Seizing the Numerical Semigroups of Kuntz Polyhedra

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June 2019

Abstract

The connection between numerical semigroups and their corresponding faces on the Kunz polyhedron is still largely an open area of research. In this paper we attempt to categorize the families of numerical semigroups living on various faces of arbitrary dimensions. We do this by examining the Kunz poset structures, by looking both at familiar families and going to the faces as well as finding what other numerical semigroups live on the same face, should any others exist.

Contents

1 Classifying Apery Posets of Generalized Arithmetic Semigroups

Let S be a numerical semigroup. If $S = \langle a, ah + d, ah + 2d, ..., ah + kd \rangle$ for $a, h, d, k \in \mathbb{N}$ with $\gcd(a, d) = 1$, then we define S as a *generalized arithmetic semigroup.* For this section we explore the structure of the Kunz poset for this family of numerical semigroups and seek to find what other undiscovered numerical semigroups might live within this family.

1.1 Poset Structure for Generalized Arithmetic Semigroups

We conjecture the following poset structure for generalized arithmetic semigroups:
Conjecture 1.1.

1. *The Poset is graded. It follows that every element in a certain row is the sum of the same amount of indistinct atoms and that no two elements from different rows are the sum of the same amount of indistinct atoms.*

2. *For elements i, j, such that i and j are in rows with sequential heights, we find that $i \preccurlyeq j$ if and only if:*

   - $j - i \in [d, kd] \cap d\mathbb{Z}$ *when* $i < j$

   - $j - i + a \in [d, kd] \cap d\mathbb{Z}$ *when* $i > j$

We can use the following machinery to examine our conjecture for the relevant numerical semigroups. In particular, we use the division algorithm and a membership criterion.

- Consider $n_1, n_2 \in \mathrm{Ap}(S, a)$ such that $S = \langle a, a{+}d, \ldots, a{+}kd \rangle$ and $\gcd(a, d) = 1$. By the division algorithm, we can write $n_1 = q_1 a + r_1 d$ and $n_2 = q_2 a + r_2 d$

where $0 \leq r_1 \leq a-1$ and $0 \leq r_2 \leq a-1$. Applying the division algorithm to r_1 and r_2 yields $r_1 = l_1k + r_1'$ and $r_2 = l_2k + r_2'$ where $1 \leq r_1' \leq k$ and $1 \leq r_2' \leq k$. It follows that $n_1 = q_1a + (l_1k + r_1')d$ and $n_2 = q_2a + (l_2k + r_2')d$.

- According to Omidali and Rahmadi, $n \in S = \langle a, ah+d, \dots ah+kd \rangle$ if and only if $\lceil \frac{r}{k} \rceil h \leq q$ [2]. We call this statement a *membership criterion*. The membership criterion becomes $n \in S$ if and only if $r \leq kq$ for arithmetic numerical semigroups with $h = 1$.

We now use the division algorithm and membership criterion to prove our conjecture for arithmetic numerical semigroups.

Proposition 1.2. *: The Apèry poset of S is graded, and the height of some element $n_i \in \mathrm{Ap}(S)$ is equal to the amount of indistinct atoms that are added together to yield n_i. We call the amount of indistinct atoms l_i.*

Proof. Without loss of generality, $n_2 > n_1$. Since elements in consecutive rows of a graded Apèry poset differ by exactly one atom, we want $n_2 - n_1 \in [a+d, a+kd] \cap (a + d\mathbb{Z})$ if the heights of the rows for n_2 and n_1 differ by 1. We want to show that $n_2 - n_1 \notin [a+d, a+kd] \cap (a+d\mathbb{Z})$ if the heights of the rows for n_2 and n_1 are either the same or differ by a value greater than 1. Our first case is the former and our second case is the latter.

Case 1: We will show that for n_1 and n_2 occupying the same row of our poset, then $n_1 \preccurlyeq n_2$ implies $n_2 - n_1 = 0 \notin [a+d, a+kd] \cap (a+d\mathbb{Z})$. We then will be able to conclude that two elements occupying the same row are only comparable if we are actually comparing an element to itself. Recall that $n_1 = q_1a + r_1d$ and $n_2 = q_2k + r_2d$. It follows from Omidali and Rahmadi's Proposition 2.6 [2] that that $n \in Ap(S, a)$ if and only if $q - 1 < \lceil \frac{r}{k} \rceil h \leq q$ for any $h \in \mathbb{Z}_{\geq 1}$. Since $\lceil \frac{r}{k} \rceil h \in \mathbb{Z}_{\geq 1}$ and $q \in \mathbb{Z}$, we have that $\lceil \frac{r}{k} \rceil h = q$ for any $n \in Ap(S, a)$. Therefore, we can write $n_1 = \lceil \frac{r_1}{k} \rceil ha + r_1d$ and $n_2 = \lceil \frac{r_2}{k} \rceil ha + r_2d$. Without loss of generality, let $n_2 \geq n_1$ and note that $r_2{}' \geq r_1{}'$ follows. Taking the difference of n_1 and n_2 yields

$$n_2 - n_1 = \left(\left\lceil \frac{r_2}{k} \right\rceil - \left\lceil \frac{r_1}{k} \right\rceil \right) ha + (r_2 - r_1)d.$$

Suppose that $n_2 - n_1 \in S$. By our membership criterion,

$$\left\lceil \frac{r_2 - r_1}{k} \right\rceil h \leq \left(\left\lceil \frac{r_2}{k} \right\rceil - \left\lceil \frac{r_1}{k} \right\rceil \right) h.$$

Dividing by h, we have,

$$\left\lceil \frac{r_2 - r_1}{k} \right\rceil \leq \left(\left\lceil \frac{r_2}{k} \right\rceil - \left\lceil \frac{r_1}{k} \right\rceil \right),$$

and adding $\left\lceil \frac{r_1}{k} \right\rceil$ to each side yields

$$\left\lceil \frac{r_1}{k} \right\rceil + \left\lceil \frac{r_2}{k} - \frac{r_1}{k} \right\rceil \leq \left\lceil \frac{r_2}{k} \right\rceil.$$

Furthermore, by properties of the ceiling function, we find

$$\left\lceil \frac{r_2}{k} \right\rceil \leq \left\lceil \frac{r_1}{k} \right\rceil + \left\lceil \frac{r_2}{k} - \frac{r_1}{k} \right\rceil.$$

Taking the previous two inequalities together shows

$$\left\lceil \frac{r_2}{k} \right\rceil = \left\lceil \frac{r_1}{k} \right\rceil + \left\lceil \frac{r_2}{k} - \frac{r_1}{k} \right\rceil,$$

and subtracting $\left\lceil \frac{r_1}{k} \right\rceil$ from both sides yields

$$\left\lceil \frac{r_2}{k} \right\rceil - \left\lceil \frac{r_1}{k} \right\rceil = \left\lceil \frac{r_2}{k} - \frac{r_1}{k} \right\rceil.$$

It follows that

$$\begin{aligned}
\left\lceil \frac{{r_2}' - {r_1}'}{k} \right\rceil &= \left\lceil \frac{{r_2}'}{k} - \frac{{r_1}'}{k} \right\rceil \\
&= \left\lceil l_1 + \frac{{r_2}'}{k} - l_1 - \frac{{r_1}'}{k} \right\rceil \\
&= \left\lceil l_2 + \frac{{r_2}'}{k} - l_1 - \frac{{r_1}'}{k} \right\rceil \\
&= \left\lceil \frac{(l_2k + {r_2}') - (l_1k + {r_1}')}{k} \right\rceil \\
&= \left\lceil \frac{r_2 - r_1}{k} \right\rceil \\
&= \left\lceil \frac{r_2}{k} - \frac{r_1}{k} \right\rceil \\
&= \left\lceil \frac{r_2}{k} \right\rceil - \left\lceil \frac{r_1}{k} \right\rceil \\
&= \left\lceil \frac{l_2k + {r_2}'}{k} \right\rceil - \left\lceil \frac{l_1k + {r_1}'}{k} \right\rceil \\
&= \left\lceil l_2 + \frac{{r_2}'}{k} \right\rceil - \left\lceil l_1 + \frac{{r_1}'}{k} \right\rceil \\
&= \left\lceil l_2 + \frac{{r_2}'}{k} \right\rceil - \left\lceil l_1 + \frac{{r_1}'}{k} \right\rceil \\
&= l_2 + \left\lceil \frac{{r_2}'}{k} \right\rceil - l_1 - \left\lceil \frac{{r_1}'}{k} \right\rceil \\
&= l_1 + \left\lceil \frac{{r_2}'}{k} \right\rceil - l_1 - \left\lceil \frac{{r_1}'}{k} \right\rceil \\
&= \left\lceil \frac{{r_2}'}{k} \right\rceil - \left\lceil \frac{{r_1}'}{k} \right\rceil .
\end{aligned}$$

Since $1 \leq {r_1}' \leq k$ and $1 \leq {r_2}' \leq k$, we have $\left\lceil \frac{{r_1}'}{k} \right\rceil = \left\lceil \frac{{r_2}'}{k} \right\rceil = 1$, and thus

$$\left\lceil \frac{{r_2}' - {r_1}'}{k} \right\rceil = 0$$

Since we have ${r_2}' \geq {r_1}'$, we conclude ${r_1}' = {r_2}'$, so $n_1 = n_2$. Therefore $n_2 - n_1 = 0$.

Case 2: Let $|l_2 - l_1| > 1$ and without loss of generality $l_2 > l_1$. It follows that $l_2 - l_1 > 1$, and we must show that $n_2 - n_1 \notin [a + d, a + kd] \cap (a + d\mathbb{Z})$. Since

$$n_2 - n_1 = (q_2 - q_1)a + ((l_2 - l_1)k + r'_2 - r'_1)d,$$

we can show that either either $q_2 - q_1 \neq 1$ or $(l_2 - l_1)k + r_2{}' - r_1{}' \notin [1, k] \cap \mathbb{Z}$. Since $1 \leq r_1' \leq k$ and $1 \leq r_2' \leq k$, we know that $1 - k \leq r_2' - r_1' \leq k - 1$. By substituting this inequality into our expression for the coefficient of d, we find $(l_2 - l_1)k + r_2{}' - r_1{}' \geq (l_2 - l_1)k + (1 - k)$, and since $l_2 - l_1 > 2$, we have $(l_2 - l_1)k + r_2{}' - r_1{}' \geq 2k + (1 - k) = k + 1 > k$. Since $(l_2 - l_1)k + r_2{}' - r_1{}' \notin [1, k] \cap \mathbb{Z}$, we know $n_2 - n_1 \notin [a + d, a + kd] \cap (a + d\mathbb{Z})$ as required. $\square$

Proposition 1.3. *: For elements i, j, such that i and j are in rows with sequential heights, we find that $i \preccurlyeq j$ if and only if:*

- *$j - i \in [d, kd] \cap d\mathbb{Z}$ when $i < j$*
- *$j - i + m \in [d, kd] \cap d\mathbb{Z}$ when $i > j$*

Proof. We have $i, j \in [1, a - 1] \cap \mathbb{Z}$ which correspond to $r_1 d$ and $r_2 d$ in the expressions for n_1 and n_2, respectively. We know $i \preccurlyeq j$ if and only if $n_1 \preccurlyeq n_2$ where i, j are in the Kunz poset and n_1, n_2 are the corresponding elements in the Apéry poset. Thus, we aim to prove that $n_1 \preccurlyeq n_2$ if and only if $n_2 - n_1 \in [a + d, a + kd] \cap (a + d\mathbb{Z})$.

Assume $n_1 \preccurlyeq n_2$. Since n_1 and n_2 are sequential heights and the poset is graded, it then follows that there must be an edge connecting the vertices of n_1 and n_2. By the construction of posets, any edge is defined by an atom that's not the multiplicity of the numerical semigroup. Therefore, $n_2 - n_1 \in [a + d, a + kd] \cap (a + d\mathbb{Z})$.

Conversely, if $n_2 - n_1 \in [a + d, a + kd] \cap (a + k\mathbb{Z})$, then they differ by an atom of the numerical semigroup that is not the multiplicity. Since they are of sequential height, by the construction of posets an edge would then be drawn between them. Therefore, $n_1 \preccurlyeq n_2$.

$\square$

1.2 A New Family of Numerical Semigroups: The Pessimistic Arithmetic Numerical Semigroup

As it turns out there are more numerical semigroups that yield posets that resemble the poset structure of generalized arithmetic numerical semigroups. In particular consider the semigroup S generated by $\langle 11, 12, 14, 16, 18, 20 \rangle$. By constructing the Kunz poset for S we find that it looks like Figure 1.

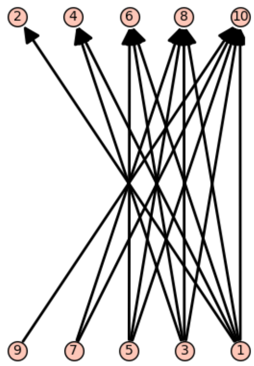

Figure 1: Kunz Poset for $S = \langle 11, 12, 14, 16, 18, 20 \rangle$

At first glance this numerical semigroup doesn't quite look like a generalized arithmetic numerical semigroup. However, if we rearrange the generators to write them as

$$S = \langle 11, 20, 18, 16, 14, 12 \rangle \,,$$

we can then let $a = 11$ and $d = -2$ to see that

$$S = \langle a, ah + d, ah + 2d, ah + 3d, ah + 4d, ah + 5d \rangle \,.$$

Traditionally, this would not be considered a generalized arithmetic semigroup. We will show that despite that fact, semigroups whose generators can be written as this form belong to a larger family of semigroups which we will refer to as *super-generalized arithmetic numerical semigroups*. For the family that lives within the super-generalized arithmetic family where d is negative, we shall then refer to them as *pessimistic arithmetic numerical semigroups*.

It is important to note that for the Pessimistic Arithmetic family, some additional requirements are required to be a numerical semigroup in this family. We still have, for clear reasons, that $\gcd(a, d) = 1$ in order for S to be a numerical semigroup. We then further have that the value of d, k are dependant on the value of a as we need to guarantee that the multiplicity of the numerical semigroup remains a. To see this we can look at our previous example and see that if we continued just one more step we would end up with a numerical semigroup S' generated by $\langle 11, 20, 18, 16, 14, 12, 10 \rangle$. As the multiplicity is now 10 the resulting Kunz poset takes the form as in Figure 2.

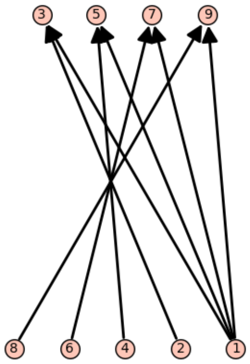

Figure 2: Kunz Poset for $S' = \langle 10, 11, 12, 14, 16, 18, 20 \rangle$

Although S' is a numerical semigroup, we do not include it in the super-generalized arithmetic family since $a = 11$ is no longer our multiplicity. So in the pessimistic arithmetic numerical semigroup family, we keep in mind that d must always be chosen such that $a < ah + kd$ is still satisfied.

We can now show that the membership criterion proved by Omidali and Rahmadi still holds for pessimistic arithmetic numerical semigroups.

Proposition 1.4. *Let S be a numerical semigroup generated by*

$$\langle a, ah + d, ah + 2d, ..., ah + kd \rangle$$

with $m(S) = a, h \in \mathbb{Z}_{\geq 2}$, and $d \in \mathbb{Z}^-$. Further, for any $n \in \mathbb{N}$, let $n = qa + rd$ where $q, r \in \mathbb{N}_0$ and $0 \leq r \leq a - 1$. Then, $n \in S$ if and only if $\left\lceil \frac{r}{k} \right\rceil \leq q$.

Proof. First assume $n \in S$. Then for some $\alpha, \beta_i \in \mathbb{N}_0$ with $1 \leq i \leq k$, it follows that $n = \alpha a + \sum_{i=1}^{k} \beta_i (ah + id)$. Let

$$q = \alpha + h \sum_{i=1}^{k} \beta_i + \left\lceil \frac{\sum_{i=1}^{k} \beta_i i}{a} \right\rceil d$$

and

$$r = \left(\sum_{i=1}^{k} \beta_i i \mod a \right).$$

Then,

$$\begin{aligned} qa + rd &= \alpha a + ah \sum_{i=1}^{k} \beta_i + a \left\lceil \frac{\sum_{i=1}^{k} \beta_i i}{a} \right\rceil d + \left(\sum_{i=1}^{k} \beta_i i \mod a \right) d \\ &= \alpha a + \sum_{i=1}^{k} ah\beta_i + \sum_{i=1}^{k} \beta_i id \\ &= \alpha a + \sum_{i=1}^{k} \beta_i (ah + id) \\ &= n. \end{aligned}$$

Since

$$\frac{r}{k} = \frac{\left(\sum_{i=1}^{k} \beta_i i \mod a \right)}{k} \le \frac{\sum_{i=1}^{k} \beta_i i}{k} = \sum_{i=1}^{k} \beta_i \frac{i}{k} \le \sum_{i=1}^{k} \beta_i,$$

we can then conclude that

$$\left\lceil \frac{r}{k} \right\rceil h \le h \sum_{i=1}^{k} \beta_i \le q.$$

Therefore, if $n \in S$, then $\left\lceil \frac{r}{k} \right\rceil h \le q$.

Now assume that $\left\lceil \frac{r}{k} \right\rceil \le q$ for some $n = aq + rd$ with $q, r \in \mathbb{Z}_0$ and $0 \le r \le a - 1$. By induction on r we can then show that $n \in S$. First, when $r = 0, n = aq$ which clearly implies that $n \in S$. We can then consider some r where $0 < r \le k$. By the bounds on r we have that $\left\lceil \frac{r}{k} \right\rceil = 1$, thus $h \le q$. Then

$$n = aq + rd = (q - h + h)a + rd = (q - h)a + ha + rd.$$

Since $r \le k$, it follows that $ah + rd$ is a generator of S and is thus in S. Since $q - h$ is a non-negative integer, it also is clear that $(q - h)a \in S$. Thus, $n = (q - h)a + ah + rd \in S$.

We can now consider when $k < r$. Then,

$$n = qa + rd = (q - h + h)a + (r - k + k)d = (ha + kd) + [(q - h)a + (r - k)d].$$

Since $ah + kd$ is a generator of S is it clear that $ah + kd \in S$. Since $0 < k < r \le a - 1$

it follows that $r - k \leq a - 1$. Further,

$$\left\lceil \frac{r-k}{k} \right\rceil h = \left\lceil \frac{r}{k} - \frac{k}{k} \right\rceil h = \left\lceil \frac{r}{k} \right\rceil h - h \leq q - h.$$

By induction, it then follows that $(q-h)a + (r-k)d \in S$. Thus, $n = (ha + kd) + [(q-h)a + (r-k)d] \in S$. Therefore, we can conclude that if $\left\lceil \frac{r}{k} \right\rceil h \leq q$, then $n \in S$. $\square$

Corollary 1.5. *Pessimistic arithmetic numerical semigroups have the same poset structure as generalized arithmetic numerical semigroups.*

Proof. Since pessimistic arithmetic numerical semigroups have the same membership criterion as generalized arithmetic numerical semigroups and the proofs presented for the structure of posets corresponding to generalized arithmetic numerical semigroups did not rely on d being positive, it follows that the claims still holds for values of d which are negative. Therefore, pessimistic arithmetic numerical semigroups have the same poset structure as generalized arithmetic numerical semigroups. $\square$

1.3 Characterizing the Kunz Poset Structure Corresponding to Super-Generalized Arithmetic Numerical Semigroups

We observed that poset diagrams with a particular structure correspond to either generalized arithmetic or pessimistic arithmetic numerical semigroups unless the embedding dimension is equal to the multiplicity or one less than the multiplicity. For instance, Figure 3 is the poset diagram that corresponds to the numerical semigroup $S = \langle 8, 9, 10, 11, 12, 13 \rangle$ and Figure 4 is the poset diagram that corresponds to the numerical semigroup $S = \langle 27, 28, 29, 30, 31, 32 \rangle$.

Both of these numerical semigroups contain an element that covers all minimal elements and one element that covers all but one of the elements. While both of these are arithmetic semigroups, we can see the same pattern in generalized arithmetic semigroups and pessimistic arithmetic semigroups. For example, Figure 5 is the poset diagram that corresponds to the numerical semigroup $S = \langle 7, 23, 25, 27, 29 \rangle$ and Figure 6 is the poset diagram that corresponds to the numerical semigroup $\langle 11, 20, 18, 16, 14, 12 \rangle$. Note that $a = 7, h = 3$, and $d = 2$

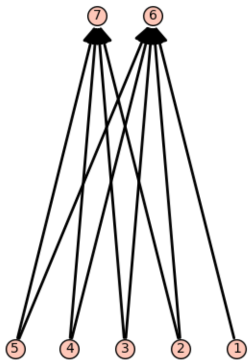

Figure 3: Kunz Poset for
$S = \langle 8, 9, 10, 11, 12, 13 \rangle$

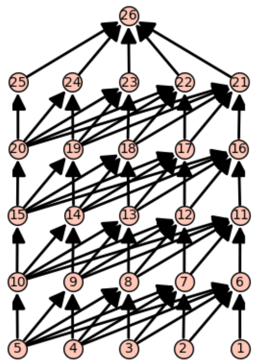

Figure 4: Kunz Poset for
$S = \langle 27, 28, 29, 30, 31, 32 \rangle$

in the first numerical semigroup, and $a = 11, h = 2$, and $d = -2$ in the second numerical semigroup.

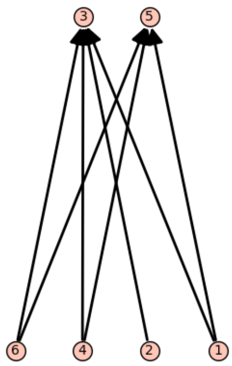

Figure 5: Kunz Poset for
$S = \langle 7, 23, 25, 27, 29 \rangle$

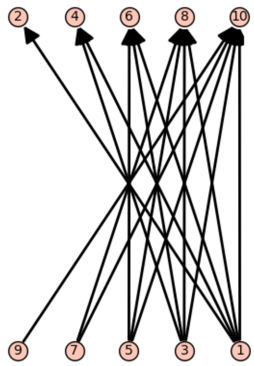

Figure 6: Kunz Poset for
$S = \langle 11, 20, 18, 16, 14, 12 \rangle$

Theorem 1.6. *Given a numerical semigroup which corresponds to a Kunz poset with k minimal elements that are consecutive multiples of $d \in \mathbb{Z}_{k+1}$, such that one element covers all the minimal elements and one element covers all but one of the elements, the corresponding numerical semigroup is either a generalized arithmetic semigroup or a pessimistic arithmetic semigroup.*

Proof. We break this proof into two cases. Our first case is that k is odd and our second case is that k is even.

Case 1: First assume that k is odd. Consider the poset P with elements of each row labeled from least to greatest such that $a_{\frac{k+1}{2}} = a_i$, which is the median. In particular, the minimal elements of the poset are $a_1 < a_2 < \cdots < a_{i-1} < a_i < a_{i+1} < \cdots < a_{i+(\frac{k-1}{2})} = a_k$. By our initial statement, there exists $\alpha \in P$ such that α covers $\{a_1, \ldots a_k\}$. Since k is odd and the minimal elements are distinct, we know that $\alpha = 2e$ for some $e \in \{a_1, \ldots a_k\}$. If we let $e = a_i$, then the remaining minimal elements are partitioned into pairs that sum to α. In particular, $\alpha = 2a_i = 2a_{(\frac{k+1}{2})} = a_{k+1}$. As such, we find that $\alpha = 2a_i = a_j + a_{k+1-j}$ for $j \in \left[1, \frac{k-1}{2}\right] \cap \mathbb{Z}$. Therefore, we can also write $a_i = \frac{a_{k+1-j}+a_j}{2}$ for the same j. It follows that we can rewrite $a_{i-j} < a_i$ as $a_{i-j} = a_i - d_j$ and $a_{i+j} > a_i$ as $a_{i+j} = a_i + d_j$ for the same j where $d_j \in \mathbb{Z}$ represents the distance from a_{i+j} to a_i and from a_{i-j} to a_i, as a_i is the midpoint of a_{i-j} and a_{i+j}.

Now consider $\beta \in P$ such that $\deg \beta = k-1$ and $\beta > \alpha$. Since the smallest element a_1 was paired with the largest element a_k to make α, it follows that a_1 is the only element not connected to β because $\beta > \alpha = a_1 + a_k$ and there exists no a_n such that $a_1 + a_n > a_1 + a_k$. There must be a pairing of the $k-1$ elements such that $\beta = a_j + a_{k+2-j} = \beta$ for $j \in \left[1, \frac{k-1}{2}\right] \cap \mathbb{Z}$. It follows $a_{i-1} + a_{i+2} = a_i + a_{i+1}$. Since $a_{i-j} = a_i - d_j$ for $a_{i-j} < a_i$ and $a_{i+j} = a_i + d_j$ for $a_{i+j} > a_i$, we can substitute these expressions and find $d_2 = 2d_1$. We now use strong induction to show that $d_j = jd_1$.

Hypothesis: We claim that $d_n = nd_1$ for all n.

Base Case: It is clear that $d_1 = 1 \cdot d_1 = d_1$ as required.

Induction: Suppose that $d_m = md_1$ for all $m \in [1, n] \cap \mathbb{Z}$. Note that $a_{i-(n-1)} + a_{i+n} = a_{i-(n-2)} + a_{i+(n-1)}$. We can substitute our expressions in terms of a_i and d_j to find that

$$a_i - d_{n-1} + a_i + d_{n+1} = a_i - d_{n-2} + a_i + d_n.$$

By our inductive hypothesis, we can simplify this expression to

$$a_i - (n-1)d_1 + a_i + d_{n+1} = a_i - (n-2)d_1 + a_i + nd_1.$$

Simplifying both sides yields $d_{n+1} = (n+1)d_1$. We have now shown that each minimal element differs by exactly d as required.

Case 2: Now assume that k is even. Consider the poset P with elements of each row labeled from least to greatest. In particular, the minimal elements of the poset are $a_1 < a_2 < \ldots < a_k$. Consider $\alpha \in P$ such that $\deg \alpha = k$. Since there are an even number of minimal elements, it follows that the element can be partitioned into pairs that sum to α. In particular, we find that $\alpha = a_j + a_{k-j}$ for $j \in \left[1, \frac{k-1}{2}\right] \cap \mathbb{Z}$.

Now consider $\beta \in P$ such that $\deg \beta = k-1$. Since there are an odd amount of elements covered by β and $a_{\left(\frac{k-1}{2}\right)} + a_{\left(\frac{k+1}{2}\right)} = \alpha$, we find $\alpha < \beta = 2e$ for some $e \in \{a_2, \ldots, a_k\}$ such that $a_2 < a_3 \ldots a_{i-1} < a_i < a_{i+1} < \ldots a_k$. If we let $e = a_i$, which is the median element, the remaining elements can be partitioned into pairs that sum to β. As such we find that $\beta = 2e = 2a_i = a_j + a_{k+2-j} = \beta$ for $j \in \left[2, \left(\frac{k}{2} - 1\right)\right] \cap \mathbb{Z}$. Therefore, we can write $a_i = \frac{a_j + a_{k+2-j}}{2}$ for the same j. It follows that we can rewrite $a_{i-j} < a_i$ as $a_{i-j} = a_i - d_j$ and $a_{i+j} = a_i + d_j$ for the same j.

Since we know that $a_{i-1} + a_{i+2} = a_i + a_{i+1}$, we can substitute expressions in terms of a_i and d_j to find $d_2 = 2d_1$. Similarly to the odd case, we can use induction to find that $d_j = jd_1$ for all $j \in \left[2, \frac{k}{2} - 1\right] \cap \mathbb{Z}$. Therefore, we have shown that every consecutive pair of elements from a_2 to a_k differ by d. We now show that a_1 and a_2 also differ by d. Recall that $\alpha = a_1 + a_k = a_2 + a_{k-1}$. Since $d_j = jd_1$, we find that $a_1 + a_i + \left(\frac{k}{2} - 1\right)d = a_i - \left(\frac{k}{2} - 2\right)d + a_i + \left(\frac{k}{2} - 2\right)$. Therefore, $a_1 = a_i - \left(\frac{k}{2} - 1\right)$. Hence, we find that $a_2 - a_1 = \left(a_i - \left(\frac{k}{2} - 2\right)d\right) - \left(a_i - \left(\frac{k}{2} - 1\right)d\right) = d$. We have now shown that each minimal element differs by d as required.

Since the elements of our poset correspond to exact multiples of d (mod a), it follows that S is of the form $\langle a, ah+d, ah+2d, .., ah+kd\rangle$. □

We can note that this handles the case where there are at least 2 non-minimal elements, i.e. $a - e(S) \geq 2$. For the cases where $a = e(S)$ or $a = e(S) - 1$, we find that there exists numerical semigroups with the same poset structure as the family of super-generalized arithmetic numerical semigroups but do not belong into that family themselves. It is easy to find an example with the maximum embedding dimension such as $S_1 = \langle 6, 8, 10, 13, 15, 17\rangle$ which has the Kunz poset as seen in Figure 7. As an example for a numerical semigroup that is not in the family of super-generalized arithmetic numerical semigroups we can consider $S_2 = \langle 6, 15, 16, 19, 20\rangle$ which has the Kunz poset as seen in Figure 8.

Figure 7: Kunz Poset for $S_1 = \langle 6, 8, 10, 13, 15, 17\rangle$

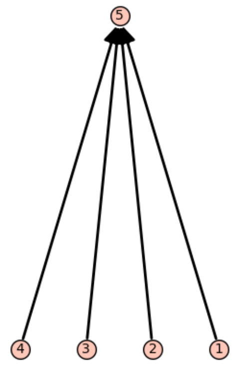

Figure 8: Kunz Poset for $S_2 = \langle 6, 15, 16, 19, 20\rangle$

Despite the limitations brought when the embedding dimension is no more than 1 difference from the multiplicity, we can now conclusively say that the generalized arithmetic Kunz posets are fully classified by the super-generalized arithmetic numerical semigroups.

1.4 Frobenius Number for the Pessimistic Arithmetic Numerical Semigroup

With the introduction of a new family of numerical semigroups it is natural to ask about further characteristics of numerical semigroups that live within the family. The first most basic question is can we derive the Frobenius number of such numerical semigroups.

Theorem 1.7. *For a pessimistic arithmetic numerical semigroup* $S = \langle a, ah + d, \ldots, ah + kd\rangle$*, the Frobenius number is given by* $F(S) = \lceil \frac{a-1}{k} \rceil (ah + kd) + (1 - k)d - a$.

Proof. Recall that the Frobenius number of a numerical semigroup can be calculated by subtracting a from $\max(\mathrm{Ap}(S; a))$, where $\max(\mathrm{Ap}(S; a))$ denotes the largest element of the maximal row of the Apèry poset. By the poset properties we have described in Section 1.3, row f contains elements $(f + 1)(ah) + gd$ for

$g \in [1, a-1] \cap \mathbb{Z}$. For the maximal row, we have $f+1 = \lceil \frac{a-1}{k} \rceil$, since we know each non-maximal row contains exactly k points and we have $a-1$ total vertices. Since d is negative, it follows that the element in the maximal row with the smallest coefficient for d will be $\max(\mathrm{Ap}(S;a))$. We now determine the value of this smallest coefficient. The coefficients for d for the elements of the maximal row are given by $a-n$, where $n \in [1, p] \cap \mathbb{Z}$ and p denotes the number of elements in the maximal row. Since we know we have $a-1$ total vertices in our poset, and we have $\lceil \frac{a-1}{k} \rceil - 1$ non-maximal rows containing exactly k elements each, the expression $a - 1 - (\lceil \frac{a-1}{k} \rceil - 1)k$ gives the number of elements in the maximal row. Thus, $a - (a - 1 - (\lceil \frac{a-1}{k} \rceil - 1)k)$ is the smallest coefficient for d of all elements in the maximal row. So $\max(\mathrm{Ap}(S;a)) = a\lceil \frac{a-1}{k} \rceil h + (a - (a - 1 - (\lceil \frac{a-1}{k} \rceil - 1)k))d$, and then $F(S) = \max(\mathrm{Ap}(S;a)) - a = a\lceil \frac{a-1}{k} \rceil h + (a - (a - 1 - (\lceil \frac{a-1}{k} \rceil - 1)k))d - a$. Algebraic simplification yields $F(S) = \lceil \frac{a-1}{k} \rceil (ah + kd) + (1-k)d - a$. □

Below an alternative proof is provided that uses properties of the same poset structure in a slightly different way.

Proof. Recall that $F(S) = \max(\mathrm{Ap}(S;a)) - a$. Since we characterized our elements to be in increasing order, we find that every row's largest element is the rightmost element. As such, the largest element in the entire poset is the rightmost element of the maximal row. Since there $a-1$ vertices in our poset with exactly k elements in each full row, we know that the number of rows is $\lfloor \frac{a-1}{k} \rfloor$. It follows that the minimal possible element in the maximal row is $\lfloor \frac{a-1}{k} \rfloor (ah + kd)$ because $(ah+kd)$ is the smallest element in the poset. Since there are k elements in each row we can add d $k-1$ times to reach the rightmost element. Therefore, the rightmost element of the maximal row, which is the maximum element of the poset, is $\lfloor \frac{a-1}{k} \rfloor + (1-k)d$. We subtract a to find $F(S) = \lfloor \frac{a-1}{k} \rfloor + (1-k)d - a$ as required. □

1.5 Other Extensions of the Super-Arithmetic Numerical Semigroup?

A natural way to extend these results would be to examine semigroups that take one of the more general forms given by $\langle a, a+s+d, a+s+2d, \ldots, a+s+kd \rangle$ and $\langle a, ah - k_1 d, ah - (k_1 - 1)d, \ldots, ah - d, ah + d, \ldots, ah + (k_2 - 1)d, ah + k_2 d \rangle$. The posets in Figure 9 and Figure 10 represent two semigroups of the form $\langle a, a+s+$

$d, a+s+2d, \ldots, a+s+kd\rangle$ generated by $S = \langle 23, 144, 218, 292, 366, 440, 415, 588\rangle$ and $S = \langle 24, 193, 330\rangle$ respectively.

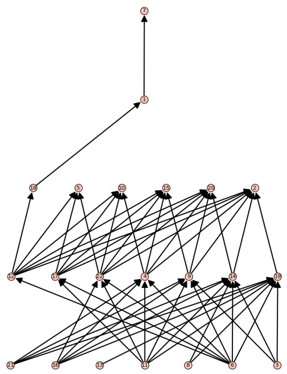

Figure 9: Kunz Poset for
$S = \langle 23, 144, 218, 292, 366, 440, 514, 588\rangle$

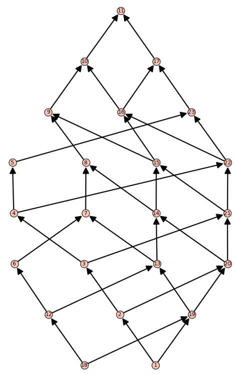

Figure 10: Kunz Poset for
$S = \langle 24, 193, 330\rangle$

We then have two posets of numerical semigroups of the form $\langle a, ah-k_1d, ah-(k_1-1)d, \ldots, ah-d, ah+d, \ldots, ah+(k_2-1)d, ah+k_2d\rangle$ seen in Figure 11 and Figure 12.

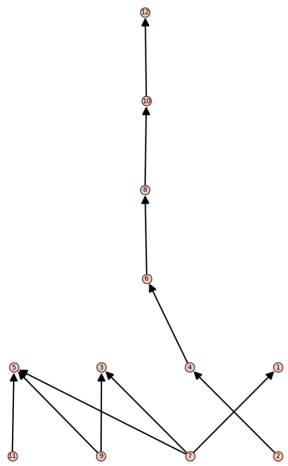

Figure 11: Kunz Poset for
$S = \langle 13, 67, 115, 139, 163\rangle$

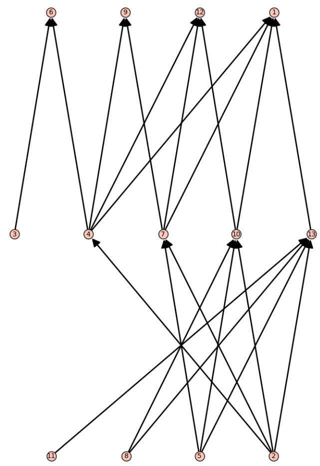

Figure 12: Kunz Poset for
$S = \langle 14, 30, 33, 36, 39, 45\rangle$

Although these seem like natural ways to extend our super-generalized arithmetic numerical semigroup family, we can see by the posets that they have a vastly different structure than what was discovered for our family. Thus, they would not be included. From here we make the following conjecture which currently remains an open question.

Conjecture 1.8. *The Kunz Generalized Arithmetic posets with the labeling removed are completely classified by the generalized arithmetic numerical semigroups and the pessimistic numerical semigroups of the super-generalized arithmetic numerical semigroup family.*

1.6 The Kunz Polyhedra Face to the Super-Generalized Arithmetic Numerical Semigroup

Now that we have fully classified the super-generalized arithmetic numerical semigroup family and the Kunz generalized arithmetic posets, the final concept we wish to explore in this chapter is the dimension of the face of the Kunz Polyhedra corresponding to each poset (and thus each numerical semigroup). This leads to our final claim for this family.

Proposition 1.9. *Let P be a Kunz generalized arithmetic poset of a super-generalized arithmetic numerical semigroup with embedding dimension $e(s) = k+1$ and multiplicity a. Then:*

1. *If $k+1 = a$, then the numerical semigroups live within the interior of the Kunz polyhedra.*

2. *If $k+1 \leq a-2$, then the numerical semigroup lives on 2-dimensional face of the Kunz polyhedra.*

3. *If $k+1 = a-1$, then the numerical semigroups live on a face of the Kunz polyhedra of dimension $\left\lfloor \frac{k}{2} \right\rfloor + 1$.*

Proof. (1) Trivial. (2) Since $k+1 \leq a-2$, and we have a Kunz generalized arithmetic poset, we know by Theorem 1.6 that the corresponding numerical semigroup belongs to the family of super-generalized arithmetic numerical semigroups. If we only had one minimal element there would be no way to tell how big the "jumps" are between elements, i.e. our value for d. Thus, we need at least two minimal elements. Suppose we had the minimal elements that have one and two edges coming out of them which we can call a_1, a_2 respectively. Since by the structure of our poset these are adjacent terms in our poset, $a_2 - a_1$ give us the difference between adjacent non-mulitplicity generators. Further, should our numerical semigroup be a generalized arithmetic numerical semigroup $a_1 = a_2 - a_1$. Otherwise, it must be the case that we have a pessimistic arithmetic numerical semigroup. Therefore, the face of the Kunz polyhedron corresponding to our generalized arithmetic poset has dimension at most 2. Coming from the reverse direction, consider the generalized arithmetic semigroups given by $S_0 = \langle a, ah+d, \ldots, ah+kd\rangle$, $S_1 = \langle a, ah+(d+a), \ldots, ah+k(d+a)\rangle$, and $S_2 = \langle a, a(h+1)+d, \ldots, a(h+1)+kd\rangle$, all of which have the same labeled poset and thus live on the same face of the Kunz polyhedron. We can calculate the Kunz tuples of these semigroups, which are given by $w_0 = (h + \lfloor \frac{d}{a} \rfloor, h + \lfloor \frac{2d}{a} \rfloor, \ldots, h + \lfloor \frac{kd}{a} \rfloor)$, $w_1 = (h+1+\lfloor \frac{d}{a} \rfloor, h+2+\lfloor \frac{2d}{a} \rfloor, \ldots, h+k+\lfloor \frac{kd}{a} \rfloor)$, and $w_2 = (h+1+\lfloor \frac{d}{a} \rfloor, h+1+\lfloor \frac{2d}{a} \rfloor, \ldots, h+1+\lfloor \frac{kd}{a} \rfloor)$, corresponding to S_0, S_1, and S_2 respectively. We can think of w_0, w_1, and w_2 as distinct integer points in the face of the Kunz polyhedron corresponding to our generalized arithmetic poset. Taking pairwise differences yields the vectors $w_1 - w_0 = (1, 2, \ldots, k)$ and $w_2 - w_0 = (1, 1, \ldots, 1)$. Clearly these vectors are not collinear as one cannot be written as an integer multiple of the other. Thus we have found two linearly independent vectors in our face, so its dimension must be at least 2. Therefore, the face of the Kunz polyhedron corresponding to our generalized arithmetic poset is exactly 2 when $k + 1 \leq a - 2$.

(3) Since $k + 1 = a - 1$, every element of our Kunz poset is minimal except for one. Then, by the structure of Kunz generalized arithmetic posets the single non-minimal element has an edge between itself and every minimal element. We then know by Theorem 1.6, there exists unique pairs of elements that sum to the maximum element (with one of the minimal elements being added to itself in the case where k is odd). It then follows that in order to fully construct an unlabeled poset we need one element from each pair that sums to the single maximal element plus the second element for one of the pairs of distinct elements adding to the maximal element to know what the maximal element is. Therefore, it follows that we need at least $\lfloor \frac{k}{2} \rfloor + 1$ minimal elements from the poset which must then live in a face with dimension at most $\left(\lfloor \frac{k}{2} \rfloor + 1\right)$. Coming from the reverse direction, we can consider the same S_0, S_1, and S_2 from (2) with $k = a - 2$, so we know we have at least 2 linearly independent vectors in our face. We construct more linearly independent vectors by adjusting pairs of generators in our generalized arithmetic semigroup $S = \langle a, ah+d, ah+2d, \ldots, ah+(a-2)d\rangle$ to create the sequence of semigroups ${S_1}' = \langle a, ah+d-a, ah+2d, \ldots, ah+(k-1)d, ah+kd+a\rangle$, ${S_2}' = \langle a, ah+d, ah+2d-a, \ldots, ah+(k-2)d, ah+(k-1)d+a, ah+kd\rangle, \ldots, {S_{\lfloor \frac{k}{2} \rfloor - 1}}' = \langle a, ah+d, ah+2d, \ldots, ah+(\lfloor \frac{k}{2} \rfloor - 1)d - a, \ldots, ah+(k - \lfloor \frac{k}{2} \rfloor + 2)d + a, \ldots, ah+kd\rangle$, all of which have the same labeled poset and

thus live on the same face of the Kunz polyhedron. The corresponding Kunz tuples are given by $w_1{}' = (h - 1 + \lfloor \frac{d}{a} \rfloor, h + \lfloor \frac{2d}{a} \rfloor, \ldots, h + \lfloor \frac{(k-1)d}{a} \rfloor, h + 1 + \lfloor \frac{kd}{a} \rfloor)$, $w_2{}' = (h + \lfloor \frac{d}{a} \rfloor, h - 1 + \lfloor \frac{2d}{a} \rfloor, \ldots, h + 1 + \lfloor \frac{(k-1)d}{a} \rfloor, h + \lfloor \frac{kd}{a} \rfloor), \ldots, w_{\lfloor \frac{k}{2} \rfloor - 1}{}' = (h + \lfloor \frac{d}{a} \rfloor, h + \lfloor \frac{2d}{a} \rfloor, \ldots, h - 1 + \left\lfloor \frac{(\lfloor \frac{k}{2} \rfloor - 1)d}{a} \right\rfloor, \ldots, h + 1 + \left\lfloor \frac{(k - \lfloor \frac{k}{2} \rfloor + 2)d}{a} \right\rfloor, \ldots, h + \lfloor \frac{(k-1)d}{a} \rfloor, h + \lfloor \frac{kd}{a} \rfloor)$, respectively. We can subtract w_0 from each of these tuples to obtain the vectors $(-1, 0, \ldots, 0, 1)$, $(0, -1, \ldots, 1, 0), \ldots, (0, 0, \ldots, -1, \ldots, 1, \ldots, 0, 0)$, which are clearly pairwise linearly independent. So we have found a total of $2 + (\lfloor \frac{k}{2} \rfloor - 1) = \lfloor \frac{k}{2} \rfloor + 1$ linearly independent vectors in our face, meaning its dimension is at least $\lfloor \frac{k}{2} \rfloor + 1$. Therefore the Kunz polyhedron face corresponding to the generalized arithmetic poset is exactly $\lfloor \frac{k}{2} \rfloor + 1$ when $k + 1 = a - 1$. $\square$

1.7 Containing Rays of Kunz Polyhedra Faces of Super-Generalized Arithmetic Numerical Semigroups

Finally we were explore the bounding rays of the 2-dimensional Kunz polyhedra faces containing super-generalized arithmetic numerical semigroups.

Theorem 1.10. *The first ray for the face containing the 2-dimensional Generalized Kunz posets corresponds to a total ordered poset corresponding to the numerical semgigroup generated by* $\langle a, a + (d \pmod a) \rangle$.

Proof. We begin by showing that every face of a Generalized Kunz Poset has a containing ray whose poset is a total ordering with minimal element $d \pmod a$. For simplicity, we can use the fact that every such kunz poset has a arithmetic semigroup of the form $S = \langle a, a + d, a + 2d, ..., a + kd \rangle$ where a, k, d are positive integers and $\gcd(a, d) = 1$. We can further restrict d so that $1 \leq d \leq a - 1$ by taking it $\bmod a$ without changing the poset structure and thus the face we are on.

Now let T be the numerical semigroup generated by $\langle a, a + d \rangle$. Since T has the same multiplicity as S they live in the same Kunz Polyhedron. Further, it is clear that T lives on a 1-dimensional ray of the Kunz Polyhedron. Since the poset for T is a total ordering it is easy to write all the cover relations it satisfies as

$$d \preccurlyeq 2d \preccurlyeq 3d \preccurlyeq ... \preccurlyeq (a - 2)d \preccurlyeq (a - 1)d.$$

By the structure defined for the poset structure of S it is clear that all the same relations are satisfied. Therefore, the ray that T lives on must be one of the containing rays for the face of S. $\square$

Theorem 1.11. *The second containing ray for the face containing the 2 dimensional Generalized Kunz poset of a super-generalized arithmetic numerical semigroup S takes one of the following forms:*

1. *If $k \mid (a-1)$, a totally ordered ray corresponding to the numerical semigroup generated by $\langle a, a + (kd \pmod a) \rangle$.*

2. *If $k \mid a$, a beta ray in the direction of*

$$(q_1, q_2, \ldots, q_k, q_1, q_2 \ldots, q_{k-1}),$$

where for all $(ah+id) \in \mathcal{A}(S) \setminus m(S)$, $q_{id \pmod k} = (k-i)$, repeated $\frac{a}{k}$ times with the final q_k excluded. Alternatively, this can be written as

$$\big(q_1, 2q_1, 3q_1, \ldots, (k-1)q_1, 0, q_1, 2q_1, \ldots, (k-1)q_1\big),$$

where $q = \big((k-1)\cdot d^{-1}\big) \pmod k$ and every k^{th} component of the direction vector is a 0 before repeating the pattern.

3. *Some type of "quasi-total ordering" that still needs to be fully classified, but is beyond the scope to fully do during the time we have left.*

Proof. For these proofs for simplicity we will again let S be the arithmetic numerical semigroup generated by $\langle a, a+d, a+2d, \ldots, a+kd \rangle$, where a, d, k are positive integers, $\gcd(a,d) = 1$ and $1 \le d \le a-1$.

(1) Let T be the numerical semigroup generated by $\langle a, a+kd \rangle$. Then, T clearly lives on a ray of the same Kunz polyhedron has S. Let $\frac{a-1}{k} = m$. Then, m represents the number of rows of the generalized Kunz poset of S. It is then easy to see that the relations satisifed by the Kunz poset of T can be written as

$$kd \preccurlyeq 2kd \preccurlyeq \ldots \preccurlyeq mkd \preccurlyeq (k-1)d \preccurlyeq (2k-1)d \preccurlyeq \ldots \preccurlyeq (mk-1)d \preccurlyeq \ldots \preccurlyeq d \preccurlyeq \ldots \preccurlyeq \big(mk-(k-1)\big)d.$$

Since $k \mid (a-1)$ it follows the the maximal row of the Kunz poset for S has the maximal number of elements, k, that it can have. We then know by the structure of generalized Kunz posets that elements $d, (k+1)d, \dots, \big(mk-(k-1)\big)d$ only satisfies the cover relations $d \preccurlyeq (k+1)d \preccurlyeq \dots \preccurlyeq \big(mk-(k-1)\big)d$. Similarly, the next column starting with $2d$ and incrementing up by kd only satisfy each other in order as well as the previous elements from the first column. Since this continues until the last element in each row relates to everything in the row above it, it follows that all the equations that satisfy the face of S are also satisfied by the ray of T. Therefore, the ray described by T is a containing ray for the face of S.

(2) We can first note that since $k \mid a$ and $\gcd(a, d) = 1$, it follows that $\gcd(k, d) = 1$. Let $G = \mathbb{Z}_a$ and $H = \{0, k, 2k, \dots\}$. Then, G is a group and since $k \mid a, H$ is a subgroup of G. Thus, G/H is a group isomorphic to $\mathbb{Z}_k$. Kaplan and O'Neill then showed in Theorem 3.3 of **CITATION NEEDED?** that the ray living in P_k with a totally ordered Kunz poset corresponds to a beta ray in P_a of the desired form.

We can then see that for each generator of S, save for the multiplicity, $a + id$ corresponds to the $id \pmod a$ index of the Kunz tuple for S. This same index then corresponds to $q_{id \pmod k}$ in the beta ray whose component is $k - i$. For our original Face of S we then have two possible general inequalities that could be met with equality: $x_{id} + x_{jd} = x_{(i+j)d}$ and $x_{id} + x_{jd} + 1 = x_{(i+j)d-a}$. In order for our beta ray to be a containing ray for the face of S it must then satisfy the same equations. We can then see that when $x_{id} + x_{jd} = x + (i+j)d$,

$$\begin{aligned} q_{id} + q_{jd} &= (k-i) + (k-j) \\ &= 2k - (i+j) \\ &= k - (i+j) \\ &= q_{(i+j)d}, \end{aligned}$$

satisfying the first equality. For the second case where $x_{id} + x_{jd} + 1 = x_{(i+j)d-a}$, we then similarly find that

$$\begin{aligned} q_{id} + q_{jd} &= (k-i) + (k-j) \\ &= 2k - (i+j) \\ &= k - (i+j) \\ &= q_{(i+j)d} \\ &= q_{(i+j)d-a}, \end{aligned}$$

satisfying the second equation.

Since each minimal element of the poset of S must satisfy being added to every element with a coefficient of d less than or equal to it, from the above equations and the structure of genearalized Kunz posets, it then follows that in order for $(i+j)$ to remain less than or equal to k for the necessary equations that we have a natural mapping from $ah + id$ to $q_{id} = k - i$ as described in the statement of the theorem.

For the alternative form we know from the structure of beta rays that the first term determines the increment between successive terms. Since $\gcd(k,d) = 1$, regardless if $\mathbb{Z}_k$ is a field or not we are guaranteed that the multiplicative inverse d^{-1} of d exists in $\mathbb{Z}_k$. Then, $q_1 = q_{dd^{-1}} = k - d^{-1} = (k-1) \cdot d^{-1}$. Therefore, our beta ray is a containing ray for super-generalized arithmetic numerical semigroups falling into the second case.

(3) For the final case things start getting really crazy real quick. We find that the ray attempts to attempt a total ordering similar to (1) using kd as the minimal generator. However, as we are moving up the columns of the original poset for the 2D face we end up breaking cover relations which forces sections of our total ordering to move down into a new column. One such example of this is the ray for the numerical semigroup generated by $\langle 8, 9, 10, 11, 12, 13\rangle$ as in Figure 13. However, this is a nicer case since kd is relatively prime to our multiplicity a.

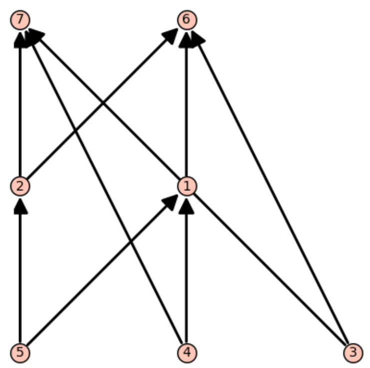

Figure 13: Bounding Ray for Face Containing $S = \langle 8, 9, 10, 11, 12, 13\rangle$

When it is not we find that we get multiple similar "towers" of "quasi-total ordering," such as in the rays for the 2D faces containing the numerical semigroups generated by $\langle 16, 23, 30, 37, 43, 50, 57\rangle$ and $\langle 22, 43, 64, 85, 106, 127, 148\rangle$ respectively as seen in Figure 14 and Figure 15. This towers then have numerous relations between them which although seem to be structured would take quite some time to fully classify everything that is happening. $\square$

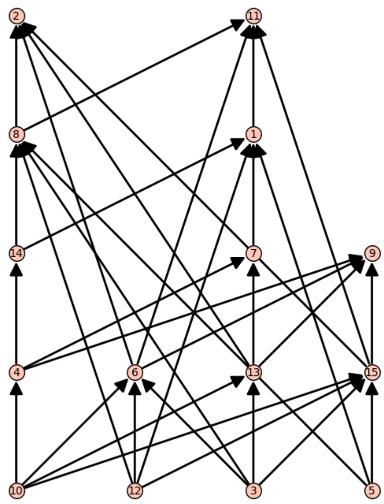

Figure 14: Bounding Ray for Face Containing $S = \langle 16, 23, 30, 37, 43, 50, 57\rangle$

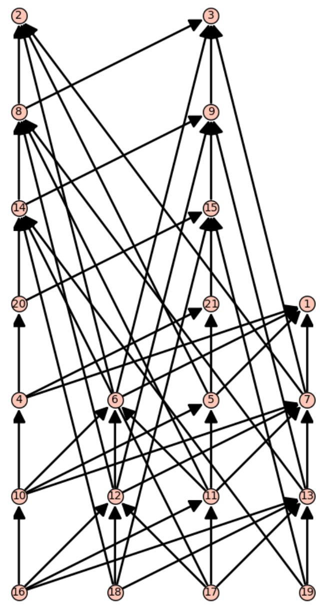

Figure 15: Bounding Ray for Face Containing $S = \langle 22, 43, 64, 85, 106, 127, 148\rangle$

2 Describing Apery Posets of Compound Sequence Semigroups

Let S be a numerical semigroup over a compound sequence (NSCS) if $S = \langle A \rangle$ such that $A = \{n_0, n_1 \ldots, n_p\}$ such that $n_i = b_1 \ldots b_i a_{i+1} \ldots b_p$ for natural numbers $a_1, a_2, \ldots a_p, b_1, b_2, \ldots, b_p$ that satisfy the following properties.

1. $2 \leq a_i < b_i$ for each $i \in [1, p]$

2. $\gcd(a_i, b_j)$ for all $i, j \in [1, p]$ with $i \geq j$

When $a_1 = a_2 = \ldots = a_p$ and $b_1 = b_2 = \ldots b_p$, we find a numerical semigroup generated by a geometric series. In particular, S is a numerical semigroup over a geometric series if $S = \langle A \rangle$ where $A = \{n_0, n_1 \ldots, n_p\}$ such that $n_i = ar^i$ for some $r \in \mathbb{Q}$.

For any numerical semigroup $S = \langle n_0, n_1 \ldots n_p \rangle$, there exists a monoid homomorphism $\phi : \mathbb{N}^{p+1} \to S$ such that $\phi(x_0, x_1, \ldots x_p) = \sum_{i=0}^{p} x^i n_i$. In particular, ϕ maps a *factorization* of an element of S to the element of S itself. As such, ϕ^{-1} maps an element of S to a variety of possible factorizations. By Kiers, O'Neil, and Ponomarenko, let the unique the i-normal factorization of x be $u \in \phi^{-1}(x)$ such that $0 \leq u_j < b_{j+1}$ for all $j < i$ and $0 \leq u_j < a_j$ for all $j > i$. In the following proof, let $\phi^{-1}(s)$ be equal to the i-normal factorization u.

Theorem 2.1. *Let S be a NSCS generated by some $A = \{n_0, n_1 \ldots, n_p\}$ where $n_i = b_1 \ldots b_i a_{i+1} \ldots b_p$. We find that the amount of elements in the k^{th} row is given by the coefficient of x^k in the following polynomial expansion*

$$\prod_{i=1}^{p} \left(\sum_{j=0}^{a_i - 1} x^j \right).$$

Proof. By Kiers, O'Neil, and Ponomarenko, it turns out that $\mathrm{Ap}(S, n_0) = \{\phi(u) : u = (u_0, u_1 \ldots u_p) \in S$ such that $S = \{u \in \mathbb{N}_0^{p+1} : u_0 = 0, u_1 < a_1, \ldots, u_p < a_p\}$ [1]. Now note that $n_i = \phi(u)$ such that $u_j = 0$ for all $j \neq i$ and $u_i = 1$ for $j = i$. It follows that $a_i n_i = \phi(u')$ such that $u'_j = 0$ for all $j \neq i$ and $u'_i = a_i$ for $j = i$. Since $u_i < a_i$ for all members of the Apery Set, it follows that $a_i n_i \notin \mathrm{Ap}(S, m(S))$. Therefore, the atom n_i can occur a maximum of $a_i - 1$ times. We can encode this information in a generating function such that the exponent of any of the terms in the polynomial $(1 + x + x^2 + \ldots x^{a_i - 1})$ represents the amount of times n_i is included in the factorization of the element. Note that we stop at x^{a_i-1} since we just showed that n_i cannot occur a_i times. Therefore, the product of these polynomials yields another polynomial where the degree of any term represents the cardinality of indistinct atoms added together. It follows that the coefficient of x^k counts linear combinations with k indistinct atoms.

We now show that this generating function counts all the elements of the poset. We can count the amount of terms that this generating function counts by finding the sum of the coefficients. Since the sum of the coefficients can be found by substituting $x = 1$, we find that the generating function counts

$$\prod_{i=1}^{p} \left(\sum_{j=0}^{a_i - 1} 1 \right) = \prod_{i=1}^{p} (a_i) = n_0$$

elements. Since this is exactly the amount of elements in the poset, we know that all these combinations must exist and that the generating functions accurately count the number of elements in each row as required. It follows that the Hasse diagram for S has dimensions $a_1 \times a_2 \ldots \times a_n$.

$\square$

Theorem 2.2. *Let S be a NSCS. We know that $s \in S$ if and only if we can write as $s = \phi(u)$ such that $u_0 \geq 0$ and $0 \leq u_i < a_i$.*

Proof. ($\Longrightarrow$): Suppose that $s \in S$. Recall that the elements of $\mathrm{Ap}(S)$ represent all equivalence classes modulo a, the multiplicity. Since every $s \in S$ is a representative of some equivalence class modulo a and the Apery Set consists of the minimum element for each equivalence class, we can write it $s = n + ak$ for some $n \in \mathrm{Ap}(S)$ and $k \in \mathbb{N}_0$. Now recall that $n = \phi(u)$ for some $u = (u_0, u_1 \ldots u_n)$ such that $u_0 = 0$ and $0 \leq u_i < a_i$ for all $i \in [1, n] \cap \mathbb{Z}$ by the result from O'Neil, Ponomarenko, and Kiers. Therefore, $s = n + ak = \phi(u')$ for some u' such that $u'_0 = k \geq 0$ and $0 \leq u'_i < a_i$ for $i \in [1, n] \cap \mathbb{Z}$. Hence, we can write s in the desired

form as required.
($\Longleftarrow$): Suppose that $s = \phi(u)$ such that $u_0 \geq 0$ and $0 \leq u_i < a_i$ such that $u_i \neq 0$. Since s is a linear combination of strictly atoms, we know that it must exist in S as required.

$\square$

3 Telescopic Gluings

Consider a numerical semigroup S. We let T be a telescopic gluing of S if $T = \langle\{\alpha\} \cup \beta \cdot \mathcal{A}(S)\rangle$ such that $\alpha \in S$, $\alpha \notin \mathcal{A}(S)$, $\beta \in \mathbb{N}_{\geq 2}$, and $\gcd(\alpha, \beta) = 1$. -

• I should have commented on this notation earlier: either write $T = \alpha\langle 1\rangle + \beta S$ or $T = \langle\{\alpha\} \cup \beta \cdot \mathcal{A}(S)\rangle$. As written, T is not a numerical semigroup.

3.1 Apery Set and Poset

Theorem 3.1. *We claim that a telescopic gluing $T = \langle\{\alpha\} \cup \beta \cdot \mathcal{A}(S)\rangle$, such that $m(T) = \beta \cdot m(S)$, has Apery Set*

$$\mathrm{Ap}(T, m(T)) = \{t_{b,s} | s \in \mathrm{Ap}(S), b \in [0, \beta - 1] \cap \mathbb{Z}\}$$

such that $t_{b,s} = b\alpha + s\beta$.

Proof. We start by showing that all elements in the set $G = \{t_{b,s} | s \in \mathrm{Ap}(S), b \in [0, \beta-1] \cap \mathbb{Z}\}$ are contained in $\mathrm{Ap}(T)$. Suppose that there exists some $t_{b,s} \in G \subseteq T$ such that $t_{b,s} \notin \mathrm{Ap}(T)$. It follows that $t_{b,s} - m(T) = t_{b,s} - \beta m(S) = t_{b,s-m(S)} \in T$. Since $s \in \mathrm{Ap}(S)$, we also know that $s - m(S) \notin S$.

If $t_{b,s} - m(T) = t_{b,s-m(S)} \in T$, it follows that $t_{b,s-m(S)} = k \cdot \alpha + \beta \cdot n$ for $n = \left(\sum_{i=0}^{|\mathcal{A}(S)-1|} c_i g_i\right)$ for $c_i, n \in \mathbb{N}_0$ and $g \in \mathcal{A}(S)$. Recall that $\alpha \in S$ and $k = q \cdot \beta + r$ for $q \in \mathbb{Z}$ and $r \in [0, \beta - 1] \cap \mathbb{Z}$ by the division algorithm. Since $b \cdot \alpha = (q \cdot \beta + r) \cdot \alpha$, we can consume $q\beta\alpha$ inside n and rewrite our equality as $t_{b,s-m(S)} = r \cdot \alpha + \beta n'$ where $n' = n + q\alpha$. As such, we find that $b \cdot \alpha + (s - m(S))\beta = r \cdot \alpha + \beta n'$. Hence, $(b - r) \cdot \alpha = (n' - (s - m(S))\beta$. Since $\gcd(\alpha, \beta) = 1$, we need $\beta | (r - b)$. However, this is only possible when $r = b$ because $r, b \in [0, \beta - 1] \cap \mathbb{Z}$. Hence, both sides must be equal to 0. However, $s - m(S) \neq n'$ because $n' \in S$ but $s - m(S) \notin S$ because $s \in \mathrm{Ap}(S)$. Therefore, we have a contradiction and hence $t_{b,s} \in \mathrm{Ap}(T)$ as required.

It follows that $G \subseteq \mathrm{Ap}(T)$. Since $|\mathrm{Ap}(T)| = m(T) = \beta m(S)$ and $|G| = \beta |\mathrm{Ap}(S)| = \beta m(S)$, we find that G must be exactly the set $\mathrm{Ap}(T)$ as required. $\square$

Theorem 3.2. *Consider a telescopic gluing $T = \langle\{\alpha\} \cup \beta \cdot \mathcal{A}(S)\rangle$ where $m(T) =$*

$\beta \cdot m(S)$. *We claim that* t_{b_1,s_1} *and* t_{b_2,s_2} *have a relation if and only if one the following rules is satisfied.*

• This appears to have not been updated to account for "all relations" instead of "cover relations" only.

1. $s_1 = s_2$ *and* $b_1 - b_2 = 1$

2. $s_1 - s_2 \in \mathcal{A}(S)$ *and* $b_1 = b_2$

3. $b_1 = 0$, $b_2 = \beta - 1$, *and* $s_1 - s_2 = \alpha$.

Proof. Recall that t_{b_1,s_1} relates t_{b_2,s_2} with $t_{b_2,s_2} \preccurlyeq t_{b_1,s_1}$ if and only if $t_{b_1,s_1} - t_{b_2,s_2} = \alpha$ or $t_{b_1,s_1} - t_{b_2,s_2} = \beta \cdot a$ for some $a \in \mathcal{A}(S)$. We show that these conditions exactly describe the rules above.

First we suppose that $t_{b_1,s_1} - t_{b_2,s_2} = \alpha$. It follows that $t_{b_1,s_1} - t_{b_2,s_2} = \alpha = t_1$. It follows that $t_{b_1-b_2,s_1-s_2} = t_{1,0}$. Therefore, $t_{0,s_1-s_2} = t_{1-(b_1-b_2),0}$. Since $\gcd(\alpha,\beta) = 1$, we need $\alpha|(s_1-s_2)$ and $\beta|(1-(b_1-b_2))$ to be true. Since $b_1, b_2 \in [0, \beta-1] \cap \mathbb{Z}$, we find the inequality $-(\beta-1) < b_1 - b_2 < \beta - 1$. It follows that $1 - (b_1 - b_2) = 0$ or β only. Hence, $t_{0,s_1-s_2} = t_{(1-(b_1-b_2),0} = 0$ or $t_{0,s_1-s_2} = t_{(1-(b_1-b_2),0} = \beta$. We consider these two cases separately.

Suppose that $t_{0,s_1-s_2} = t_{1-(b_1-b_2),0} = 0$. It follows that $1 - (b_1 - b_2) = 0$. Therefore, we know that $b_1 - b_2 = 1$. Furthermore, note that $s_1 = s_2$. Hence, this case describes our first family of cover relations. Now suppose that $t_{0,s_1-s_2} = t_{1-(b_1-b_2),0} = \beta$. It follows that $b_1 - b_2 = -(\beta - 1)$. Furthermore, $s_1 - s_2 = \alpha$ in order to make both sides equal to $\alpha\beta$. Since $b_1, b_2 \in [0, \beta - 1] \cap \mathbb{Z}$, we see that $b_1 = 0$ and $b_2 = \beta - 1$. Hence, this case describes our third family of cover relations.

Now suppose that $t_{b_1,s_1} - t_{b_2,s_2} = a\beta$ for $a \in \mathcal{A}(S)$. It follows that $t_{b_1,s_1} - t_{b_2,s_2} = t_{0,a}$. It follows that $t_{(b_1-b_2),(s_1-s_2)} = t_{0,a}$, and therefore $t_{(b_1-b_2),0} = t_{0,a-(s_1-s_2)}$. Since $\gcd(\alpha,\beta)$, we need $\beta|(b_1-b_2)$. However, this is not possible unless $b_1 = b_2$ because $-(\beta-1) < b_1 - b_2 < \beta - 1$ because $b_1, b_2 \in [0, \beta-1] \cap \mathbb{Z}$. Hence, both sides must be equal to 0 and it follows that $a - (s_1 - s_2) = 0$. As such, $s_1 - s_2 = a \in \mathcal{A}(S)$. Therefore, this case describes our second family of cover relations.

We have shown that the criterion for a cover relation exactly corresponds to the three families of relationships that are described above as required.

□

We describe our Apery Set as $\mathrm{Ap}(T, m(T)) = \{t_{b,s} | s \in \mathrm{Ap}(S), b \in [0, \beta - 1] \cap \mathbb{Z}\}$ because it allows to express the Apery set as a Cartesian product of the Hasse diagram for S and some $b \in [0, \beta - 1] \cap \mathbb{Z}$. Indeed, the poset structure can be geometrically interpreted as such a Cartesian product with a few additional edges. For example consider the following poset diagrams for S and T seen in Figures 16 and 17, where $T = \langle 15, 18, 20, 27 \rangle = 20 \cup 3\langle 5, 6, 9 \rangle$ and $S = \langle 5, 6, 9 \rangle$.

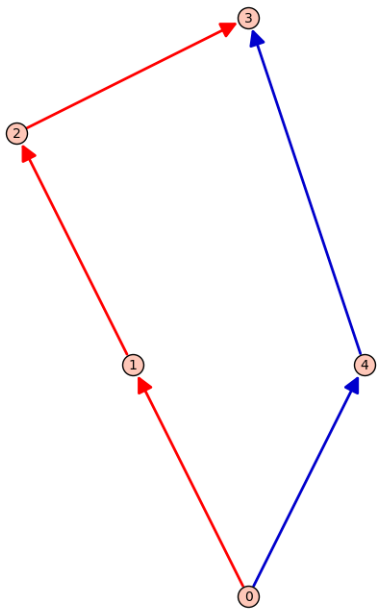

Figure 16: Kunz Poset for $S = \langle 5, 6, 9 \rangle$

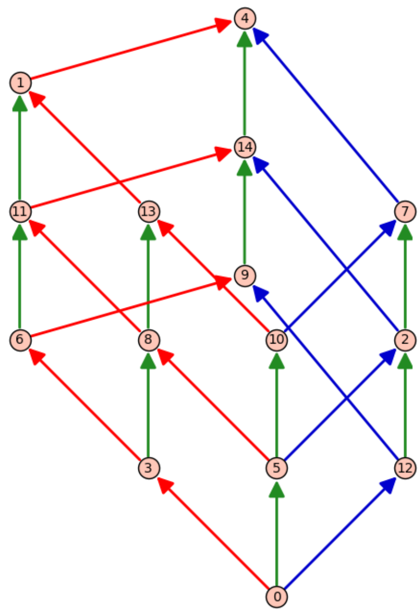

Figure 17: Kunz Poset for $T = \langle 15, 18, 20, 27 \rangle$

Let two poset diagrams be isomorphic if they are described by the exact same cover relations. We can see 3 isomorphic copies of the poset structure for S appear in the poset structure for T. In general, there will be β isomorphic copies of S that are included in T. Furthermore, the green edges connect the elements of β that have the same placement on the isomorphic copies of S. We describe this structure as a "slicing," where each copy of a S is a "slice."

Note that our first rule, where $s_1 = s_2$ and $b_1 - b_2 = 1$, describe the green cover relations between elements that are on different slices but have the same position on the slices. As described in the proof, these elements differ by α. Our second rule, where $s_1 - s_2 \in \mathcal{A}(S)$ and $b_1 = b_2$, describe the remaining cover relations that correspond to the cover relations in the original poset structure S.

While this poset diagram doesn't have additional cover relations, there exist additional cover relations when there are $s_1, s_2 \in \text{Ap}(S)$ such that $s_1 - s_2 = \alpha$. Furthermore, these cover relations start from the first copy of the original poset diagram and extend to the last copy of the original poset diagram as demonstrated by the third rule that shows $b_1 = 0$ and $b_2 = \beta - 1$.

3.2 Occurrence of additional edges in Kunz Poset

At previously mentioned there are cases where extra edges exist from the last slice to the first slice. For example, consider the poset diagrams for S and T, where $T = \langle 25, 28, 35\rangle = 28 \cup 5\langle 5, 7\rangle$ and $S = \langle 5, 7\rangle$. This poset has an additional red cover relation from $112 \equiv 12 \pmod{25}$ to $140 \equiv 15 \pmod{25}$ as seen in Figures 18 and 19.

Now note that $112 = \beta s_1 = 7 \cdot 16$. Furthermore $140 = \beta \cdot s_1 = 5 \cdot 28$ and $112 = \alpha \cdot (\beta - 1) + \beta \cdot s_1 = 28 \cdot 4 + 5 \cdot 0$. Hence, $s_1 - s_2 = 28 = \alpha$ as required. In general, we can find all such extra edges by finding all s_1, s_2 such that $s_1 - s_2 = \alpha$ and then including cover relations between $t_{s_1,0}$ and $t_{s_2,\beta-1}$.

It is important to note that posets with the same slicing structure don't necessarily have the extra edges. For example, consider the poset diagrams for S and T, where $T = \langle 25, 128, 35\rangle = 128 \cup 5\langle 5, 7\rangle$.

In this case, the s_1 and s_2 values in the decomposition of 25 and 128 don't differ by α. It follows that the Kunz posets aren't necessary the same. While the Apery Poset has an extra edge for any pair $s_1, s_2 \in \text{Ap}(S)$ such that $s_1 - s_2 = \alpha$, Kunz posets that correspond to the slicing structure of the telescopically glued semigroup do not necessarily have that extra edge. We can prove that either all or none of the edges occur. In particular, we can characterize when all of the edges occur.

Theorem 3.3. *Suppose that there exist $t_{s_1,b_1}, t_{s_2,b_2} \in \text{Ap}(T)$ such that $s_1 - s_2 \equiv \alpha \pmod{m(S)}$, $b_1 = 0$ and $b_2 = \beta - 1$. Then $t_{s_1,b_1} - t_{s_2,b_2} = \alpha$ iff $\alpha \in \text{Ap}(S)$ and*

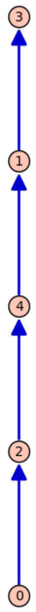

Figure 18: Kunz Poset for $S = \langle 5, 7 \rangle$

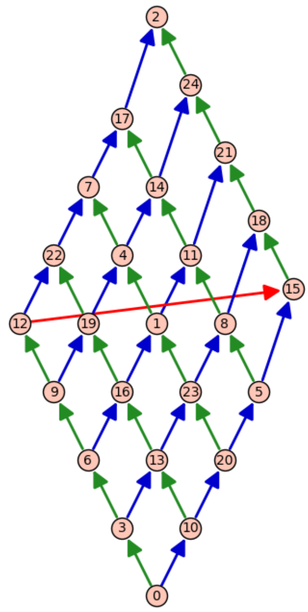

Figure 19: Kunz Poset for $T = \langle 25, 28, 35 \rangle$

$s_1 - s_2 = \alpha$.

• This should also be updated to account for all extra relations, not just the cover relations. A separate result can prove that the "extra" cover relations are precisely the extra relations that reach from the last slice back to the first.

Proof. ($\Longrightarrow$): Suppose that $t_{s_1,b_1} - t_{s_2,b_2} = \alpha$. It follows that $\beta(s_1 - s_2) + \alpha(b_1 - b_2) = \alpha$. We can substitute our values of b_1 and b_2 to find $\beta(s_1 - s_2) + \alpha(-(\beta - 1)) = \alpha$. Hence, $s_1 - s_2 = \alpha$ as required. Recall that $\alpha \in S$ by the definition of a telescopic gluing. Suppose $\alpha \notin \mathrm{Ap}(S)$. it follows that $\alpha - m(S) \in S$. Hence, $s_1 - s_2 - m(S) \in S$, and it follows by additive closure that $s_1 - m(S) \in S$. However, that contradicts our claim that $s_1 \in \mathrm{Ap}(S)$. Hence, $\alpha \in \mathrm{Ap}(S)$ as required.

($\Longleftarrow$): Suppose that $\alpha \in \mathrm{Ap}(S)$ and $s_1 - s_2 = \alpha$. Then $t_{s_1,b_1} - t_{s_2,b_2} = \beta(s_1 - s_2) + \alpha(b_1 - b_2) = \beta\alpha + \alpha(-(\beta - 1)) = \alpha$ as required.

Remark: Since $\alpha \in \mathrm{Ap}(S)$ when we have additional edges, it follows that $\alpha\beta \in \mathrm{Ap}(T)$. Therefore, we have a minimal trade in $\mathrm{Ap}(T)$. □

For all relations, we can generalize that extra relations from the last slice to the first slice exist iff $\alpha \in \mathrm{Ap}(S)$.

Theorem 3.4. *$t_{b_2,s_2} \preceq t_{b_1,s_1}$ such that $b_1 < b_2$ iff $\alpha \in \mathrm{Ap}(S)$.*

Proof. Suppose that $\alpha \in \mathrm{Ap}(S)$ and $t_{b_1,s_1} - t_{b_2,s_2} = t_{b_3,s_3} \in \mathrm{Ap}(T)$. It follows that $\beta(s_1 - s_2) + (b_1 - b_2)\alpha = \beta \cdot s_3 + \alpha \cdot b_3$. Then if we let $s_1 = \alpha, s_2 = 0, b_1 = 0, b_2 = \beta - 1$, we find that $t_{b_1,s_1} - t_{b_2,s_2} = \beta\alpha + (-\beta + 1)\alpha = 1 \cdot \alpha$ and $1 \in [0, \beta - 1]$. Hence, the relation exists as required.
Suppose $t_{b_2,s_2} \preccurlyeq t_{b_1,s_1}$. Then $t_{b_1,s_1} - t_{b_2,s_2} = t_{b_3,s_3} \in \mathrm{Ap}(T)$. Hence $\beta(s_1 - s_2) + (b_1 - b_2)\alpha = \beta \cdot s_3 + \alpha \cdot b_3$. It follows that $\beta(s_1 - s_2 - s_3) = \alpha(b_3 + b_2 - b_1)$. Since $\gcd(\alpha, \beta) = 1$, it follows that $\alpha|(s_1 - s_2 - s_3)$ and $\beta|(b_3 + b_2 - b_1)$. □

Theorem 3.5. *Consider $T = \{\alpha\} \cup \beta\mathcal{A}(S)$ such that $m(T) = \beta m(S)$. Suppose that $\dim(S) = n$. We show that the upper bound of the dimension is $\dim(T) = n + 1$ when there are no extra edges and $\dim(T) = n$ when there are extra edges.*

Proof. We first find the upper bound for the dimension by counting how many elements we need to choose so that we can determine the entire poset.

First we consider the case where there are no extra edges that satisfy the third condition in Theorem 3.2. By the poset structure we determined earlier, we must choose all the elements that determine the poset of the slice. That gives us n elements. After that we need to choose an additional element that is connected to an element in the original slice by a difference of α so that we can determine α and use that to compute all the other elements in the poset. It follows that our upper bound on the maximum dimension of T is $n + 1$ when there are no extra edges as required.

□

3.3 Dimension of a Telescopic GULAG for any Beta

• What on earth is a GULAG???

In this section, we use the properties of the extremal rays that bound faces of the Kunz polyhedron to determine the dimension of a face resulting from a telescopic gluing.

• This map does not require a semigroup, it maps *all* rays of .

Definition 3.6. Consider a ray that bounds a certain face of the Kunz polyhedron and consider a telescopic gluing such that $\gcd(\alpha, \beta) = 1$. Let R_1 denote the set of rays that bound the original face and R_2 denote the set of rays that bound the new face. We will define G_1 and G_2 to be the sets of corresponding ray generators for the sets of extremal rays R_1 and R_2, which bound the faces of the Kunz polyhedron. In particular, R_1 is the set of rays that bound F and R_2 is the set of rays that bound F'. We augment each $\vec{w} \in G_1$ and each $\vec{x} \in G_2$ by inserting 0 as the first coordinate (w_0 or x_0, respectively) in order to simplify our mapping. We claim that there exists a bijective map $f_{\alpha,\beta} : G_1 \to G_2$, such that

• The "[]" and "()" notation here seems asymmetric. . .

• I would name the map $f_{\alpha,\beta}$ so that we get a different map for each fixed α and β that sends rays to rays.

$$f_{\alpha,\beta}(w_0, w_1, \dots, w_{m-1}) = (x_0, x_1, \dots x_{\beta m-1}),$$

where $(w_0, w_1, \dots, w_{m-1}) \in G_1$ and $(x_0, x_1, \dots x_{\beta m-1}) \in G_2$. Furthermore, we let $\alpha \equiv \rho \pmod{\beta m}$ such that $\rho \in [1, \beta m - 1] \cap \mathbb{Z} \setminus \{m\}$ and $gcd(\rho, \beta) = 1$. Furthermore, for all $k \in \mathbb{Z}_\beta$, we let

$$x_{\beta i + k\rho} = \frac{\beta w_i + k w_\rho}{\gcd(\beta, w_\rho)}.$$

Since the construction of the new ray relies entirely on the elements of the previous ray, it follows that this mapping is well-defined. Furthermore, we treat the indices of the coordinates of $\vec{w}$ as elements of $\mathbb{Z}_m$, and the indices of the coordinates of $\vec{x}$ as elements of $\mathbb{Z}_{\beta m}$.

We now prove some of the properties of this mapping.

Theorem 3.7. *The map provided in Definition 3.6 is injective.*

Proof. Suppose we have $\vec{w}, \vec{w}' \in G_1$ and some fixed α, β. For $\vec{w}$ we then have that for any index i, that

$$x_{\beta i + k\rho} = \frac{\beta w_i + k w_\rho}{\gcd(\beta, w_\rho)}.$$

As all of our variables are now "fixed" except for i this is simply a linear function. The same holds for $\vec{w'}$. Since two lines either intersect at every point, exactly one point, or never, it then follows that in order for $\vec{w}, \vec{w'}$ to map to the same ending ray, they must have been the same ray to begin with. Therefore, our mapping is injective. $\square$

We now show that the new ray does not live in the middle of a higher dimensional face by using the facet equations of F to show that one coordinate uniquely determines the rest of the coordinates.

Theorem 3.8. *The face containing $\vec{x}$ in its interior has dimension 1.*

• If you define G_2 to be a set of ray generators, this theorem has no content. Do you mean for G_2 to be the image of G_1 under f?

• I don't understand "ray generator of the bounding ray".
In general, the theorem statements are way to verbose. Make a precise mathematical claim, e.g. "The face containing $\vec{x}$ in its interior has dimension 1", not a declaration of what you intend to do or how you intend to do it (that part goes in the proof).

Proof. Let G_2 be the image of G_1 under f and let G_2 contain $\vec{x}$ in its interior. Suppose that we are given x_ρ. By Definition 3.6, we know that $\beta x_\rho = x_{\beta\rho}$. We know that w has dimension 1, meaning that all of its coordinates can be uniquely determined from any w_i. It follows that we can determine $x_{\beta i + k\rho}$ by simply finding the appropriate values of i and k, and then substituting the value for w_ρ.

We now show that every element of the ray is generated with $\beta i + k\rho$. Since $k \in \mathbb{Z}_\beta$ and $i \in [0, m-1] \cap \mathbb{Z}$, there are a total of $|\mathbb{Z}_\beta||[0, m-1] \cap \mathbb{Z}| = \beta \cdot m = m\beta$ possibilities.

We now show that $\beta i' + k'\rho = \beta i + k\rho$ if and only if $i = i'$ and $k = k'$. Suppose that $\beta i' + k'\rho = \beta i + k\rho$. Hence $\beta(i' - i) = \rho(k' - k)$. Since $\gcd(\beta, \rho) = 1$, we know that $\beta|(k - k')$. However, this is impossible unless $k - k' = 0$ because $k, k' \in \mathbb{Z}_\beta$. Hence $k = k'$ and it follows that $i = i'$. Since every pair of i and k gives a distinct value, it follows that all $m\beta$ elements of the ray are covered as required.

Since all elements of the new ray can be found by knowing just one element of $\vec{w}$, and $\vec{w}$ has dimension 1, it follows that the new dimension is also 1 as required. ☐

We will now use the poset structure and the corresponding Kunz inequalities to show that all of the relations of the face F' are satisfied by all of the rays. This proof shows that the posets of the face are contained within the posets of the rays.

• I don't understand this last sentence.

Theorem 3.9. *All facet equations satisfied by F' are satisfied by $\vec{x} = f_{\alpha,\beta}(\vec{w})$ for all $\vec{w}$ that bound F.*

• Is the previous theorem supposed to be a special case of this one?

Proof. We start by finding all facet equations satisfied by F'. The set of equations satisfied by the poset of a telescopic gluing with extra edges can be partitioned into two groups. The first group represents the relations between elements in the same slice, or from an element in some slice to an element in a higher slice. The second group represents relations that occur between some element of a higher slice and an element in a lower slice. Now we show they are satisfied by all $\vec{x}$ contained in the interior of F'.

• Large equations should be centered, *especially* if they have big fractions.

We can characterize the relations in the first group by $x_{\beta a+k\rho} + x_{\beta b+(j-k)\rho} = x_{\beta(a+b)+j\rho}$ for $k, j \in [0, \beta-1] \cap \mathbb{Z}$ such that $k \le j$. Applying the map given in Definition 3.6 to the left side yields

$$\frac{(\beta w_a + kw_\rho)}{\gcd(\beta, w_\rho)} + \frac{(\beta w_b + (j-k)w_\rho)}{\gcd(\beta, w_\rho)};$$

applying the same mapping to the right side yields

$$\frac{\beta w_{a+b} + jw_\rho}{\gcd(\beta, w_\rho)}.$$

We see that $(\beta w_a + kw_\rho) + (\beta w_b + (j-k)w_\rho) = \beta w_{a+b} + jw_\rho$ if and only if $w_a + w_b = w_{a+b}$.

We characterize the relations in the second group by

$$x_{\beta a+k\rho} + x_{\beta b+j\rho} = x_{\beta(a+b)+(j+k)\rho}$$

where $j + k \ge \beta$, so we can rewrite the right side as $x_{\beta(a+b+\rho)+(j+k-\beta)\rho}$. Applying our map to the left hand side yields

$$\frac{(\beta w_a + kw_\rho)}{\gcd(\beta, w_\rho)} + \frac{(\beta w_b + jw_\rho)}{\gcd(\beta, w_\rho)};$$

applying the map to the right hand side gives

$$\frac{\beta w_{a+b+\rho} + (j+k-\beta)w_\rho}{\gcd(\beta, w_\rho)}.$$

We seek to determine the conditions needed for

$$(\beta w_a + kw_\rho) + (\beta w_b + jw_\rho) = \beta w_{a+b+\rho} + (j+k-\beta)w_\rho$$

to be true. We can subtract $(j+k)w_\rho$ from each side and divide by β to yield $w_a + w_b + w_\rho = w_{a+b+\rho}$, which is true precisely when $w_a + w_b = w_{a+b}$ and

$w_{a+b}+w_\rho = w_{a+b+\rho}$. Thus an equation satisfied by the poset of a telescopic gluing with extra edges is satisfied by our new ray x if and only if $a \preccurlyeq a+b \preccurlyeq a+b+\rho$, which covers all the relations in our original ray w. □

Corollary 3.10. *The relation $\beta x_\rho = x_{\beta\rho}$ holds for any $\vec{x} \in G_2$.*

Proof. We can rewrite βx_ρ as $x_\rho + x_\rho + \ldots + x_\rho$, where there are β many terms in the expression. We collapse the expression, taking pairwise sums from the left using the first set of relations described in the proof above with $a = b = 0$, so $x_\rho + x_\rho + \ldots + x_\rho = x_{2\rho} + x_\rho + \ldots + x_\rho = \cdots = x_{(\beta-1)\rho} + x_\rho$. For the final sum we must use the second set of relations from the proof above since $j + k \geq \beta$, so the expression reduces to $x_{(\beta-1)\rho} + x_\rho = x_{\beta\rho}$. Thus we have that $\beta x_\rho = x_{\beta\rho}$ exists as a relation for any $\vec{x} \in G_2$. □

By showing that linear independence or dependence is preserved by the mapping, we can conclude that $\dim F \leq \dim F'$.

Theorem 3.11. *We claim that a collection of old rays is linearly independent if and only if their images are linearly independent.*

• Old sentence?

Proof. ($\Longrightarrow$) : Suppose that the collection of old rays in linearly independent. In general, suppose that a set of vectors $W = \{w_1, w_2, \ldots w_n\}$ is linearly independent. It follows that

$$\sum_{i=0}^{n} c_i w_i = 0$$

if and only if $c_i = 0$. It follows that $kW = \{kw_1, kw_2 \ldots kw_n\}$ is also linearly independent because

$$\sum_{i=0}^{n} c_i k w_i = 0 \iff \sum_{i=0}^{n} c_i w_i = 0.$$

Therefore, it follows that $c_i = 0$ for all i again. Now we show that $X =$

$\{x_1, x_2, \ldots x_n\}$ is also linearly independent. Suppose there exists some set of constants k_i such that

$$\sum_{i=0}^{n} k_i x_i = \vec{s} = 0$$

It follows that $s_i = 0$ for all $i \in [1, n] \cap \mathbb{Z}$. Furthermore, when $\beta | k$, we have $s_k = w_{\frac{k}{\beta}}$. Since kW is, a linearly independent set of vectors and vectors that are made from the components s_k are exactly kW,it follows that all the coefficients must be 0. Hence, the set of new rays is linearly independent as required.

• Again, using symbols is *way* more clear. Extended descriptions should only be used alongside symbols, as clarification.

($\Longleftarrow$): We show the contrapositive. Suppose that the original set of rays is linearly dependent and furthermore recall that $x_{\beta i + k\rho} = \frac{\beta w_i + k w_\rho}{\gcd(\beta, w_\rho)}$. Since the $\gcd(\beta, w_\rho)$ is fixed, it follows that we are left with the linear equation $x_{\beta i + k\rho} = \beta w_i + k w_\rho$. Since this is linear and injective, it follows that the linearly dependent vectors are mapped to linearly dependent vectors as required. $\square$

• Where do we write down explicitly what the beta ray is? This should be done *before* using that phrase.

Let our $\beta - ray$ be given by $\vec{b}$. Let $b_{k\rho} = n$, where n is the minimal representative of the equivalence class that corresponds to k modulo β and for $k \in \mathbb{Z}$. ‘

By showing that the β ray does not live in the span of the new ray, we can conclude that $\dim F + 1 \leq \dim F'$.

Theorem 3.12. *The β ray does not live in the span of the new ray.*

Proof. By Kaplan and O'Neill, we find that $G/H = \mathbb{Z}_\beta$. It follows that our "beta-ray" is such that the coordinate value for all indices that are a multiple of β (starting with 0) are 0.

Suppose that our β-ray denoted as $\overline{s}$ lives in the span of the new rays. Then there exist coefficients such that for $\overline{x}_h \in R_2$,

$$\sum_{h=0}^{n} c_h x_h = \vec{s}$$

such that $s_{\beta i} = 0$ for any $i \in [0, m] \cap \mathbb{Z}$. Since $x_{h_{\beta i}} = kw_{h_i}$ for any $x_h \in R_2$ and any corresponding $v_h \in R_1$, it follows that there exist coefficients such that

$$\sum_{h=0}^{n} c_h w_h = 0.$$

Since we can write $\vec{x}$ as a linear transformation of $\vec{w}$, such that $T(\vec{v}) = \vec{x}$, applying the linear transformation to the equation gives

$$T\left(\sum_{h=0}^{n} c_h w_h\right) = \sum_{h=0}^{n} c_h T(w_h) = \sum_{h=0}^{n} c_h x_h = T(\vec{0}) = \vec{0}.$$

However, $\vec{s} \neq \vec{0}$, and we have a contradiction as required.

□

When there are no extra edges, we can show that the dimension increases by 1 and conclude that $\dim F' \leq \dim F + 1$, which shows that $\dim F' = \dim F + 1$. However, when there are extra edges in the poset structure (see Thm 3.6), we can only show that $\dim F' \leq \dim F$.

• SYMBOLS!!!!!

Theorem 3.13. $\dim F' \leq \dim F + 1$, *where F is bounded by R_1 and F' is bounded by R_2.*

Proof. By the poset structure we determined earlier, we must choose all the elements that determine the poset of the slice. That gives us n elements. After that we need to choose an additional element that is connected to an element in the original slice by a difference of α so that we can determine α and use that to compute all the other elements in the poset. It follows that our upper bound on the maximum dimension of the glued poset is $n + 1$ when there are no extra edges as required.

When we have an extra edge, we know that $t_{s1,b1} - t_{s2,b2} = \alpha$. Hence, we can rewrite this equation as $\beta(s_1 - s_2) + \alpha(b_1 - b_2) = \alpha$. Since we can find $s_1, s_2, \beta, b_1, b_2$

from the slicing structure, we know that α is determined. As such, we don't need to choose an additional element that is connected to an element in the original slice by a difference of α. Hence, the upper bound on the maximum dimension of T is still n when there are extra edges as required. □

3.4 Generalizing Slicing Structures

Suppose that we have a poset with a slicing structure. We want to show that posets with slicing structures are only produced by telescopic gluings.

Theorem 3.14. Chris Perp's Version *Given a face of a Kunz polyhedron that contains a numerical semigroup that is a Telescopic gluing, any other numerical semigroup on that face will also be a telescopic gluing.*

• Probably better to save x_i's for Kunz coordinates rather than generators.

Proof. Let $S = \langle m, l_1m + r_1, l_2m + r_2, ..., l_nm + r_n\rangle$ be a telescopic glued numerical semigroup with $l_i, r_i \in \mathbb{N}$ and $1 \leq r_i \leq m - 1$. Without loss of generality, we further assume that the set of generators is minimal. Since S is a telescopic glued numerical semigroup, there then exists a $\beta \in \mathbb{N}_{\geq 2}$ such that $\beta \mid m$ and there exists exactly one $j \in [1, n]$ such that $\beta \nmid l_jm + r_j$.

• What does "extraneous components" mean?

• I would write the generators of S in this form to start with, rather than starting with general x_i's and deriving this form.

Let $S' = \langle m, l'_1m + r_1, l'_2m + r_2, ..., l'_nm + r_n\rangle$ be another numerical semigroup on the same face as S. Let $k_i \in \mathbb{Z}$ where $k_i = l'_i - l_i$. Then,

$$l'_im + r_i = (l_i + k_i)m + r_i = l_im + r_i + k_im.$$

It then follows since $\beta \mid l_im + r_i$ and $\beta \mid m$ that $\beta \mid l'_im + r_i$ for all $i \neq j$. As S' is a numerical semigroup we then know that $\beta \nmid l'_jm + r_j$. Therefore, S' is also the result of a Telescopic gluing. □

• Clever! This will do it.

3.5 General Gluings

Theorem 3.15. *We claim that a general gluing $T = \beta S + \alpha R$, such that $m(T) = \beta \cdot m(S)$ has Apery Set*

$$\mathrm{Ap}(T, m(T)) = \{t_{b,s} | s \in \mathrm{Ap}(S), b \in \mathrm{Ap}(R, \beta)\}$$

such that $t_{b,s} = b\alpha + s\beta$.

Proof. We start by showing that the set $G = \{t_{b,s} | s \in \mathrm{Ap}(S), b \in \mathrm{Ap}(R, \beta)\}$ is contained in $\mathrm{Ap}(T, m(T))$. Suppose that there exists some element $t_{b,s} \in G$ such that $t_{b,s} \notin \mathrm{Ap}(T, m(T))$. It follows that $t_{b,s} - m(T) \in T$.

If $t_{b,s} - m(T) = t_{b,s-m(S)} \in T$, it follows that $t_{b,s-m(S)} = \beta m + \alpha n$ for $m = \left(\sum_{i=0}^{|\mathcal{A}(S)-1|} c_i g_i\right)$ and $n = \left(\sum_{i=0}^{|\mathcal{A}(T)-1|} k_i h_i\right)$ for constants $c, k \in \mathbb{Z}$, $g_i \in \mathcal{A}(S), h_i \in \mathcal{A}(R)$. It follows that

$$b \cdot \alpha + (s - m(S))\beta = \alpha \cdot n + \beta \cdot m.$$

Simplifying this equation yields

$$(b - n)\alpha = \beta \cdot (m - (s - m(S)).$$

Since $\gcd(\alpha, \beta) = 1$, we find that $\beta | (b - n)$ and $\alpha | (m - (s - m(S))$. Now suppose that $b - n = k\beta$ for $k \in \mathbb{Z}_{>0}$. Then $b = k\beta + n$, which means that $b - \beta = (k-1)\beta + n \in R$ since $\beta, n \in R$. However, that is a contradiction because $b - \beta \notin R$ by definition of $b \in \mathrm{Ap}(R, \beta)$. It follows that $t_{b,s} \in \mathrm{Ap}(T)$ and therefore $G \subseteq \mathrm{Ap}(T)$ as required.

It follows that $G \subseteq \mathrm{Ap}(T)$. Since $|\mathrm{Ap}(T)| = m(T) = \beta m(S)$ and $|G| = \beta |\mathrm{Ap}(S)| = \beta m(S)$, we find that G must be exactly the set $\mathrm{Ap}(T)$ as required. □

Theorem 3.16. *Consider a gluing $T = \alpha S + \beta R$, where $m(T) = \beta \cdot m(S)$. We claim that the poset structure contains all cover relations that represent a Cartesian product of the original two posets. In particular, t_{b_1,s_1} and t_{b_2,s_2} definitely have a relationship if*

- $b_1 = b_2$ *and* $s_1 - s_2 = g \in \mathcal{A}(S)$

- $s_1 = s_2$ *and* $b_1 - b_2 - h \in \mathcal{A}(R)$

Furthermore, additional relations exist if and only if there exist values of $s_1, s_2 \in \mathrm{Ap}(S), h \in \mathcal{A}(R), b_1, b_2 \in \mathrm{Ap}(R, \beta)$ *such that* $\frac{s_1 - s_2}{\alpha} = \frac{h-(b_1-b_2)}{\beta}$.

Proof. Case 1: Suppose that $t_{b_1,s_1} - t_{b_2,s_2} = \beta \cdot g$ for $g \in \mathcal{A}(S)$. It follows that $\beta(s_1 - s_2) + \alpha(b_1 - b_2) = \beta \cdot g$. Therefore,

$$\beta(s_1 - s_2 - g) = \alpha(b_2 - b_1).$$

Since $\gcd(\alpha, \beta) = 1$, we know that $\beta|(b_2 - b_1)$. However, since $b_1, b_2 \in \mathrm{Ap}(R, \beta)$ they represent different equivalence classes modulo β. As such, b_1 and b_2 must be the same, and it follows that we need $b_1 = b_2$ and $s_1 - s_2 = g \in \mathcal{A}(S)$. As such, these cover relations represent slices of the poset structure that corresponds to S.

Case 2: Suppose that $t_{b_1,s_1} - t_{b_2,s_2} = \alpha \cdot h$ for $h \in \mathcal{A}(R)$. It follows that $\beta(s_1 - s_2) + \alpha(b_1 - b_2) = \alpha \cdot h$ for $h \in \mathcal{A}(R)$. Therefore,

$$\beta(s_1 - s_2) = \alpha(h - (b_1 - b_2)).$$

Since $\gcd(\alpha, \beta) = 1$, we know that $\alpha|(s_1 - s_2)$ and $\beta|(h - (b_1 - b_2))$. When $s_1 = s_2$, then $b_1 - b_2 \in h \in \mathcal{A}(R)$. As such, these cover relations represent slices of the poset structure that correspond to R.

However, it is not necessary for $s_1 = s_2$. In particular, $s_1 - s_2 = k\alpha$. In order to preserve the equality, we also need for $h - (b_1 - b_2) = k\beta$. It follows that our additional relationships occur for t_{b_1,s_1} and t_{b_2,s_2} such that

$$\frac{s_1 - s_2}{\alpha} = \frac{h - (b_1 - b_2)}{\beta}$$

as required.

Theorem 3.17. *There are no additional relations if $\alpha \notin \mathrm{Ap}(S)$.*

Proof. Suppose $\alpha \notin \mathrm{Ap}(S)$ and $\alpha \in S$. Since $t_{b_1,s_1} \preccurlyeq t_{b_2,s_2}$, we know that $t_{b_1,s_1} - t_{b_2,s_2} = t_{b_3,s_3}$. It follows that

$$\beta(s_1 - s_2 - s_3) = \alpha(b_3 - b_1 + b_2)$$

Since $\gcd(\alpha, \beta) = 1$, it follows that $\alpha | (s_1 - s_2 - s_3)$. Since $s_1 - s_2 = s_4 \in \mathrm{Ap}(S)$. It follows that there exists $k \in \mathbb{N}$ such that $k\alpha + s_3 \in \mathrm{Ap}(S)$ because $s_2 \preccurlyeq s_1$. Since $\alpha \notin \mathrm{Ap}(S)$, we know that $\alpha - m(S) = \alpha' \in S$. Then $(\alpha' + m(S))k + s_3 = s_4 \in \mathrm{Ap}(S)$ However, this is a contradiction because $s_4 - m(S) = \alpha' k + (k-1)m(S) + s_3 \in S$. Therefore, $\alpha \not| (s_1 - s_2 - s_3)$, and there is no additional relationship as required. □

Theorem 3.18. *There are additional relations if $\alpha \in \mathrm{Ap}(S)$.*

Proof. Recall that additional relationships where $t_{b_1,s_1} \preccurlyeq t_{b_2,s_2}$ satisfy

$$\beta(s_1 - s_2 - s_3) = \alpha(b_3 - b_1 + b_2).$$

Let $s_1 = \alpha, s_2 = 0, s_3 = 0$. It suffices to show that there exist $b_1, b_2, b_3 \in \mathrm{Ap}(R, \beta)$ such that $b_3 + b_2 - b_1 = \beta$.

□

□

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