Kunz cone, symmetric numerical semigroups also exist within the Kunz cone. Specifically, symmetric numerical semigroups live on faces of the Kunz cone that are represented by posets with a unique maximal element. We call these faces with symmetric numerical semigroups Ripley faces and denote them as $Q_{m,b}$, where m is the multiplicity and b is the maximal element. Recall that faces of the Kunz cone are defined by equalities of the form $x_i + x_j = x_k$ where $i + j \equiv k \mod m$. All of the equalities that define $Q_{m,b}$ are of the form $x_i + x_j = x_b$ as b is the maximal element of the poset representing $Q_{m,b}$. When $b \neq 0$, the poset representing $Q_{m,b}$ is shown in Figure 2. Otherwise, $Q_{m,0}$ is the 0-dimensional face of the Kunz cone i.e. the vertex of the Kunz cone.

Theorem 3.0.1. *Given m, b, the dimension of $Q_{m,b}$ is given by*

$$\dim Q_{m,b} = \begin{cases} \lfloor \frac{m}{2} \rfloor - 1 & m \text{ is even, } b \text{ is even} \\ \lfloor \frac{m}{2} \rfloor & \text{else.} \end{cases}$$

Proof. To prove dimension, we will first count the number of equations that define $Q_{m,b}$, then prove that those equations are linearly independent. Every linearly independent equation that defines $Q_{m,b}$ brings the dimension down by one, allowing us to then determine the dimension of $Q_{m,b}$. When counting the number of equations in $Q_{m,b}$, we must consider cases based on the parity of m and b.

1. Let m be odd. The equations that define $Q_{m,b}$ either sum two distinct coordinates to $b \bmod m$ or sum the same coordinate twice to $b \bmod m$. We will show there exists exactly one equation that sums the same coordinate twice to equal $b \bmod m$.

   Since m is odd, there exists at least one number, i, such that

   $$2i \equiv b \bmod m.$$

   If $2i = b$, then b is an even number, and if $2i = b + m$, b is an odd number. Since $i \in \mathbb{Z}_m$, b and $b + m$ are the only possible sums and since these produce unique values for b, there is only one coordinate, x_i, such that $2x_i = x_b$.

   By definition of $Q_{m,b}$, every atom sums with another atom to b. Thus, for every atom $j \in \mathbb{Z}_m \backslash \{0, i, b\}$ there exists $k \in \mathbb{Z}_m \backslash \{0, i, b\}$ such that $x_j + x_k = x_b$. Note that $j \neq k$ since we have omitted i which we showed above is the only coordinate index such that $2x_i = x_b$. We have also omitted 0 and b because they are not atoms. Since there are $m - 3$ possible values for j and the equations sum pairs of j and k, there are $\frac{m-3}{2}$ face equality equations of $Q_{m,b}$ in addition to $2x_i = x_b$. Therefore there are

   $$\frac{m-3}{2} + 1 = \frac{m+1}{2} - 1 = \left\lceil \frac{m}{2} \right\rceil - 1$$

   face equality equations of $Q_{m,b}$.

2. Let m be even. When m is even, we must consider the parity of b as well.

   (a) First let us consider when when b is odd. If there were an i such that $2i \equiv b \bmod m$, then $m \mid 2i - b$. This cannot hold because m is even and $2i - b$ is odd. Therefore, all $m - 2$ atoms must sum with distinct atoms to b. This results in

   $$\frac{m-2}{2} = \left\lceil \frac{m}{2} \right\rceil - 1$$

   equations.

   (b) When m and b are both even, there exist two index values, $\frac{b}{2}$ and $\frac{b}{2} + \frac{m}{2}$, that when doubled are congruent to b. If there were a third such index, it would be expressed as $\frac{b}{2} + \frac{m}{2} + \frac{m}{2} = \frac{b}{2} + m$. However this is congruent to $\frac{b}{2} \bmod m$, so we have already considered it. Thus, there are only two such indices.

Since we cannot use 0 or b, this leaves $m-4$ atoms with which to construct equations that sum two distinct elements, thus there are $\frac{m-4}{2}$ such equations. This brings the total number of equations in this case to

$$\frac{m-4}{2}+2=\frac{m}{2}=\left\lceil\frac{m}{2}\right\rceil.$$

Now, we will show that the face equations of $Q_{m,b}$ are linearly independent. First, we will verify that no x_i, for some $i \in \mathbb{Z}_m \backslash \{0, b\}$, appears in more than one equation. Once we choose x_i, x_j in the equation $x_i + x_j = x_b$ is uniquely determined. Specifically, if $i > b$ then $j = m + b - i$ otherwise $i < b$ then $j = b - i$. Since every x_i appears in only one equation, no equation can be a linear combination of other face equality equations of $Q_{m,b}$. Thus, the face equations of $Q_{m,b}$ are linearly independent.

This allows us to conclude that the dimension of $Q_{m,b}$ is $\dim C_m$ minus the number of face equations that define $Q_{m,b}$. Recall that the dimension of C_m is $m-1$. So, when m and b are both even, the dimension of $Q_{m,b}$ is $(m-1)-\lceil\frac{m}{2}\rceil=\lfloor\frac{m}{2}\rfloor-1$. Otherwise, the dimension of $Q_{m,b}$ is $(m-1)-(\lceil\frac{m}{2}\rceil-1)=\lfloor\frac{m}{2}\rfloor$. □

Example 3.0.2. Consider the Kunz Cone C_{12} and the symmetric face $Q_{12,11}$. We define $Q_{12,11}$ by the following face equalities

$$\begin{aligned} x_1 + x_{10} &= x_{11} \\ x_2 + x_9 &= x_{11} \\ x_3 + x_8 &= x_{11} \\ x_4 + x_7 &= x_{11} \\ x_5 + x_6 &= x_{11}. \end{aligned}$$

We can see that these five equations are linearly independent, thus the dimension is five less than the dimension of C_{12}, so the dimension of $Q_{12,11}$ is $\dim(C_{12}) - 5 = 11 - 5 = 6 = \lfloor\frac{m}{2}\rfloor$.

Now let us consider the face $Q_{12,10}$ within the Kunz cone C_{12}. Its face equalities are

$$\begin{aligned} x_1 + x_9 &= x_{10} \\ x_2 + x_8 &= x_{10} \\ x_3 + x_7 &= x_{10} \\ x_4 + x_6 &= x_{10} \\ 2x_5 &= x_{10} \\ 2x_{11} &= x_{10}. \end{aligned}$$

Here we can see the instance of two equations doubling an element when m and b are both even. We can also see that all six equations are linearly independent and conclude that the dimension is $\dim(C_{12}) - 6 = 11 - 6 = 5 = \lfloor\frac{m}{2}\rfloor - 1$.

Theorem 3.0.3. *The equalities stated in the definition of $Q_{m,b}$ comprise its H-description as a face of C_m.*

Proof. To prove this, we will locate a point x on the interior of $Q_{m,b}$ - namely it satisfies the additional equalities of $Q_{m,b}$, and no others. We claim that $x \in \mathbb{R}^{m-1}$ with $x_i = 1$ for $i \neq b$ and $x_b = 2$ is such a point. Notice that if $i + j \equiv b \bmod m$ and $i, j \in [1, m-1]$, then $x_i + x_j = 1 + 1 = 2 = x_b$. If $i + j \not\equiv b \bmod m$, then $x_i + x_j = 1 + 1 > x_{i+j} = 1$, so we have equality only for those specified by $Q_{m,b}$. □

Next we will investigate the existence of numerical semigroups on $Q_{m,b}$.

Theorem 3.0.4. *There is a numerical semigroup S with $S \in Q_{m,b}$ iff one of m, b is odd.*

Proof. Here we will construct example numerical semigroups that live on the interior of $Q_{m,b}$. Suposing m, b have different parities, we have the following construction: let $x_b = 3m + b$. For $\frac{b}{2} \leq i \leq \frac{m}{2} + \frac{b}{2}$, set $x_i = m + i$. Otherwise, let $x_j = x_b - x_i$ whenever $j + i \equiv b \bmod m$. To see that the set $S = \{m\} \cup \{x_i : i \in [1, m-1] \setminus \{b\}\}$ generates the numerical semigroup with the desired property, notice that

$$\begin{aligned}
i + j \equiv b \bmod m &\implies x_i + x_j = x_i + (3m + b - x_i) = 3m + b = x_b \\
i + j \not\equiv b \bmod m \wedge i + j < \frac{b}{2} &\implies x_i + x_j \geq 2m + i + j > 2m + (i + j \bmod m) \geq x_{i+j} \\
i + j \not\equiv b \bmod m \wedge i + j > \frac{b}{2} &\implies x_i + x_j \geq 2m + i + j > 2m \geq x_{i+j}.
\end{aligned}$$

We see that there is equality if and only if the equation corresponds to $Q_{m,b}$. Hence, S generates the desired numerical semigroup.

Suppose m and b are both odd. Notice that in this case that the solution to $2t \equiv b \bmod m$ has $2x = m + b$. Let $x_b = 3m + b$, and set $x_i = m + b$ for $\frac{b}{2} \leq i \leq \frac{m+b}{2}$, and set $x_j = x_b - x_i$ for j yet unassigned, where $i + j \equiv b \bmod m$. By choice, we have

$$\begin{aligned}
i + j \equiv b \bmod m &\implies x_i + x_j = x_i + (3m + b) - x_i = 3m + b = x_b \\
i + j \not\equiv b \bmod m \wedge i + j < \frac{b}{2} &\implies x_i + x_j \geq 2m + i + j
\end{aligned}$$

Suppose m and b are both even. Both

$$2x_{\frac{b}{2}} = x_b \quad \text{and} \quad 2x_{\frac{b}{2}+\frac{m}{2}} = x_b$$

are equations specified by $Q_{m,b}$, which forces $x_{\frac{b}{2}} = x_{\frac{b}{2}+\frac{m}{2}}$, though they must be different modulo m. □

Remark 3.0.5. The above proof constructs a numerical semigroup which lives on the interior of $Q_{m,b}$. Lemma 1 in [3] constructs numerical semigroups living on $Q_{m,b}$ for b odd, but generally not on the interior of $Q_{m,b}$.

Example 3.0.6. Looking at the symmetric face $Q_{12,11}$, we can construct a semi group from the previous proof with the case when m, b have different parities. We obtain the semigroup

$$S = \langle 12, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34 \rangle$$

which has the correct poset structure. See Figure 3.

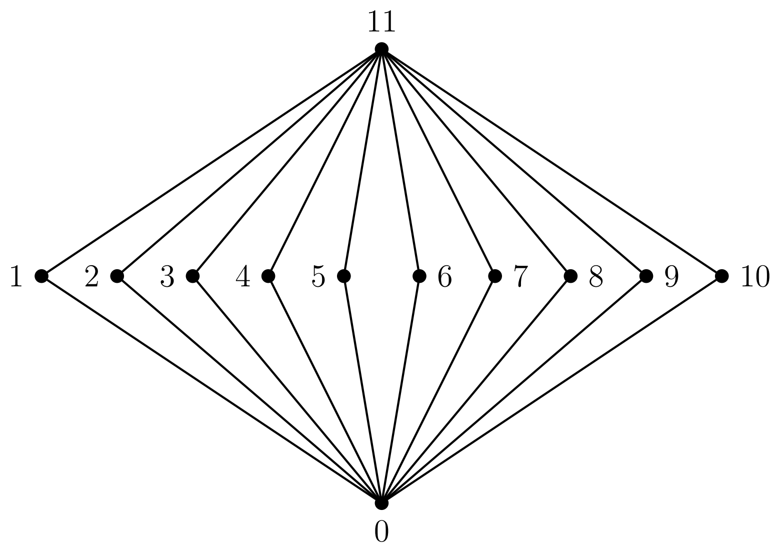


Figure 3: Kunz poset corresponding to $Q_{12,11}$

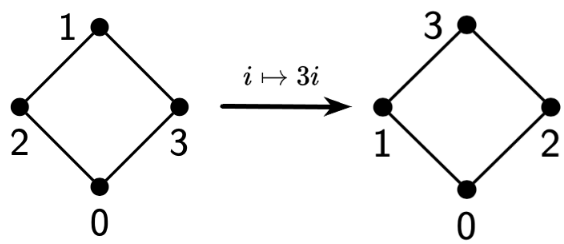


Figure 4: The poset of $Q_{4,1}$ maps to the poset of $Q_{4,3}$

4 Relationships

It is clear to see that the action of $(\mathbb{Z}_m)^* \cong \mathrm{Aut}(\mathbb{Z}_m)$ takes the group cone onto itself. We know hence that any of these actions take faces to faces, and analagously the action maps Kunz Posets to Kunz Posets with the same structure but permuted elements. For example, the action by $3 \in (\mathbb{Z}_m)^*$ maps the poset of $Q_{4,1}$ to the poset of $Q_{4,3}$. See Figure 4.

Theorem 4.0.1. *If* $\gcd(m, a) = \gcd(m, b)$*, then* $Q_{m,a}$ *and* $Q_{m,b}$ *are combinatorially isomorphic.*

Proof. Identify $Q_{m,a}, Q_{m,b}$ with the tuples $(H_a, \preceq_a), (H_b, \preceq_b)$ respectively, as guaranteed by Theorem 3.4. It is simple to see that both H_a, H_b are trivial (as long as neither $a, b = 0$). Let $x \in (\mathbb{Z}_m)^*$ be such that $ax = b$, as guaranteed by the hypothesis. Suppose that $\bar{i} \preceq_a \bar{i} + \bar{j} = \bar{a}$, we wish to show that $\overline{xi} \preceq_b \overline{xi} + \overline{xj} = \overline{xa} = \bar{b}$, which follows since $x \in (\mathbb{Z}_m)^*$. Thus, $Q_{m,a}$ and $Q_{m,b}$ are combinatorially isomorphic. $\square$

Notation 4.0.2. For simplification, we will adopt the convention of prepending a "0" entry to each point in $\mathcal{C}(G)$, indexed by the identity element of G. More precisely, we write each

$(x_1, x_2, \dots) \in \mathcal{C}(G)$ in the form $(x_0, x_1, x_2, \dots)$ with $x_0 = 0$, effectively replacing $\mathcal{C}(G)$ with $\{0\} \times \mathcal{C}(G)$.

Next, we will discuss injections from C_d to C_m. Suppose $d \mid m$, and define $\phi_{d,m} : C_d \to C_m$ by $w_i = x_{\bar{i}}$ where $x \in C_d$ maps to $w \in C_m$. It was seen in Corollary 3.7 [2] that the map $\phi_{d,m}$ gives an injection on the face lattices $C_d \to C_m$. We explore the action of this map on $Q_{d,b}$.

Theorem 4.0.3. *Given that $d \mid m$, the injection $\phi_{d,m} : C_d \to C_m$ when $d \mid m$ has $\phi(Q_{d,\bar{b}}) = \phi(C_d) \cap Q_{m,b}$.*

Proof. Notice that both $\phi(Q_{d,\bar{b}}, \phi(C_d) \cap Q_{m,b}$ are faces of C_m, so we identify them with $(H_1, \preceq_1), (H_2, \preceq_2)$ respectively. It is clear that $\phi(Q_{d,\bar{b}}) \subseteq \phi(C_d)$, so we first show that $\phi(Q_{d,\bar{b}}) \subseteq Q_{m,b}$.

Suppose that $i + j = b$ for $i, j \in \mathbb{Z}_m$. Take $w = \phi(x) \in \phi(Q_{d,\bar{b}})$. By construction,

$$w_i + w_j = w_{\bar{i}} + w_{\bar{j}} = x_{\bar{i}} + x_{\bar{j}} = x_{\bar{b}}$$

as $\bar{i} + \bar{j} = \bar{b}$ since $d \mid m$. This shows that $\preceq_1$ is a refinement of $\preceq_2$.

Next, we will see that $H_1 = H_2$. It is clear that $H_1 = \langle d \rangle$, and that $\langle d \rangle \subseteq H_2$ as $\phi_{d,m}(C_m) \cap Q_{m,b} \subseteq \phi_{d,m}(C_m)$. As we saw $\phi_{d,m}(Q_{d,\bar{b}}) \subseteq \phi(C_d) \cap Q_{m,b}$, we have $\langle d \rangle \supseteq H_2$ and hence $H_1 = H_2$.

Finally, we must show that $\preceq_2$ is a refinement of $\preceq_1$. Suppose that $\bar{i} \preceq_1 \bar{i} + \bar{j} = \bar{b}$ is a relation of $\preceq_1$. Take $x \in \phi(C_d) \cap Q_{m,b}$. Since $x \in Q_{m,b}$ we have $x_{\bar{i}} + x_{b-\bar{i}} = x_b$. Then, because $d \mid m$ and $H_2 = \langle d \rangle$, it follows that $\overline{b - \bar{i}} = \bar{j}$ and $\bar{i} \preceq_2 \bar{i} + \bar{j} = \bar{b}$. $\square$

Example 4.0.4. The rays of $Q_{8,3}$ with a zero in the middle index are $(1, 2, 3, 0, 1, 2, 3)$ and $(1, 0, 1, 0, 1, 0, 1)$. The former appears from the injection $\phi_{4,8}(Q_{4,3})$, where $(1, 2, 3)$ is a ray of $Q_{4,3}$, and the latter appears from the injection $\phi_{2,8}(Q_{2,1})$ where (1) is a ray of $Q_{2,1}$.

Example 4.0.5. The rays of C_{10} with a zero coordinate are

$(1, 2, 3, 4, 0, 1, 2, 3, 4)$	$(2, 4, 1, 3, 0, 2, 4, 1, 3)$	$(4, 3, 2, 1, 0, 4, 3, 2, 1)$
$(4, 3, 2, 6, 0, 4, 3, 2, 6)$	$(6, 2, 3, 4, 0, 6, 2, 3, 4)$	$(3, 1, 4, 2, 0, 3, 1, 4, 2)$
$(2, 4, 6, 3, 0, 2, 4, 6, 3)$	$(3, 6, 4, 2, 0, 3, 6, 4, 2)$	$(1, 0, 1, 0, 1, 0, 1, 0, 1)$

which come from injections from C_2 and C_5. The rays of C_5 are:

$(1, 2, 3, 4)$	$(2, 4, 1, 3)$
$(4, 3, 2, 1)$	$(3, 1, 4, 2)$
$(4, 3, 2, 6)$	$(6, 2, 3, 4)$
$(2, 4, 6, 3)$	$(3, 6, 4, 2)$

and the ray on C_2 is (1). Notice some of these rays are permuted versions of others - this comes from Theorem 4.0.1.

Corollary 4.0.6. *Given* $Q_{m,b}$ *and* $Q_{m,b'}$*, let* $d = \gcd(b - b', m)$*. Then* $Q_{m,b} \cap Q_{m,b'} = \phi_{d,m}(Q_{d,\bar{b}})$ *where* $\phi_{d,m} : C_d \mapsto C_m$*.*

Proof. By application of the previous theorem, we have the following.

$$\phi_{d,m}(Q_{d,\bar{b}}) = Q_{m,b} \cap Q_{m,b'} \cap \mathrm{Im}\ \phi_{d,m}$$

Note that $\overline{b} = \overline{b'}$, so b and b' can be assigned arbitrarily. It suffices to prove that $Q_{m,b} \cap Q_{m,b'} \subseteq \mathrm{Im}\ \phi$. We will show that the subgroup H of $Q_{m,b} \cap Q_{m,b'}$ is $\langle d \rangle$.

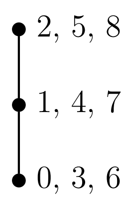
We know that any point $x \in Q_{m,b} \cap Q_{m,b'}$ must satisfy both $x_b + x_{b-b'} = x_{b'}$ and $x_{b'} + x_{b-b'} = x_b$. Substituting x_b we get $x_{b-b'} + x_{b'-b} = 0$. Since coordinates are non-negative integers, we have that $x_{b-b'} = x_{b'-b} = 0$. The subgroup H is comprised of the coordinates whose values are zero, consequently, $b - b' \in H$ and $\langle b - b' \rangle \subseteq H$. Now we will show $\langle b - b' \rangle = \langle \gcd(b - b', m) \rangle$.

Let $s = b - b'$. Since $\gcd(s, m) \mid s$, we know that $\langle s \rangle \subseteq \langle \gcd(s, m) \rangle$. To show the other direction of containment, note that, by Bezout's Theorem $\gcd(s, m) = n_1 s + n_2 m$ where n_1, n_2 are integers. Thus, $\gcd(s, m) \equiv n_1 s \bmod m$ and so $\langle s \rangle \supseteq \langle \gcd(s, m) \rangle$. Since we have shown both directions of containment, $\langle s \rangle = \langle \gcd(s, m) \rangle$. By Theorem 2.0.1 since our subgroup is $\langle \gcd(s, m) \rangle$, the intersection of $Q_{m,b}$ and $Q_{m,b'}$ is equal to $\phi_{d,m}(Q_{d,\bar{b}})$. $\square$

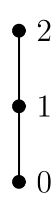
Example 4.0.7. Let us consider the intersection of the faces $Q_{12,1}$ and $Q_{12,9}$ in the Kunz cone C_{12}. Take the face equalities $x_3 + x_8 = x_2$ from $Q_{9,2}$ and $x_2 + x_6 = x_8$ from $Q_{9,8}$. Substituting in x_2, we get

$$x_3 + x_8 + x_6 = x_8 \text{ and } x_3 + x_6 = 0.$$

Since all coordinates in the Kunz cone are non-negative integers, $x_3 = x_6 = 0$. Thus, the subgroup of coordinates whose values are zero is $\{3, 6\}$. By Theorem 2.0.1, the poset resulting from the subgroup $\{3, 6\}$ is the following:



The poset above is injective to the poset $Q_{3,2}$ below.



5 Facets of Ripley faces

Definition 5.0.1. Define $F_{i,j,k}$ to be the intersection of all equations of $Q_{m,b}$ with the additional equations

$$\begin{aligned} x_i + x_j &= x_{i+j} \\ x_j + x_k &= x_{j+k} \\ x_i + x_k &= x_{i+k} \end{aligned} \tag{1}$$

for every $i, j, k \in \{\mathbb{Z}_m\} \setminus \{0, b\}$ with $i + j + k \equiv b \bmod m$.

Theorem 5.0.2. *Each $F_{i,j,k}$ is a face of $Q_{m,b}$, and any given $F_{i,j,k}$ is a facet if and only if the following two conditions hold:*

1. *If, without loss of generality, there exists i such that $2i \equiv b \bmod m$, then $j = k$.*
2. *If $3 \mid m$ and $3 \mid b$, then without loss of generality, no more than one of the following can be true: $i = \frac{b}{3}$, $j = \frac{m+b}{3}$, and $k = \frac{2m+b}{3}$.*

Proof. We will first prove that $F_{i,j,k}$ is indeed a face of $Q_{m,b}$ by identifying a point in the face that satisfies all face equations with equality and all other Kunz inequalities of C_m with strict inequality. We have two cases.

First suppose that $2i \not\equiv b \bmod m$, $2j \not\equiv b \bmod m$, and $2i \not\equiv b \bmod m$. Consider the point $x \in \mathbb{R}^{m-1}$ with

$$\begin{aligned} x_i = x_j = x_k &= 2 \\ x_{i+j} = x_{i+k} = x_{j+k} &= 4 \\ x_b &= 6, \end{aligned}$$

and all other coordinates equal to 3. For each of the equations in (1), we have equality with $2 + 2 = 4$. Since $2i \not\equiv b \bmod m$, the equation $2x_i = x_{2i}$ holds only when $x_{2i} = x_{i+j}$ or $x_{2i} = x_{j+k}$. If $x_{2i} = x_{i+j}$, then $i = j$ and thus this is the equation $x_i + x_j = x_{i+j}$, which has already been addressed. If $x_{2i} = x_{j+k}$, this implies $2i \equiv j + k$, which is not a valid choice by the second condition. By symmetry, the same holds for $2j$ and $2k$. Otherwise if x_{2i} is any other coordinate, we have strict inequality with $2(2) > 3$. Keeping in mind that $i+j+k = b$, we also have the following equations involving x_i, x_j, and x_k:

$$\begin{aligned} x_i + x_{j+k} &= x_b \\ x_j + x_{i+k} &= x_b \\ x_k + x_{i+j} &= x_b. \end{aligned} \tag{2}$$

It can easily be seen that each of these are indeed satisfied with equality since $2 + 4 = 6$. Any other equations of $F_{i,j,k}$ involve two coordinates other than $x_i, x_j, x_k, x_{i+j}, x_{i+k}$, and x_{j+k} summing to x_b. That is, we have $3+3 = 6$ and so we again have equality. Finally, note that the list of inequalities for C_m not describing $Q_{m,b}$ all have two coordinates summing to some coordinate other than x_b. However, the possible integer sums of 2, 3, 4, and 6 are 4, 5, 6, 8, 9, 10, and 12. Since the sum cannot be x_b, and no coordinates are 5, the only actually possible chance of equality is with a sum of 4. But that forces us to consider exactly the equations in (1). Therefore all other inequalities of C_m must be satisfied with strict inequality. So $x \in F_{i,j,k}$.

Next suppose, again without loss of generality, that $2i \equiv b \bmod m$. By the first condition, we know $j = k$. As a consequence, we find $2j = 2k = i$. For the duration of this paragraph, we will use j to represent both j and k. Consider the point $x \in \mathbb{R}^{m-1}$ with $x_i = 6$, $x_j = 3$, $x_{i+j} = 9$, $x_b = 12$, and all other coordinates are equal to 6. The first and third equations in (1) are equivalent and so the two distinct equations become $6+3 = 9$ and $3+3 = 6$ and are clearly satisfied with equality. The last two equations in (2) are equivalent so we have the two equations $6+6 = 12$ and $3+9 = 12$, both again clearly satisfied with equality. The inequality $2x_i \geq x_{2i} = x_b$ becomes $2(6) = 12$, and the inequality $2x_j \geq x_{2j} = x_i$ becomes $2(3) = 6$, and thus both hold with equality. Any other equations of $F_{i,j,k}$ involve two coordinates other than x_i, x_j, and x_{i+j}, summing to x_b. That is, we have $6 + 6 = 12$ and so we again have equality. Finally, we consider the list of inequalities for C_m not describing $Q_{m,b}$. These all have two coordinates summing to some coordinate other than x_b. The possible integer sums of 3, 6, 9, and 12 are 6, 9, 12, 15, 18, 21, and 24. Since the sum cannot be x_b and x_b is the largest coordinate, the available sums for equality are 6 and 9. A sum of 6 arises from the equation $2x_j \geq x_i$, which has already been considered. A sum of 9 arises from $x_i + x_j \geq x_{i+j}$, which has also already been considered. Thus any other equations must be satisfied with strict inequality. So $x \in F_{i,j,k}$ and we can confirm that $F_{i,j,k}$ is indeed a face of $Q_{m,b}$.

Now we suppose $F_{i,j,k}$ is a facet and we will show that the given conditions must apply. We will proceed by contradiction. First, let us assume that there exists i such that $2i \equiv b \bmod m$ and $j \neq k$. Then we know

$$\begin{aligned} x_i + x_j + x_k &= x_b \\ 2x_i = x_{2i} &= x_b \\ x_j + x_k = x_{j+k} &= x_i. \end{aligned}$$

Further, once we set those equalities, four more equalities come out when making the poset for this face.

$$\begin{aligned} 2x_j &= x_{2j} \\ 2x_k &= x_{2k} \\ x_j + x_{2k} = x_{2k+j} &= x_{i+k} \\ x + k + x_{2j} = x_{2j+k} &= x_{i+j}. \end{aligned}$$

Since the existing equations defining $Q_{m,b}$ are already known to be linearly independent (see proof of Theorem 3.0.1), we can construct a submatrix of the H-description with only the equations involving x_i, x_j, x_k, x_{2j}, x_{2k}, x_{i+j}, x_{i+k}, and x_b. Our list of relevant equations then, is:

$$\begin{array}{ccc} 2x_i = x_b & x_i + x_j = x_{i+j} & x_j + x_{2k} = x_{i+k} \\ x_j + x_{i+k} = x_b & x_i + x_k = x_{i+k} & x_k + x_{2j} = x_{i+j} \\ x_k + x_{i+j} = x_b & x_j + x_k = x_i & 2x_j = x_{2j} \\ x_{2j} + x_{2k} = x_b & & 2x_k = x_{2k}. \end{array}$$

Note that the first four come from $Q_{m,b}$, the middle three from the definition of $F_{i,j,k}$, and the remaining four from the poset structure. Forming a matrix with these equations, we have

$$\begin{array}{c}
\begin{matrix} x_i & x_j & x_k & x_{2j} & x_{2k} & x_{i+j} & x_{i+k} & x_b \end{matrix} \\
\begin{bmatrix}
2 & & & & & & & -1 \\
 & 1 & & & & & 1 & -1 \\
 & & 1 & & & 1 & & -1 \\
 & & & 1 & 1 & & & -1 \\
1 & 1 & & & & -1 & & \\
1 & & 1 & & & & -1 & \\
-1 & 1 & 1 & & & & & \\
 & 1 & & & 1 & & -1 & \\
 & & 1 & 1 & & 1 & & \\
 & 2 & & -1 & & & & \\
 & & 2 & & -1 & & &
\end{bmatrix}
\end{array}.$$

After row reduction, we obtain

$$\begin{bmatrix}
2 & & & & & & & -1 \\
 & 1 & & & & & 1 & -1 \\
 & & 2 & & & & -2 & 1 \\
 & & & 1 & & & 2 & -2 \\
 & & & & 1 & & -2 & 1 \\
 & & & & & 2 & 2 & -3
\end{bmatrix}.$$

This matrix has rank 6, so of the eleven original equations, there exist a set of six that are linearly independent. Note that only four linearly independent equations came from $Q_{m,b}$, so the remaining two linearly independent equations came from the seven new equations. Therefore the dimension of $F_{i,j,k}$ is codimension 2 with respect to $Q_{m,b}$. Since this is a lower dimension than codimension 1, this contradicts the assumption that $F_{i,j,k}$ is a facet. So if there exists i such that $2i \equiv b \bmod m$, we must have $j = k$.

Next, let us assume that $3 \mid m$ and $3 \mid b$. We will again proceed by contradiction. First, note that for the equation $i+j+k = b$ with fixed b, choosing two of i, j, k uniquely determines the third. As a consequence, since

$$\frac{b}{3} + \frac{m+b}{3} + \frac{2m+b}{3} \equiv b \bmod m,$$

choosing two of

$$i = \frac{b}{3} \qquad j = \frac{m+b}{3} \qquad k = \frac{2m+b}{3} \tag{3}$$

to be true is equivalent to all of them being true. So we will suppose they are all true.

Notice

$$
\begin{aligned}
i+j &\equiv \frac{b}{3}+\frac{m+b}{3} \equiv \frac{m+2b}{3}+\frac{3m}{3} \equiv 2\left(\frac{2m+b}{3}\right) \equiv 2k \\
j+k &\equiv \frac{m+b}{3}+\frac{2m+b}{3} \equiv \frac{3m+2b}{3} \equiv 2\left(\frac{b}{3}\right) \equiv 2i \\
i+k &\equiv \frac{b}{3}+\frac{2m+b}{3} \equiv 2\left(\frac{m+b}{3}\right) \equiv 2j,
\end{aligned}
$$

so $i+j \equiv 2k \bmod m$, $j+k \equiv 2i \bmod m$, and $k+i \equiv 2j \bmod m$. Combining these three equations with the three equations from the definition of $F_{i,j,k}$ and the three relevant equations from $Q_{m,b}$ (listed in (2)), we have the list

$$
\begin{array}{lll}
x_i + x_{j+k} = x_b & x_i + x_j = x_{i+j} & 2x_i = x_{j+k} \\
x_j + x_{i+k} = x_b & x_i + x_k = x_{i+k} & 2x_k = x_{i+j} \\
x_k + x_{i+j} = x_b & x_j + x_k = x_{j+k} & 2x_j = x_{i+k},
\end{array}
$$

which comprises the H-description for $F_{i,j,k}$. Forming a matrix from these equations, we have

$$
\begin{array}{c}
\begin{array}{ccccccc} x_i & x_j & x_k & x_{i+j} & x_{i+k} & x_{j+k} & x_b \end{array} \\
\begin{bmatrix}
1 & & & & & 1 & -1 \\
 & 1 & & & 1 & & -1 \\
 & & 1 & 1 & & & -1 \\
1 & 1 & & -1 & & & \\
1 & & 1 & & -1 & & \\
 & 1 & 1 & & & -1 & \\
2 & & & & & -1 & \\
 & 2 & & & -1 & & \\
 & & 2 & -1 & & &
\end{bmatrix}.
\end{array}
$$

Using row reducing operations we can transform our original matrix into the following

$$
\begin{bmatrix}
3 & & & & & & -1 \\
 & 3 & & & & & -1 \\
 & & 3 & & & & -1 \\
 & & & 3 & & & -2 \\
 & & & & 3 & & -2 \\
 & & & & & 3 & -2
\end{bmatrix}.
$$

The above matrix has rank 6, so there are six linearly independent equations from the original nine. Three of these come from $Q_{m,b}$ and are known to be linearly independent, but the three additional equations reduce the dimension of $F_{i,j,k}$ to codimension 3, which is less than the required codimension 1 for a facet. This contradicts the assumption that $F_{i,j,k}$ is a facet. Thus when $3 \mid m$ and $3 \mid b$, no more than one of (3) can hold.

Now suppose that the two conditions apply, and we will show $F_{i,j,k}$ is a facet of $Q_{m,b}$. Let α be the number of linearly independent equalities describing $Q_{m,b}$. The dimension of $Q_{m,b}$ is equal to $\dim(C_m) - \alpha = (m-1) - \alpha$. We will construct a matrix for the H-description of $F_{i,j,k}$. Knowing that the equalities representing $Q_{m,b}$ are linearly independent, the only equations in which dependence could arise are those involving x_i, x_j, x_k, x_{i+j}, x_{i+k}, and x_{j+k}. So we can create a submatrix involving only these equations. Further, because of the conditions applied, we know that the only additional equations are those listed in (1). We have

$$\begin{array}{c} \begin{array}{ccccccc} x_i & x_j & x_k & x_{i+j} & x_{i+k} & x_{j+k} & x_b \end{array} \\ \begin{bmatrix} 1 & & & & & 1 & -1 \\ & 1 & & & 1 & & -1 \\ & & 1 & 1 & & & -1 \\ 1 & 1 & & -1 & & & \\ 1 & & 1 & & -1 & & \\ & 1 & 1 & & & -1 & \end{bmatrix} \end{array}.$$

Using row reducing operations, we can transform our original matrix into the following

$$\begin{bmatrix} 1 & & & & & 1 & -1 \\ & 1 & & & 1 & & -1 \\ & & 1 & 1 & & & -1 \\ & 1 & & -1 & & -1 & 1 \\ & & 1 & & -1 & -1 & 1 \\ & 1 & 1 & & & -1 & \end{bmatrix} \sim \begin{bmatrix} 1 & & & & & 1 & -1 \\ & 1 & & & 1 & & -1 \\ & & 1 & 1 & & & -1 \\ & & & -1 & -1 & -1 & 2 \\ & & 1 & & -1 & -1 & 1 \\ & & 1 & & -1 & -1 & 1 \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & & & & & 1 & -1 \\ & 1 & & & 1 & & -1 \\ & & 1 & 1 & & & -1 \\ & & & -1 & -1 & -1 & 2 \\ & & & -1 & -1 & -1 & 2 \\ & & & -1 & -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & & & & & 1 & -1 \\ & 1 & & & 1 & & -1 \\ & & 1 & 1 & & & -1 \\ & & & 1 & 1 & 1 & -2 \\ & & & & & & \\ & & & & & & \end{bmatrix}.$$

Clearing the pivot columns, we obtain our final matrix:

$$\begin{bmatrix} 1 & & & & & 1 & -1 \\ & 1 & & & 1 & & -1 \\ & & 1 & & -1 & -1 & 1 \\ & & & 1 & 1 & 1 & -2 \\ & & & & & & \\ & & & & & & \end{bmatrix}. \tag{4}$$

The above matrix has rank 4 indicating that there is one additional linearly independent equality in addition to the three linearly independent equalities inherited from $Q_{m,b}$. That means there are $\alpha + 1$ linearly independent equalities, so the dimension of F is

$$\dim(C_m) - (\alpha + 1) = (m-1) - \alpha - 1,$$

which is one less than the dimension of $Q_{m,b}$, implying the face is a facet. □

Theorem 5.0.3. *For each facet F on $Q_{m,b}$ with at least one of m, b odd, there is a numerical semigroup S living on F.*

Proof. Take $F_{i,j,k}$, and write $i+j+k = sm+b$, for $s \in \{0,1,2\}$. We will consider three cases.

1. Suppose $i, j, k \neq \frac{b}{2}$ and m, b are of different parities. We have two subcases.

   (a) When m is odd, b is even, and $s \neq 1$, or when m is even and b is odd, we put

$$\begin{aligned} x_i &= 2m+i & x_{i+j} &= 4m+i+j & x_b &= (6+s)m+b \\ x_j &= 2m+j & x_{i+k} &= 4m+i+k & x_l &= \left(3+\left\lfloor \frac{s}{2} \right\rfloor\right)m+l \\ x_k &= 2m+k & x_{j+k} &= 4m+j+k & x_p &= x_b - x_l \end{aligned}$$

   with

$$\frac{b}{2} \leq l \leq \frac{b}{2}+\frac{m}{2}, \quad p \neq i,j,k,b,i+j,i+k,j+k,l, \quad \text{and} \quad p+l \equiv b \bmod m.$$

   It is a quick check to see each of the $Q_{m,b}$ equalities hold; notice that

$$x_i + x_{j+k} = x_j + x_{i+k} = x_k + x_{i+j} = x_b$$

   since $i+j+k \equiv b \bmod m$. The equalities

$$x_i + x_j = x_{i+j} \quad x_j + x_k = x_{j+k} \quad x_i + x_k = x_{i+k}$$

   are all satisfied directly by construction. To see that no other equalities hold we check that given r, s such that $r+s \not\equiv b \bmod m$, and not both $r, s \in \{i,j,k\}$, we have that

$$x_r + x_s > (2+3+\lfloor s/2 \rfloor)m + r + s > 5m + r + s > 5m > x_{r+s},$$

   so we have a numerical semigroup on $F_{i,j,k}$.

   (b) When m is odd, b is even, and $s = 1$. Put

$$\begin{aligned} x_i &= 3m+i & x_{i+j} &= 7m+i+j & x_b &= 12m+b \\ x_j &= 4m+j & x_{i+k} &= 7m+i+k & x_l &= 6m+l \\ x_k &= 4m+k & x_{j+k} &= 8m+j+k & x_p &= x_b - x_l \end{aligned}$$

   with

$$\frac{b}{2} \leq l \leq \frac{m}{2}+\frac{b}{2}, \quad p \neq i,j,k,b,i+j,i+k,j+k,l \quad \text{and} \quad l+p \equiv b \bmod m.$$

   It is similar as above to check these equalities are the only ones satisfied.

2. When, without loss of generality, $i = \frac{b}{2}$, $j = k$. Here $i+j+k = sm+b$ for $s \in \{0, 1, 2, 3\}$. Put

$$x_k = 3m + k \qquad x_{i+k} = 9m + 3k x_l = 6m + l$$
$$x_i = 2x_k = 6m + 2k \qquad x_b = (12 + s)m + b x_p = x_b - x_l$$

with
$$\frac{b}{2} < l < \frac{m}{2} + \frac{b}{2} \quad \text{and} \quad l + p \equiv b \bmod m.$$
As before, it is easy to check that the desired equalities hold. To see that none other hold, we check that given r, s such that $r + s \not\equiv b \bmod m$, and not both $r, s \in \{i, j, k\}$, we have that
$$x_r + x_s > 3m + k + 6m + s > 9m > x_{r+s},$$
so a numerical semigroup exists on $F_{i,j,k}$.

3. When m, b both odd, since any facet of $Q_{m,m-b}$ falls in the previous cases, and since $Q_{m,b} \cong Q_{m,m-b}$ by Theorem 4.0.1, we conclude that every facet of $Q_{m,b}$ must contain a numerical semigroup on its interior.

We conclude that for each facet with at least one of m, b odd, there is a numerical semigroup S living on F. $\square$

Example 5.0.4. Figure 5 is a poset representation of a facet of $Q_{9,2}$. The set i, j, k of this facet is $\{5, 7, 8\}$ and $s = 2$, so this facet is covered by Case $1a$, and the numerical semigroup constructed is $S = \langle 9, 37, 23, 25, 26\rangle$.

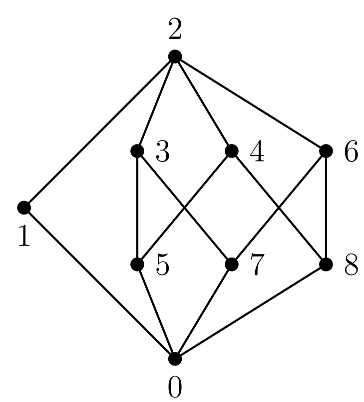


Figure 5: Kunz poset corresponding to a facet of $Q_{9,2}$ with set $\{5, 7, 8\}$

Example 5.0.5. The poset in Figure 6 represents another facet of $Q_{9,2}$, with a set $\{1, 5, 5\}$ and because $\frac{b}{2} = 1$, this facet is covered by Case 2. The numerical semigroup constructed for this facet is $S = \langle 9, 32, 57, 58, 70, 71\rangle$.

We will now consider the number of facets of $Q_{m,b}$, by appealing to the triple characterization given by 5.0.2. We will give a counting for $b = 1$, and for $m \geq 3$.

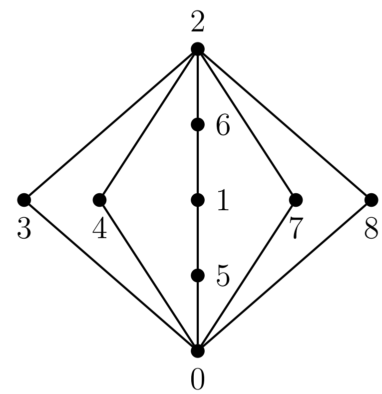


Figure 6: Kunz poset corresponding to a facet of $Q_{9,2}$ with set $\{1,5,5\}$

Theorem 5.0.6. *For $m \geq 2$, the number of facets of $Q_{m,1}$ is given by*

$$\begin{cases} \frac{1}{6}(m^2-3m) & m \equiv 0 \bmod 6 \\ \frac{1}{6}(m^2-6m+17) & m \equiv 1 \bmod 6, 5 \equiv \bmod 6 \\ \frac{1}{6}(m^2-3m+2) & m \equiv 2 \bmod 6, 4 \equiv \bmod 6 \\ \frac{1}{6}(m^2-6m+15) & m \equiv 3 \bmod 6 \end{cases}$$

Proof. Given m, b arbitrary, write

$$t = \#\{x \in [m-1] : 2x \equiv b \bmod m\} = \#C_{\frac{b}{2}}$$

$$q = \#\{x \in [m-1] : 2x = m\} = \#C_{\frac{m}{2}}$$

$$r = \#\{x \in [m-1] : 3x \equiv b \bmod m\} = \#C_{\frac{b}{3}}$$

To count the number of triples, we will count those with and without duplicates separately. To count those with duplicates, notice each such triple takes the form $(i, i, b-2i)$ where $i \notin \{0, b\} \cup C_{\frac{b}{2}} \cup C_{\frac{m}{2}}$. Hence there are

$$m - 2 - q - t$$

valid triples of this form. To count the number of triples with distinct entries - we will proceed by counting the number of pairs (i, j) with:

1. $i \neq j$, as this would give a triple with duplicates which has been counted already
2. $i, j \neq 0, b$
3. $i, j \notin C_{\frac{b}{2}}$, as if one of these were it would two elements of the triple to be the same
4. $i + j \neq b$, as this would force the third element of triple $k = 0$.
5. $i + j \neq m$, as this would force the third element of the triple $k = b$.

6. $k = b - i - j \notin C_{\frac{b}{2}}$, as this would force $i = j$.

7. $k = b - i - j$ must not be equal to i, j, otherwise we get a duplicate which we already counted.

The number of pairs (i, j) satisfying the first three conditions is easily seen to be $(m - 2 - t)(m - 3 - t)$. Of these, we must remove the pairs not satisfying item 4, namely that $i + j = b$. It is clear to see that any choice $i \notin \{0, b\} \cup C_{\frac{b}{2}}$ will work, as if $x \in C_{\frac{b}{2}}$, then $j = b - x = x$ which was not a counted distinct pair. Thus, there are $m - 2 - t$ of these pairs to remove.

To count the number of pairs (i, j) with distinct entries satisfying $i + j = m$, we must disallow $i, j \in \{0, b\} \cup C_{b/2}$. Hence we cannot have $i = m - b$ or $m - x$ for $x \in C_{b/2}$. We see that $i = \frac{m}{2}$ is also disallowed, since $j = m - i = i$. If it were true that $b = \frac{m}{2}$, then we would have double counted this pair. Since $m \geq 3$, this is not the case. In total, the number of such pairs is

$$m - 1 - 2 - 2t - q.$$

To count the number of pairs (i, j) with distinct entries having $i + j \notin C_{\frac{b}{2}}$, we will count the number of i such that $(i, \frac{b}{2} - i)$ doesn't take values in the set $\{0, b\} \cup C_{\frac{b}{2}} \cup C_{\frac{b}{4}}$. If, without loss of generality, $i \in C_{\frac{b}{4}}$, then $j = i$, and there this doesn't correspond to a distinct pair we've already counted - so these do not need to be subtracted off. However, i can indeed take values in any of the other specified values to get a bad pair. The other thing to check is that $\frac{b}{2} - i$ is in the specified values, and this occurs for $i = m - \frac{b}{2}$ or $\frac{m}{2}$. Hence, there are

$$t(m - 2 - 3t)$$

additional pairs to be subtracted off from our count. Of course, if $\frac{m}{2}$ exists, since $b = 1$, we have that $t = 2$, so this accounts for no appearance of q in the previous formula. The multiplication by t accounts for symmetry in the cases where $t = 2$, and $\frac{m}{2} + \frac{b}{2}$ exists.

We need only count the cases now where $b - i - j = i$ or $b - i - j = j$. Here we must ensure that we do not count pairs already subtracted off in previous cases. By symmetry, we will only count those cases where $b - i - j = i$, and $j = b - 2i$. We can count these pairs (i, j) with $b - i - j = i$ by counting the pairs $(i, b - 2i)$ where i can take values such that $i \notin \{0, b\} \cup C_{\frac{b}{2}} \cup C_{\frac{b}{4}}$, and also disallowing $b - 2i \in \{0, b\} \cup C_{\frac{b}{2}}$. $b - 2i \neq 0$ as this would imply $i \in C_{\frac{b}{2}}$ which is disallowed already. So, $b - 2i \neq b$ as this would imply $i = 0$ or $i \in C_{\frac{m}{2}}$ which is disallowed. This implies $b - 2i \notin C_{\frac{b}{2}}$ implies that $i = \frac{3b}{2}$ or $i = \frac{m}{2} + \frac{3b}{2}$, which does not exist for $b = 1$, but does for other values. Furthermore, there are r choices of i that set $i = b - 2i$, which must get subtracted off as these were not counted as part of items $1, 2, 3$. In total - this gives rise to

$$2(m - 2 - 2t - r - q)$$

such pairs we need to subtract off, where the multiplication by 2 comes from symmetry with the other condition $b-i-j=j$. Thus, this gives a total of

$$\frac{1}{6}[(m-2-t)(m-3-t)-(m-2-t)-(m-1-2-2t-q)$$
$$-t(m-2-3t)-2(m-2-2t-r-q)]+(m-2-q-t)$$

triples corresponding to facets of $Q_{m,b}$. The result follows from evaluating the quantities t, q, r for m mod 6. □

6 Code

Throughout the program, we wrote methods in Sage to aid us with our exploration into symmetric numerical semigroups. A compilation of these appears in: `https://github.com/GeorgeTsoukalas/symmetricFaceFunctions`. There are dependencies on both numsgps.sage and KunzPoset.sage. We will give some discussion of these functionalities here:

1. *generateSymmetricPosetInequalities*: Inputs integers m, b and outputs the array of Kunz Inequalities that $Q_{m,b}$ satisfies.

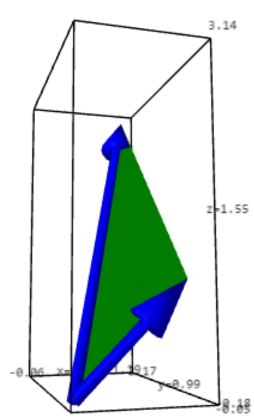
```
INPUT: generateSymmetricPosetInequalities(4,2)
OUTPUT: [[0, 2, -1, 0], [0, 1, 1, -1], [0, -1, 1, 1],
              [0, 0, -1, 2], [0, -2, 1, 0], [0, 0, 1, -2]]
```

2. *maximalSymmetricPoset*: inputs integers m, b and outputs the cover relations for the Kunz Poset of $Q_{m,b}$ to be used to construct a FinitePoset object.

```
INPUT: maximalSymmetryPoset(4,2)
OUTPUT: ([0,1,2,3,4],
        [(0, 1), (1, 3), (0, 2), (2, 3), (0, 4), (4, 3)])
```

3. *generateSymmetricFace*: inputs integers m, b and creates $Q_{m,b}$ as a Polyhedron object.

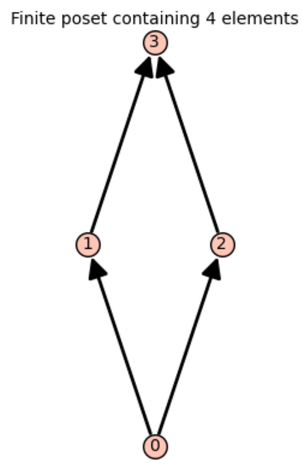
```
INPUT: generateSymmetricFace(4,3)
OUTPUT: A 2-dimensional polyhedron in QQ^3 defined as
the convex hull of 1 vertex and 2 rays (use the .plot() method to plot)
```



4. *dimensionData*: inputs integer m and outputs the dimension and f-vector of $Q_{m,b}$ for each $b \in [1, m-1]$. We ran this up to $m = 18$, and the results of those computations can be accessed here.

5. *facetEqualities*: inputs face as a Polyhedron object and integer m, returns those inequalities from the Kunz inequalities which are satisfied with equality.
```
INPUT: facetEqualities(generateSymmetricFace(4,3), 4)
OUTPUT: [[0, 1, 1, -1]]
```

6. *returnKunzPoset*: inputs face as a Polyhedron object of C_m and integer m, returns the corresponding Kunz Poset object.
```
INPUT: returnKunzPoset(face: Polyhedron, 4)
OUTPUT: KunzPoset with multiplicity 4
```



7. *findRaysForSymmetryFaces*: inputs an integer m and outputs the rays of $Q_{m,b}$ for each $b \in [1, m-1]$.
```
INPUT: findRaysForSymmetryFaces(4)
OUTPUT: Q41 has rays [(3, 2, 1), (1, 0, 1)]
        Q42 has rays [(1, 2, 1)]
        Q43 has rays [(1, 0, 1), (1, 2, 3)]
```

8. *posetElementDepth*: inputs face as a Polyhedron object and an integer m, outputs an array of integers where the index corresponds to the element of the corresponding Kunz Poset of the face and the value at that index is the element's height. Height is defined as the maximal length of a chain from 0 to the element.
```
INPUT: posetElementDepth(generateSymmetricFace(12,7).facets()[0],12)
OUTPUT: [0, 1, 1, 1, 1, 1, 1, 3, 2, 2, 1, 1]
```

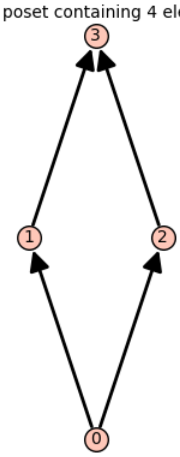
9. *getTuple*: inputs facet as a Polyhedron object and an integer m, outputs an array indexed by height of the number of elements of the facet's KunzPoset with that height.

10. *getTypesOfTuples*: inputs integers m, b and outputs the number of each facet type for each facet of $Q_{m,b}$. In order, the facet types are single-boost, double-boost, triple-boost, boost-boost, and other (we proved no "other"s exist. This method utilizes *getTuple*.

```
INPUT: getTypesOfTuples(7,1)
OUTPUT: Q7,1 with tuple (1, 2, 1, 0, 0)
```

11. *qmbFacetPosets*: inputs m, b and builds a symmetric polyhedron and prints the posets of its facets. This method utilizes *generateSymmetricFace, facetEqualities.*
```
INPUT: qmbFacetPosets(8,3)
OUTPUT: 7 KunzPosets corresponding to the 7 facets of $\Qmb$
```
12. *findTriples*: inputs m, b and returns a list of all allowed triples covered in Theorem 5.0.2.
```
INPUT: findTriples(8,3)
OUTPUT: [[1, 1, 1], [1, 4, 6], [1, 5, 5], [2, 2, 7],
        [2, 4, 5], [5, 7, 7], [6, 6, 7]]
```
13. *tripleToEq*: inputs m and a list and returns a list of each corresponding equation in the format used by KunzPoset and Polyhedron objects.
```
INPUT: tripleToEqs(8,[1, 4, 6])
OUTPUT: [[0, 1, 0, 0, 1, -1, 0, 0],
        [0, 1, 0, 0, 0, 0, 1, -1],
        [0, 0, -1, 0, 1, 0, 1, 0]]
```
14. *findTriplesDiffFormat*: inputs m, b. finds allowed triples and uses it to return a list holding each facet's list of equalities corresponding equations in the format used by KunzPoset and Polyhedron objects. Utilizes *tripleToEqs*
```
INPUT: findTriplesDiffFormat(6,1)
OUTPUT: [[[0, 0, 1, 0, -1, 0], [0, 0, 1, 1, 0, -1], [0, 0, 1, 1, 0, -1]],
        [[0, 0, -1, 1, 0, 1], [0, 0, -1, 1, 0, 1], [0, 0, 0, 0, -1, 1]],
        [[0, 0, -1, 0, 1, 0], [0, 0, 0, -1, 1, 1], [0, 0, 0, -1, 1, 1]]]
```
15. *intersectionRays*: inputs m, b, b' and returns a list of rays that define the face of the intersection of $Q_{m,b}$ and $Q_{m,b'}$. Utilizes *generateSymmetricFaces*
```
INPUT: intersectionRays(8, 3, 7)
OUTPUT: (A ray in the direction (1, 0, 1, 0, 1, 0, 1),
        A ray in the direction (1, 2, 3, 0, 1, 2, 3))
```
16. *intersectionZeroes*: inputs a list of rays and returns a list of all ray coordinates that are 0 in all rays.
```
INPUT: intersectionZeroes(intersectionRays(8,3,7)
OUTPUT: [4]
```
17. *intersectionPoset*: inputs m, b, b' and prints the poset corresponding to the intersection $Q_{m,b}$ and $Q_{m,b'}$.
```
INPUT: intersectionPoset(8, 3, 7)
OUTPUT: KunzPoset of the intersection
```

Finite poset containing 4 elements



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[3] J. C. Rosales and P. A. García-Sánchez. *Numerical semigroups*. Vol. 20. Developments in Mathematics. Springer, New York, 2009, pp. x+181. ISBN: 978-1-4419-0159-0. DOI: 10.1007/978-1-4419-0160-6. URL: https://doi.org/10.1007/978-1-4419-0160-6.