An Investigation of Single Small Atom Numerical Semigroups

Joshua Arroyo, Jackson Autry, Charlotte Crandall,
Jess Lefler, and Vadim Ponomarenko

August 9, 2019

Abstract

A numerical semigroup is a subset of the natural numbers that contains 0, is closed under addition, and is cofinite. The atoms of a numerical semigroup are its elements that cannot be expressed as a sum of other elements. An atom is small if it is less than the largest element not in the numerical semigroup, called the Frobenius number. In this paper we explore properties of numerical sets that map onto numerical semigroups with exactly one small atom, such as how many of these sets exist, what is required of their Young diagrams, and patterns to predict the number of numerical sets mapping to these types of numerical semigroups. We also investigate ways to add equivalence classes to a numerical semigroup with a single small atom to create a numerical set mapping to that semigroup.

Contents

1 Introduction

Let $\mathbb{N}_0$ denote the set of non-negative integers. A *numerical set* S is a cofinite subset of $\mathbb{N}_0$ that contains 0; cofinite means that the *gap set*, $H(S) = \mathbb{N}_0 \setminus S$, is a finite set. The *Frobenius number* of a numerical set, g, is the maximal element not in S. If a numerical set is also closed under addition it is called a *numerical semigroup*. For every numerical set, S, there exists an associated numerical semigroup, $A(S)$, called the *atomic monoid* which is defined as

$$A(S) = \{n \in S | n + S \subseteq S\}.$$

A *nyb* of a numerical set S is an element of $A(S)$. An *atom* of a numerical set is a nyb that cannot be created by adding other nybs together. The smallest atom is called the *multiplicity*. A nyb or atom is consider *small* if it less than g, the Frobenius number. When a small atom or nyb is less than $\frac{g}{2}$, then it is a *parvus* atom or nyb and when a small atom or nyb is between $\frac{g}{2}$ and g, then it is a *magna* atom or nyb. Note that if a numerical semigroup has a parvus nyb it must also have a magna nyb as each multiple of a nyb must also be a nyb.

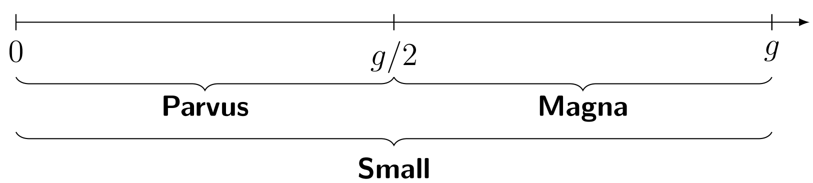


Figure 1: The ranges for small, parvus, and magna atoms or nybs.

The majority of this paper looks at numerical semigroups with exactly one small atom, particularly ones with one small nyb. The focus is one properties of the good numerical sets for these specific numerical semigroups. This investigation into the patterns of numerical semigroups with a single small nyb and their good numerical sets reveals that this group has lots of interesting properties.

Lemma 1.1. *Let T be a numerical semigroup with Frobenius number g and S a numerical set such that $A(S) = T$. Then $g \notin S$ and if T has a single small nyb k, $g - k \notin S$.*

Proof. Let T be a numerical semigroup with Frobenius number g and S a numerical set such that $A(S) = T$. Assume to the contrary that $g \in S$. Then $g \in A(S)$ also because $0 + g = g \in S$, and every integer beyond g is also in S by definition. But $g \notin T$, so $T \neq A(S)$, a contradiction. Thus, $g \notin S$. Now assume that T has small nyb k and that $g - k \in S$. Now $k \in A(S)$, and by definition of $A(S), k + S \subseteq S$. But $(g - k) + k = g \notin S$ from above, therefore $g - k$ cannot be in S if k is the only small element in T. □

When a numerical semigroup has exactly one small nyb, the numerical semigroup can be described by Figure 2. A numerical semigroup with a single small nyb contains 0 followed by a string of gaps until the small nyb, k, which is then followed by more gaps until after the Frobenius number where every element is in the numerical semigroup. The first string of gaps is called the *Mario gap* which has a length of $k - 1$ and the second string is called the *Luigi gap* which has a length $\ell = g - k$.

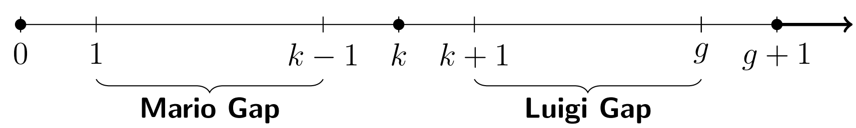


Figure 2: The two groups of gaps for a single small nyb semigroup.

A *Young diagram* is a visual representation of a numerical set or semigroup in which a step to the right represents an element in the set and a step up represents an element not in the set. For example Figure 3 shows the numerical set $S = \{0, 1, 4, 5, 6, 9 \rightarrow\}$ in Young diagram form.

The *hook lengths* of a Young diagram are, given a box b, the counts of boxes directly below and directly to the right of b, and b itself. Hook lengths are significant because the set of all hook lengths in a Young diagram for a numerical set is equal to the set of gaps in the atomic monoid. Additionally, the Frobenius number is always the hook length appearing in the upper left corner of the Young diagram. Figure

Set: $\{0, 1, 4, 5, 6, 9 \rightarrow\}$
Semigroup: $\{0, 5, 9 \rightarrow\}$
Gaps in set: $\{2, 3, 7, 8\}$

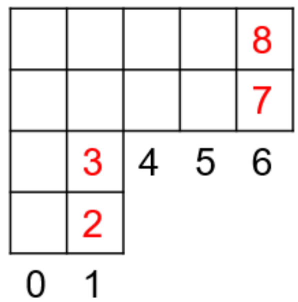
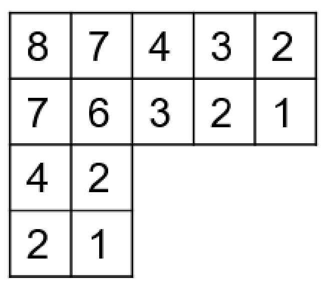


Figure 3: Example of a numerical set represented by a Young diagram.

Set: $\{0, 1, 4, 5, 6, 9 \rightarrow\}$
Semigroup: $\{0, 5, 9 \rightarrow\}$
Gaps in semigroup: $\{1, 2, 3, 4, 6, 7, 8\}$



Due to the relation between the gap set and the hook lengths, the set of hook lengths of a good numerical set for a numerical semigroup with a single small nyb is known. However, each hook length does not have to appear only once in the Young diagram. Figures 13 and 18 show the average number of hooks of a given length for the amount of good numerical sets for the given numerical semigroup. Based on this data, each hook length can be put into one of three groups. The first group is for hook lengths between 1 and ℓ where ℓ is the luigi gap. This group contains the most common hook lengths. The second group contains hook lengths with lengths between

$\ell + 1$ and $k - 1$, where k is the small nyb. These hook lengths are significantly less common than the hook lengths in the first group. The final group is hook lengths between $k + 1$ and g, the Frobenius number. This group has the least common hook lengths, specifically there only exists a single hook of length g per good numerical semigroup. The different groups result in the plots resembling a dinosaur with the first group forming the head, the second group forming the neck, k forming the legs, and the third group forming the tail, thus they are called luigi dinosaurs.

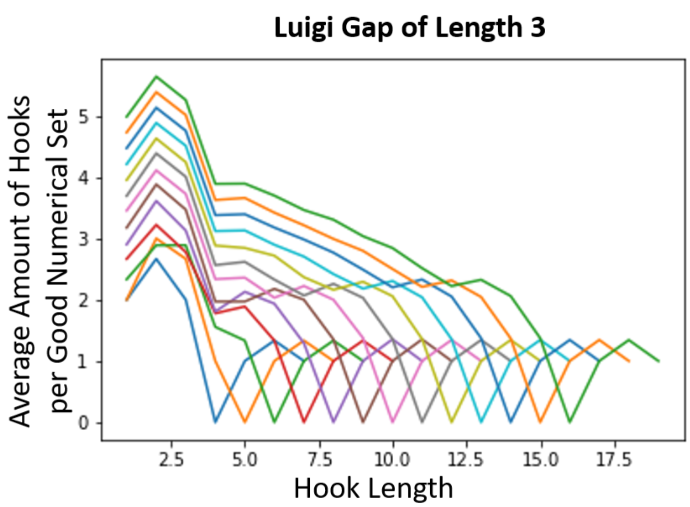


Figure 4: Luigi dinosaur for $\ell = 3$, the average number of hooks of each length per good numerical set for numerical semigroups with a single small nyb and luigi gap of 3. Each line corresponds to a different numerical semigroup with the small nyb being the value at which there are no hooks of that length.

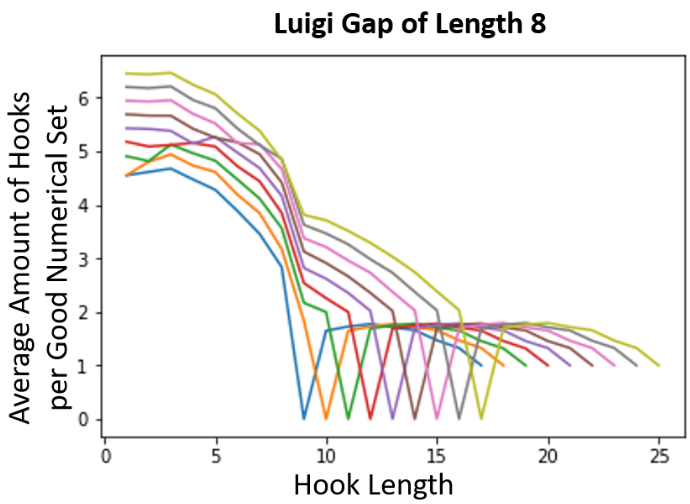


Figure 5: Luigi dinosaur for $\ell = 8$, the average number of hooks of each length per good numerical set for numerical semigroups with a single small nyb and luigi gap of 8. Each line corresponds to a different numerical semigroup with the small nyb being the value at which there are no hooks of that length.

2 Doubling Pattern

When examining the number of good numerical sets with a fixed luigi gap size ℓ, a doubling pattern appears as small nyb k and Frobenius number g increase. For $\ell = 1$, the number of good numerical sets for $g = 4, k = 3$ is twice the number of good sets for $g = 3, k = 2$. For $\ell = 2$, this same doubling occurs between $g = 5, k = 3$ and $g = 6, k = 4$. For luigi gap size 1 or 2, the doubling stops after the first pair. For larger gaps, the doubling continues to other pairs.

The smallest possible g, k for a fixed ℓ is determined by $k_0 = \ell + 1, g_0 = 2k_0 - 1 = 2\ell + 1$, since $k > \frac{g}{2}$. The second smallest g, k is $k_1 = k_0 + 1 = \ell + 2, g_1 = g_0 + 1 = 2\ell + 2 = 2k_0 = 2k_1 - 2$. These two numerical semigroups constitute the first pair for ℓ.

This first pair for a fixed ℓ has a proven doubling pattern, similar to Marzuola and Miller's paper [1].

Theorem 2.1. *The correspondence* $(G, \epsilon) \mapsto G'_\epsilon$ *is a bijection from* $\mathcal{G}(2k-1, k) \times \mathbb{Z}_2$ *to* $\mathcal{G}(2k, k+1)$.

Proof. Let $\mathcal{S}(g)$ be the set of numerical sets with Frobenius number g. Let S'_ϵ be defined for $S \in \mathcal{S}(2k-1)$ and $\epsilon \in \{0, 1\}$ as

$$S'_\epsilon = (S \cap [0, k-1]) \cup \{\epsilon k\} \cup (1 + S \cap [k, \infty)). \tag{1}$$

Let S be a numerical set. An integer i is a magna small nyb for S if and only if $1 + i$ is a magna small nyb for S'_ϵ. Let $\mathcal{G}(g, s)$ be the set of good numerical sets for the numerical semigroup with Frobenius number g and one small nyb s.

Let G be a good numerical set for $T(2k-1, k) = \{0, k, 2k, 2k+1 \rightarrow\}$. Then k is the only small nyb of G and since k is a magna small nyb, the only magna small nyb of G'_ϵ is $k+1$. Assume $n \in G$ is a parvus small nyb of G'_ϵ. Then for all $a \in \mathbb{N}_0$, an is an nyb of G'_ϵ. Let b be the smallest integer such that $k < bn \leq g + 1$. If $b > 2$ then $(b-1)n \leq k$ and $bn < 2(b-1)n \leq g+1$ which would mean that G'_ϵ has either has two magna small nybs or g is a nyb. Both of these are contradictions, therefore $b = 2$ (as

b cannot be 1 as that would mean n is a magna small nyb) and $2n = k+1$ as $2n$ is a magna small nyb of G'_ϵ. Thus, $n = \dfrac{k+1}{2}$ which means that k is odd as $n \in \mathbb{Z}$. Then, $3n = \dfrac{3}{2}(k-1)+3$. If $k \geq 3$ we then get that $3n \leq 2k$ as $\dfrac{3}{2}k + \dfrac{3}{2} \leq 2k$ when $k \geq 3$. This would mean that G'_ϵ has two magna small nybs, $2n$ and $3n$ (or that $g = 2k$ is an nyb), however it can only have one. Therefore, $k \leq 2$. If $k = 1$ then the numerical semigroup has no small nybs as there are no gaps. If $k = 2$ then the only good gaps for $T(3, 2)$ is $\{1, 3\}$ and the only good gaps for $T(4, 3)$ is $\{1, 4\}$ and $\{1, 2, 4\}$. Therefore, no such n can exists which means that if $G \in \mathcal{G}(2k-1, k)$ then $G'_\epsilon \in \mathcal{G}(2k, k+1)$.

Let G'_ϵ be a good numerical set for $T(2k, k+1) = \{0, k+1, 2k+1, 2k+2 \rightarrow\}$. Then $k+1$ is the only small nyb of G'_ϵ and since $k+1$ is a magna small nyb, the only magna small nyb of G is k. Assume $n \in G'_\epsilon$ is a parvus small nyb of G. Then for all $a \in \mathbb{N}_0$, an is an nyb of G. Let b be the smallest integer such that $k \leq bn \leq g$. If $b > 2$ then we also have that $k \leq (b+1)n \leq g$, however this implies that G either has two magna small nybs or g is an nyb. Both of these are contradictions, therefore $b = 2$ and $2n = k$. Thus, $n = k/2$ which means that k is even as $n \in \mathbb{Z}$. Then $3n = \dfrac{3}{2}k$, however this means that G has two magna small nybs, k and $\dfrac{3}{2}k$, which cannot happen. Therefore, if $G'_\epsilon \in \mathcal{G}(2k, k+1)$ then $G'_\epsilon \in \mathcal{G}(2k-1, k)$.

Thus, $G \in \mathcal{G}(2k-1, k)$ if and only if $G'_\epsilon \in \mathcal{G}(2k, k+1)$. □

This bijection can be made more intuitive by considering the corresponding Young diagrams of the numerical sets. Consider the first pair of numerical semigroups for a fixed luigi gap ℓ: $\{0, \ell+1, 2\ell+2 \rightarrow\}, \{0, \ell+2, 2\ell+3 \rightarrow\}$.

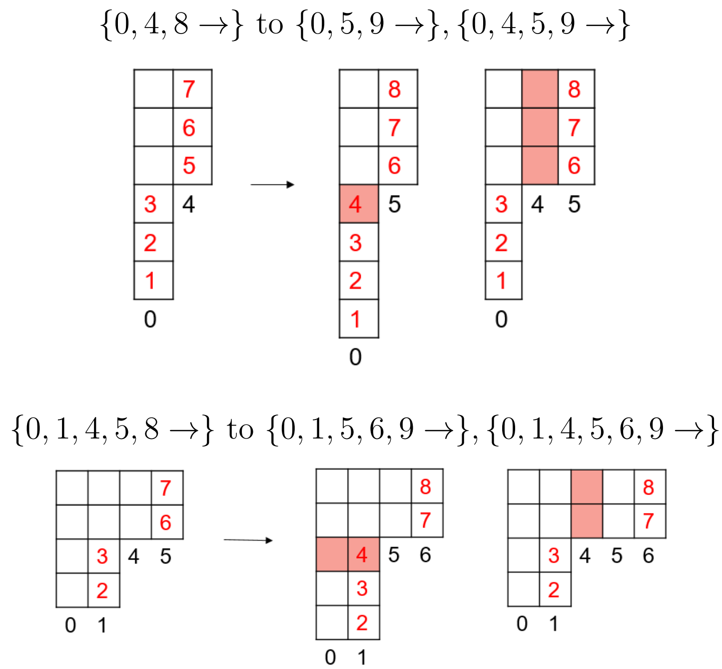


Figure 6: Young Diagram Bijection.

Let T_0 be a numerical semigroup with Frobenius number g_0 and small nyb k_0, and T_1 a numerical semigroup with Frobenius number g_1 and small nyb k_1. $g_0 = 2\ell + 1$ is always odd, and $g_1 = 2\ell + 2$ is always even. $k_0 = \ell + 1 = \frac{g_1}{2}$. Given any good numerical set for T, k_0 can be included or not, since $2k_0 = g_1$, so k_0 will always break itself and therefore will not be in the atomic monoid.

Starting with the Young diagram of a good numerical set for T_0, adding an additional row produces a good numerical set for T_1 that does not include k_0, and adding an additional column produces a good numerical set that includes k_0. This is demonstrated in Figure 6.

g	k	Total Number of Good Numerical Sets	Good % of Numerical Sets with same g
$2\ell+1$	$\ell+1$	a	$w\%$
$2\ell+2$	$\ell+2$	$2a$	$w\%$
$2\ell+3$	$\ell+3$	b	$x\%$
$2\ell+4$	$\ell+4$	$2b$	$x\%$
$2\ell+5$	$\ell+5$	c	$y\%$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2\ell+q$	$\ell+q$	d	$z\%$

Table 1: A general description of the number of numerical sets that map to a numerical semigroup with fixed ℓ.

This bijection changes after the first pair. The Young diagrams after this have more complex structures, creating more places to add a row or column.

Conjecture 2.1. *Let $k_0 = \ell + 1$ be the smallest nyb possible for a given gap size ℓ. Then the doubling pattern continues for $\left\lceil \frac{k_0 - 1}{2} \right\rceil = \left\lceil \frac{\ell}{2} \right\rceil$ pairs, and the numerical semigroup $\{0, 2k_0, 3k_0 \to\} = \{0, 2\ell + 2, 3\ell + 3 \to\}$ will be an element of the pair that breaks the pattern.*

Conjecture 2.2. *Consider the pair that breaks the pattern. Let the first element have small nyb k and a good numerical sets. For some $b, c \in \mathbb{N}$, the number of good numerical sets for the second element is:*

$$\begin{aligned} &2a - b && \text{if } k \text{ is odd} \\ &2a + c && \text{if } k \text{ is even} \end{aligned}$$

g	k	# of Good Numerical Sets
$3\ell+1$	$2\ell+1$	a
$3\ell+2$	$2\ell+2$	$2a-b$
$3\ell+2$	$2\ell+2$	a
$3\ell+3$	$2\ell+3$	$2a+c$

Table 2: The structure of the pair that breaks the doubling pattern.

Additionally, if ℓ is even, then the first semigroup of the breaking pair will be $\{0, 2\ell+1, 3\ell+2 \rightarrow\}$, and if ℓ is odd, the first semigroup of the pair will be $\{0, 2\ell+2, 3\ell+3 \rightarrow\}$.

Some examples of the doubling pattern:

g	k	Total Number of Good Numerical Sets	Good % of Numerical Sets with same g	Good % of Feasible Sets from the void poset
3	2	1	50%	100%
4	3	2	50%	100%
5	4	2	25%	50%
6	5	6	37.5%	75%

Table 3: The amount of numerical sets that map to a numerical semigroup with a single small nyb k, such that $\ell = g - k = 1$.

g	k	Total Number of Good Numerical Sets	Good % of Numerical Sets with same g	Good % of Feasible Sets from the void poset
5	3	2	25%	66.67%
6	4	4	25%	66.67%
7	5	6	18.75%	50.00%
8	6	10	15.63%	41.67%

Table 4: The amount of numerical sets that map to a numerical semigroup with a single small nyb k, such that $\ell = g - k = 2$.

g	k	Total Number of Good Numerical Sets	Good % of Numerical Sets with same g	Good % of Feasible Sets from the void poset
7	4	3	9.375%	33.33%
8	5	6	9.375%	33.33%
9	6	9	7.03%	25.00%
10	7	18	7.03%	25.00%
11	8	31	6.05%	21.53%
12	9	68	6.64%	23.61%

Table 5: The amount of numerical sets that map to a numerical semigroup with a single small nyb k, such that $\ell = g - k = 3$.

More examples can be found in the appendix.

3 Void Poset

Examining the poset generated by the gaps of a numerical semigroup can provide insight into the structure of numerical semigroups with one small nyb.

Definition 3.1 (Void Poset)**.** *The* void poset *is created from the void, $B(T) = \{n \notin T : g - n \notin T\}$, in which for $b_1, b_2 \in B(T)$, $b_1 \preceq b_2$ if $\exists t \in T$ such that $b_1 + t = b_2$.*

Flowers and seeds are parts of the poset diagram.

Definition 3.2 (Flower)**.** *A* flower *is a pair of elements a_1, a_2 such that $a_1 \preceq a_2$ in the void poset, and such that there are no other edges connecting to a_1 or a_2.*

Definition 3.3 (Seed)**.** *A* seed *is an element in the void poset with no edges.*

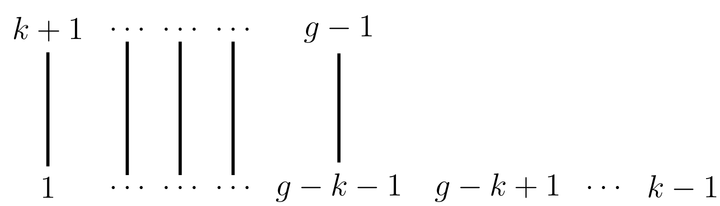


Figure 7: The void poset for numerical semigroups with one small nyb k and Frobenius number g.

Theorem 3.1. *Let T be a numerical semigroup with one small nyb k and Frobenius number g. Then $B(T) = \{1, 2, ..., g-1\} \setminus \{k, g-k\}$. Furthermore, the void poset of $B(S)$ has $g-k-1$ flowers, $2k-g-1$ seeds, and takes the form shown in* Figure 1*.*

Proof. $\forall b > g, b \in S$, so $b \notin B(T)$. $g \notin T, g - g = 0 \in S$, so $g \notin B(S)$.
$k \in T$, so $k \notin B(T)$. $g - (g-k) = k \in T$, so $(g-k) \notin B(T)$.
Let G be the set of gaps of S, $G = \{1, 2, ..., g\} \setminus \{k\}$. Let $c \in G, c < g, c \neq g-k$.
Then $0 < g - c \neq k$, so $g - c \in G$, thus $c \in B(T)$.
Therefore, $B(T) = \{1, 2, ..., g-1\} \setminus \{k, g-k\}$.

$T = \{0, k, g+1, \rightarrow\}$, so k is the only element of T such that $t \in T, b_1, b_2 \in B(T)$,

and $b_1 + t = b_2$, since $t \geq g+1$ implies $b_1 + t \geq g-1$, and $g-1$ is the largest number in $B(T)$.
$\forall n \in \{1, 2, ..., g-k-1\}, n+k \leq g-1$, so $n \preceq n+k$. Therefore, the void poset of $B(T)$ has at least $g-k-1$ flowers.
Necessarily, $\frac{g}{2} < k < g$.
Consider the case in which $g = 2k-1$. Then $g-k+1 = k \notin B(T)$. So there are no additional minimal elements.
Alternatively, let $g < 2k-1$. Then $g-k+1 \leq k-1$.
$\forall m \in \{g-k+1, ..., k-1\}, m+k \geq g-1$, so m is a minimal element with no edges, a seed.
Additionally, $\forall \ell \in \{k+1, ..., g-1\}, \ell + k > g-1$, so there is no $b \in B(T)$ such that $\ell \preceq b$, so there are no additional edges.
Therefore, the void poset of $B(T)$ has $g-k-1$ flowers.
Thus, $B(T)$ must be of the form shown in *Figure 1.* Furthermore, the number of seeds in the void poset is $2k-g-1$ since $(k-1)-(g-k+1)+1 = 2k-g-1$. □

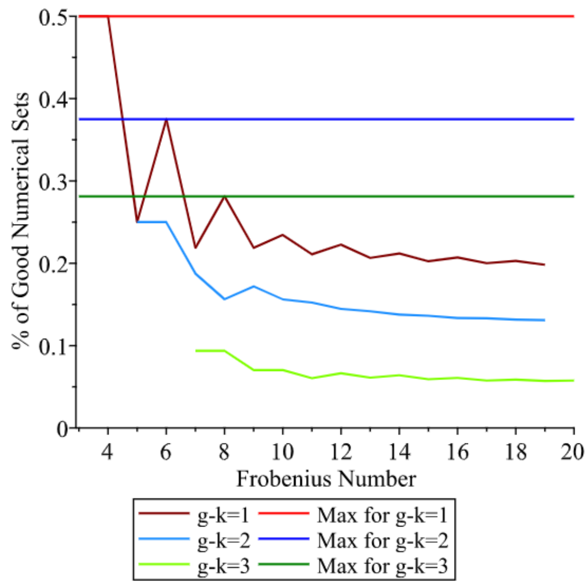
This poset allows the construction of upper and lower bounds on the number of good numerical sets for a numerical semigroup.

Theorem 3.2. *The upper bound for number of good numerical sets mapping to numerical semigroup T, given small nyb k and Frobenius number g, respectively is:*

$$U(T) = 3^{g-k-1}2^{2k-g-1} \tag{2}$$

Proof. The void poset for numerical semigroups with a single small nyb k and Frobenius number g has $g-k-1$ flowers and $2k-g-1$ seeds, by Theorem 3.1. Each seed has no connected edges, so a seed can be included or excluded in the numerical set. Thus there are 2^{2k-g-1} possible combinations of elements to include based on seeds. There are also $g-k-1$ flowers in the void poset, and 3 choices for each flower: both elements in the flower can be included, both elements in the flower can be excluded, and only the maximal element of the flower can be included. Thus, there are 3^{g-k-1} possible combinations of elements to include based on flowers.

Combining the flower and seed combinations yields an upper bound of $3^{g-k-1}2^{2k-g-1}$ possible good numerical sets for T. $\square$



This figure shows the percent of good numerical sets for a fixed luigi gap as the Frobenius number increases. The upper bound provides some information, but as g gets larger, the gap between the upper bound and actual value becomes considerable.

A graph can also be created by connecting lower elements of a pair of flowers such that the sum of those elements is $g-k$, and by connecting a pair of seeds such that their sum is g.

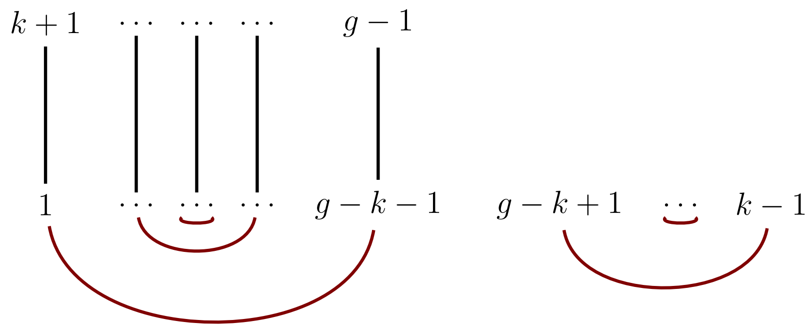


Figure 8: The void poset graph for numerical semigroups with one small nyb k and Frobenius number g.

Given these edges and some $m \in B(T)$ with $m < k$, all elements in the set $\{m, m+k, g-m, g-m-k\}$ are connected to each other in some way. This graph contributes to the construction of a lower bound.

Theorem 3.3. *The lower bound for the number of good numerical sets mapping to numerical semigroup T, given a single small nyb k and Frobenius number g, is:*

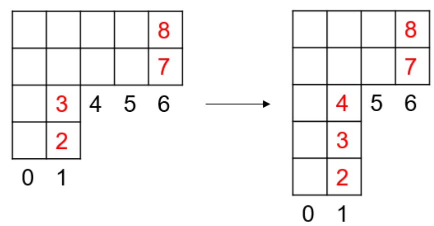
$$L(T) = 2^{\lceil (g-k-1)/2 \rceil + \lceil (2k-g-1)/2 \rceil} \tag{3}$$

Proof. Let $Q_m = \{m, m+k, g-m, g-m-k\}$ with $m < k$ and $m \neq g-k$. Let S be the union of T and any number of the sets Q_m.
$T = \{0, k, g+1 \rightarrow\}$. Take $t \in T$, then either $t = 0, t = k$ or $t > g$. Clearly $0 \in A(S)$. If $t = k$, then $t \in A(S)$ as $k+T \subseteq T \subseteq S$, $k+m = m+k \in S$, $k+m+k = m+2k \in S$ as $m+2k > g$, $k+g-m \in S$ as $g-m+k > g$, and $g-m-k+k = g-m \in S$ meaning that $k+S \subseteq S$. If $t > g$, then $t \in A(S)$ as $t+T \subseteq T \subseteq S$, $t+m \in S$, $t+m+k \in S$, $t+g-m \in S$, and $t+g-m-k \in S$ (as all four are greater than g) meaning that $t+S \subseteq S$. Thus, $T \subseteq A(S)$.
Then $m, g-m \notin A(S)$ as $m+g-m = g \notin S$ and $m+k, g-m-k \notin S$ as $m+k+g-m-k = g \notin S$. Thus, every element in S and not in T is not in $A(S)$ meaning that $A(S) \subseteq T$. Therefore $A(S) = T$.
There are $g-k-1$ options for m that are less than $g-k$, however the choices m

and $g-m-k$ give the same set Q_m, so when $m \neq g-m-k$ we are double counting that option. Thus, there are only $\lceil (g-k-1)/2 \rceil$ such unique options for m. For $g-k < m < k$ there are $2k-g-1$ options for m, however the choices m and $g-m$ produce the same set Q_m, so if $m \neq g-m$, the set Q_m is double counted. Thus, there are only $\lceil (2k-g-1)/2 \rceil$ unique options for m. For every set Q_m, we can choice whether to union it with T or to not giving us $2^{\lceil (g-k-1)/2 \rceil + \lceil (2k-g-1)/2 \rceil}$ possible choices for S. Since $A(S) = T$ for each S we have at least $2^{\lceil (g-k-1)/2 \rceil + \lceil (2k-g-1)/2 \rceil}$ good numerical sets for T. $\square$

4 Young Diagrams

The *conjugate* of a Young diagram is a reflection of a Young diagram across the line $y = -x$, with the upper left corner being the origin. Let Y be a Young diagram and Y' its conjugate. Then the number of rows of Y equals the number of columns of Y', and vice versa. Y is the conjugate of Y'. Additionally, a Young diagram can be its own conjugate.



4.1 One Small Nyb

The structure of Young diagrams provide insight into the structure of their corresponding numerical sets.

4.1.1 Conjugation Bijection

Let T be a numerical semigroup with Frobenius number g and small nyb k. For two numbers a, b such that $a, b \neq k$ and $a + b = g$, Lemmas 4.1 through 4.3 together show a bijection between good numerical sets for T that contain both a and b and good numerical sets that contain neither.

Lemma 4.1. *Let S be a numerical set with Frobenius number g, and let R be the conjugate of S. Then an element $m \in S$ if and only if $g - m \notin R$.*

Proof. Consider the Young Diagram corresponding to S.
($\Rightarrow$) Let $m \in S$. Then m corresponds to a horizontal line in the Young Diagram for S. Since R is the conjugate of S, $g - m$ corresponds to a vertical line in the Young Diagram for R, indicating that $g - m \notin R$.

($\Leftarrow$) Let $g-m \notin R$. Then $g-m$ corresponds to a vertical line in the Young Diagram for R. Since S is the conjugate of R, this means $g-(g-m)=m$ corresponds to a horizontal line in the Young Diagram for S, indicating that $m \in S$. $\square$

Lemma 4.2. *Let T be a numerical semigroup with Frobenius number g and S a good numerical set for T. Let R be the conjugate of S. Then $A(R)=T$.*

Proof. Suppose $t \notin A(R)$. Then there is some $r \in R$ such that $t+r \notin R$. Then $g-(t+r)=g-t-r \in S$. Then $g-t-r+t=g-r$, but $g-r \notin S$ since $r \in R$, meaning $t \notin T$. So $T \subseteq A(R)$. Now assume $t \notin T$, meaning $t \notin A(S)$. So there is some $s \in S$ such that $t+s \notin S$. Then $g-s \notin R$ and $g-t-s \in R$. Now $g-t-s+t=g-s$. If $t \in R$, then $t \notin A(R)$ since $g-s \notin R$. But if $t \notin R$, then $t \notin A(R)$ also. Thus, $A(R) \subseteq T$. Therefore $T=A(R)$. $\square$

Lemma 4.3. *Let T be a numerical semigroup with a single small nyb k and Frobenius number g. Let $a+b=g$. Then the number of good numerical sets that contain both a, b is equal to the number of good numerical sets that contain neither a or b.*

Proof. Let T be a numerical semigroup with a single small nyb k and Frobenius number g. Let $a+b=g$ where $a, b \neq k$. Assume $a, b \in S$, where S is an arbitrary good numerical set for T. Then S has a unique conjugate R that is also a good numerical set for T. Then $g-a, g-b \notin R$ by definition. Notice that $a=g-b$ and $b=g-a$, meaning $a, b \notin R$. Thus for every numerical set S containing both a, b that maps to T, there is a corresponding numerical set mapping to T that contain neither a or b. Thus, the number of good numerical sets that contain both a, b is equal to the number of good numerical sets that contain neither a or b. $\square$

4.1.2 L-Shapes

Another way to get information about structure of numerical sets is by focusing on L-shaped Young diagrams.

Definition 4.1 (L-Shape)**.** *An* L-shape *is a partition with exactly two unique numbers.*

L-shapes have 2 hooks of length 1, and can be segmented into a body and two arms.

Definition 4.2 (Body)**.** *The* body *of an L-shape is the $x \times y$ rectangular portion of the Young diagram starting from the top left corner where x is the size of the smallest element in the partition and y is the number of times the largest element appears in the partition.*

Definition 4.3 (Right Arm)**.** *The* right arm *of an L-shape is the $a \times y$ rectangular portion of the Young diagram that is to the right of the body. a is how much bigger the largest element in the partition is to the smallest element of the partition and y is the number of times the largest element appears in the partition.*

Definition 4.4 (Down Arm)**.** *The* down arm *of an L-shape is the $x \times b$ rectangular portion of the Young diagram that is below the body. x is the size of the smallest element in the partition and b is the number of times the smallest element appears in the partition.*

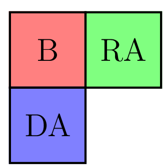


Figure 9: An example of an L-shape. The red part is the body, the blue part is the down arm, and the green part is the right arm.

The dimensions of each part of an L-shape indicate the relationship between the small nyb k and Frobenius number g.

Theorem 4.1. *Consider a numerical set with an L-shaped Young diagram whose arms and body all have the same dimensions $a \times b$. If $k = \dfrac{g+1}{2}$, there exists a good numerical set of this form.*

Proof. Let S be a numerical set with Young diagram Y such that the arms and body of Y all have the same dimensions $a \times b$. Then the numbers $\{1, \ldots, a+b-1\}$

appear as hook lengths in both arms, and the numbers $\{a+b+1,\ldots,2a+2b-1\}$ appear as hook lengths in the body. The associated numerical semigroup has the gaps $\{1,2,\ldots,a+b-1,a+b+1,\ldots,2a+2b-1\}$. This is the numerical semigroup given by $g = 2a+2b-1$ and $k = a+b$. Therefore, if $k = \dfrac{g+1}{2}$, there exists a good numerical set with the arms and body of the corresponding Young diagram being $a \times b$ rectangles with $k = a+b$. □

Theorem 4.2. *Consider a numerical set with an L-shaped Young diagram whose body and small arm have the dimensions $a \times b$, with the big arm having dimensions $c \times b$ such that $c > a$. Let $\gamma = c - a$. If $k = \dfrac{g+\gamma+1}{2}$, there exists a good numerical set of this form.*

Proof. Consider a numerical set with a Young diagram whose body and small arm have the dimensions $a \times b$, with the big arm having dimensions $c \times b$ such that $c > a$. The numbers $\{1,\ldots,a+b-1\}$ appear as hook lengths in the small arm. The numbers $\{1,\ldots,b+c-1\}$ appear as hook lengths in the big arm. Since $c > a$, the small arm hook length set is contained in the big arm hook length set. The numbers $\{b+c+1,\ldots,a+2b+c-1\}$ appear as hook lengths in the body. Therefore, the associated numerical semigroup has the gaps $\{1,\ldots,b+c-1,b+c+1,\ldots,a+2b+c-1\}$. This is the numerical semigroup given by $g = a+2b+c-1$ and $k = b+c$.

Consider the case in which the arms and body of a Young diagram all have dimensions $a \times b$. Let $\gamma = c - a$, so that γ is the number of rows or columns added to one arm in this case. Then $g = (c-\gamma)+2b+c-1 = 2c+2b-\gamma-1 = 2k-\gamma-1$. Therefore, if $k = \dfrac{g+\gamma+1}{2}$, then there exists a numerical set with the small arm and body of the Young diagram being $a \times b$ rectangles and the big arm being a $(a+\gamma)\times b$ rectangle. □

Theorem 4.3. *If a Young diagram is L-shaped and the body and small arm have different dimensions, the corresponding numerical set is bad.*

Proof. Let the body have dimensions $a \times b$, the down arm have dimensions $a \times d$, and the right arm have dimensions $c \times b$.

The numbers $\{1, \ldots, a + d - 1\}$ appear as hook lengths in the down arm. The numbers $\{1, \ldots, b + c - 1\}$ appear as hook lengths in the right arm. The numbers $\{c + d + 1, \ldots, g\}$ appear as hook lengths in the body.

Case 1: Let the big arm be equal to or larger than the body, and let the small arm be smaller than the body. WLOG, let the right arm be the big arm. Then $d < b$ and $c \geq a$. So $a+d < c+b$. The set of all hook lengths is $\{1, \ldots, c+b-1, c+d+1, \ldots, g\}$. $c + d < c + b$, so $c + d = c + b - x$, for $x > 0$. The element $c + d + 1$ can be represented as $c + b - x + 1 \leq c + b$. So for $x \geq 1$, the set of all hook lengths is $\{1, \ldots, c + b - 1\} \cup \{c + b, \ldots, g\} = \{1, \ldots, g\}$, meaning the small nyb is included as a hook length, so there is no corresponding numerical set to a Young diagram of this form.

A conjugate of this Young diagram, in which the down arm is the big arm, would produce this same problem, so there is no corresponding numerical set to a Young diagram of that form.

Case 2: Let both arms be smaller than the body. Then $d < b, c < a$. So $c + d < c + b$. $\{1, \ldots, c + b - 1\} \cup \{c + d + 1, \ldots, g\} = \{1, \ldots, g\}$. Therefore, the small nyb is included as a hook length, so there is no corresponding numerical set to a Young diagram of this form.

Case 3: Let both arms be larger than the body. Then $d > b, c > a$. So $c+d > a+d$, and $c + d > c + b$. The set of all hook lengths in the Young diagram never contains $c + d$ or $\max[a + d, c + b]$. Since at least one number is a gap in the numerical semigroup and does not appear in the hook length set, there is no corresponding numerical set to a Young diagram of this form.

So if a Young diagram is L-shaped and the body and small arm have different dimensions, the corresponding numerical set is not good. $\square$

The number of L-shapes can be determined with g and k of the numerical semigroup.

Lemma 4.4. *There are $\binom{g}{3}$ L-shaped partitions with corresponding numerical sets with Frobenius number g.*

Proof. Let the body be a $b \times c$ rectangle, the down arm be an $a \times c$ rectangle, and the right arm be a $b \times d$ rectangle. Then the top left corner has a hook length of $a+b+c+d-1$ which must also be g. Let $x_0 = a-1, x_1 = b-1, x_2 = c-1, x_3 = d-1$. Then the amount of L-shaped partitions possible is equal to the amount of solutions to $x_0 + x_1 + x_2 + x_3 = g - 3$ with $x_0, x_1, x_2, x_3 \geq 0$ which is $\binom{g}{3}$. $\square$

Not all of those are good numerical sets, however.

Theorem 4.4. *A numerical semigroup with Frobenius number g and small nyb k has $g-k$ L-shaped good numerical sets if $k = \dfrac{g+1}{2}$ and $2(g-k)$ good numerical sets if $k > \dfrac{g+1}{2}$.*

Proof. Assume $k = \dfrac{g+1}{2}$. There are $g-k$ choices for how thick the body of the L-shape is from 1 thick to $g-k$ thick. Since $k = \dfrac{g+1}{2}$ both arms must be the same size as the body, therefore there are $g-k$ such L-shapes. Then the numerical semigroup with $g' = g+a$, $k' = k+a$ has good numerical sets from extending either arm of the previous L-shapes a times, giving $2(g-k)$ such L-shapes. $\square$

4.1.3 Hooks of Length 1

Moving away from L-shapes, the number of hooks of length 1 in a Young diagram indicates the general structure of the diagram and semigroup.

It is possible to modify an existing Young diagram by adding row or column extensions to change the shape. This provides information about the relationship between numerical semigroups.

Definition 4.5 (Row Extension)**.** *A* row extension *is a mapping from one Young diagram Y to another Young diagram Z, by adding an additional row to Y. This new row must have identical length and be adjacent to an existing row in Y.*

Definition 4.6 (Column Extension)**.** *A* column extension *is a mapping from one Young diagram Y to another Young diagram Z, by adding an additional column to Y. This new column must have identical length and be adjacent to an existing column in Y.*

Theorem 4.5. *For $n \geq 3$, numerical semigroups with Frobenius number $g = 2n$ and a single small nyb $k = 2n - 1$ have 4 good numerical sets with n hooks of length 1.*

Proof. Consider a Young diagram in an upside-down staircase shape with n hooks of length 1. The Frobenius number will then be $2n - 1$. For each hook of length 1 you can pick either a row extension or a column extension. If there is no column extension or row extension on the first or last hook of length 1, no column extension on the second hook of length 1, and no row extension of the second to last hook of length 1, then the set $\{0, 2, g-3, g-1, g+1, \rightarrow\}$ is a subset of the corresponding numerical set, S. Note that $g - 3 + 0 = g - 3 \in S, g - 3 + 2 = g - 1 \in S$, and for any $s \in S, s > 3$, $s + g - 3 > g$ and is therefore in S. Since $3 \notin S$, this means that $g - 3 + S \subseteq S$ which is to say $g - 3 \in A(S)$ meaning that S is a bad numerical set for our numerical semigroup.

If the first hook of length 1 has a column extension then $1 \in S$ where S is the corresponding numerical set. Then notice that $g - 1 + 1 = g \notin S$ which means that $g - 1 \notin A(S)$. Therefore S is a bad numerical set for our numerical semigroup with small nyb $g - 1$.

If the last hook of length 1 has a row extension then $g - 1 \notin S$ where S is the corresponding numerical set. Since $g - 1$ is the small nyb S must be a bad numerical set for our numerical semigroup. If the first hook of length 1 has a row extension then the numerical set will be $S = \{0, 3, 5, \ldots, g-3, g-1, g+1, \rightarrow\}$. Each element $s \in S$ with $s < g - 1$ we have that $g - s \in S$ and since $s + g - s = g \notin S$, $s \notin A(S)$. Each element $t \in S$ with $t \geq g - 1$ we have that for all $s \in S$, $t + s = t$ or $t + s > g$. Therefore for $t \in S$ with $t \geq g - 1$, $t \in A(S)$. Thus this numerical set is good for our numerical semigroup.

If the last hook of length 1 has a column extension then the numerical set will be $S = \{0, 2, 4, \ldots, g-2, g-1, g+1, \rightarrow\}$. Each element $s \in S$ with $s < g - 1$ we have that $g - s \in S$ and since $s + g - s = g \notin S$, $s \notin A(S)$. Each element $t \in S$ with $t \geq g - 1$ we have that for all $s \in S$, $t + s = t$ or $t + s > g$. Therefore for $t \in S$ with $t \geq g - 1$, $t \in A(S)$. Thus this numerical set is good for our numerical semigroup.

If the second hook of length 1 has a column extension then the numerical set will be $S = \{0, 2, 3, 5, 7, \ldots, g-3, g-1, g+1, \rightarrow\}$. Since $2 + 2 = 4 \notin S$ we have that $2 \notin S$. Each element $s \in S$ with $2 < s < g - 1$ we have that $g - s \in S$ and since $s + g - s = g \notin S$, $s \notin A(S)$. Each element $t \in S$ with $t \geq g - 1$ we have that for all $s \in S$, $t + s = t$ or $t + s > g$. Therefore for $t \in S$ with $t \geq g - 1$, $t \in A(S)$. Thus this numerical set is good for our numerical semigroup.

If the second to last hook of length 1 has a row extension then the numerical set will be $S = \{0, 2, 4, 6, \ldots, g-6, g-4, g-1, g+1, \rightarrow\}$. Since $2 + g - 4 = g - 2 \notin S$ we have that $2 \notin S$. Each element $s \in S$ with $2 < s < g - 1$ we have that $g - s \in S$ and since $s + g - s = g \notin S$, $s \notin A(S)$. Each element $t \in S$ with $t \geq g - 1$ we have that for all $s \in S$, $t + s = t$ or $t + s > g$. Therefore for $t \in S$ with $t \geq g - 1$, $t \in A(S)$. Thus this numerical set is good for our numerical semigroup.

Thus, only 4 of the possible numerical sets are good. □

Lemma 4.5. *For a numerical semigroup with $g = 2n + 1$, $n > 1$ and a single small nyb there are no good numerical sets with more the n hooks of length 1.*

Proof. In order for a numerical set to have $n+1$ hooks of length 1 the corresponding partition must have $n + 1$ unique numbers in it. Therefore the first entry in the partition is greater than or equal to $n+1$ and the amount of entries is at least $n+1$. Therefore the Frobenius number is at least $2n + 1$ as in the Young diagram the top left corner will be at least $(n+1)+(n+1)-1 = 2n+1$. Since the Frobenius number is $2n + 1$ the first entry in the partition must be $n + 1$ are there must be $n + 1$ total entries in the partition. Therefore, the only possible partition with $n + 1$ hooks of length 1 is $P = [n + 1, n, \ldots, 3, 2, 1]$. This partition corresponds to the numerical set $S = \{0, 2, 4, \ldots, g-1, g+1, \rightarrow\}$. Then since every even number is in the set $2 \in A(S)$ and since every multiple of 4 is in the set $4 \in A(S)$. Since $2, 4 \in A(S)$ and are less than the Frobenius number the numerical semigroup $A(S)$ has at least two small nybs. Thus, S is a bad numerical set for any numerical semigroup with only one small nyb. □

4.1.4 Pure Yggle and Extensions

Considering modifying a general type of Young diagram by adding these extensions can also be helpful. By starting with an upside-down staircase shaped Young diagram and extending segments in certain places, maximizing the number of hooks of length 1 becomes intuitive.

Definition 4.7 (Pure Yggle)**.** *The* pure yggle *is a Young diagram corresponding to a set with Frobenius number* $2n+1$*, with partition* $[n+1, n, n-1, \ldots, 3, 2, 1]$ *and numerical set* $\{0, 2, 4, \ldots, g-1, g+1 \rightarrow\}$*. The pure yggle has only odd hook lengths.*

9	7	5	3	1
7	5	3	1	
5	3	1		
3	1			
1				

A pure yggle with Frobenius number 9.

Adding extensions to the pure yggle in certain places can create good numerical sets.

Lemma 4.6. *Let* T *be a numerical semigroup with Frobenius number* $g = 2n+1$ *with* $n \in \mathbb{Z}$*, and small nyb* k *such that* k *is odd. To create a good numerical set for* T *by adding column extensions (and no row extensions) to the pure yggle so it has* n *hooks of lengths 1, the column extensions must occur at* 0 *and* k*.*

Proof. Consider adding both extensions before reaching k. Then the numerical set is $S = \{0, 2, 4, \ldots, n, n+1, \ldots, m, m+1, \ldots, k, k+2, \ldots, g+1 \rightarrow\}$. $\forall a \in \{0, \ldots, n\}$ such that $a \in S$, a is even; therefore, $\forall b \in \{n+1, \ldots, m\}$ such that $b \in S$, b

is odd. Additionally, $\forall c \in \{m+1, \ldots, g+1\}$ such that $c \in S$, c is even. Since $k \in \{m+1, \ldots, g+1\}$ and $k \in S$, k is even. However, k must be odd by the original assumption, a contradiction. So it is impossible to produce a good numerical set by adding both extensions before reaching k.

Consider adding both extensions at or after k. Then the numerical set begins $\{0, 2, 4, \ldots, k, \ldots\}$. Everything in the numerical set between 0 and the first extension must be even, so k is even. However, k must be odd by the original assumption, a contradiction. So it is impossible to produce a good numerical set by adding both extensions after reaching k.

So exactly one extension must occur prior to reaching k.

Consider adding an extension at k and an extension between 0 and k. The numerical set is $S = \{0, 2, 4, \ldots, n, n+1, \ldots, k, k+1, \ldots, g-1, g+1 \rightarrow\}$. $2+(g-1) = g+1 \in S$. Therefore, $(g-1) \in A(S)$. This is only allowed if $k = g-1$, but $g-1 = 2n+1-1 = 2n$, which is even, and k must be odd by the original assumption, a contradiction. So it is impossible to produce a good numerical set by adding an extension at k and an extension between 0 and k.

Consider adding an extension at 0 and an extension after k. The numerical set is $S = \{0, 1, 3, 5, \ldots, k, k+2, \ldots, n, n+1, \ldots, g-1, g+1 \rightarrow\}$. $1, k \in S$, but $(k+1) \notin S$, so $k \notin A(S)$. Therefore, S is not a good numerical set for T. So it is impossible to produce a good numerical set by adding an extension at 0 and an extension after k.

Consider adding an extension between 0 and k, and an extension after k. The numerical set is $S = \{0, 2, 4, \ldots, n, n+1, n+3, \ldots, k, k+2, \ldots, m, m+1, m+3, \ldots, g-1, g+1 \rightarrow\}$. $2 + (g-1) = g+1 \in S$, so $(g-1) \in A(S)$. This is only allowed if $k = g-1$, but $g-1 = 2n+1-1 = 2n$, which is even, and k must be odd by the original assumption, a contradiction. So it is impossible to produce a good numerical set by adding an extension between 0 and k and an extension after k.

Therefore, adding column extensions at places other than 0 and k cannot produce a good numerical set. $\square$

Lemma 4.7. *Let T be a numerical semigroup with Frobenius number $g = 2n+1$ with $n \in \mathbb{Z}$, and small nyb k such that k is odd. A Young diagram for a good numerical* *set for T cannot be produced by adding one row and one column extension to the pure yggle so it has n hooks of length 1.*

Proof. WLOG, it is possible to add the column extension first and the row extension second, because the conjugate of the Young diagram describes adding the row extension first and the column extension second.

Consider adding both extensions before reaching k. Then the numerical set is $S = \{0, 2, 4, \ldots, n, n+1, \ldots, m, m+3, \ldots, k, k+2, \ldots, g+1 \rightarrow\}$. $\forall a \in \{0, \ldots, n\}$ such that $a \in S$, a is even; therefore, $\forall b \in \{n+1, \ldots, m\}$ such that $b \in S$, b is odd. Additionally, $\forall c \in \{m+3, \ldots, g+1\}$ such that $c \in S$, c is even. Since $k \in \{m+3, \ldots, g+1\}$ and $k \in S$, k is even. However, k must be odd by the original assumption, a contradiction. So it is impossible to produce a good numerical set by adding both extensions before reaching k.

Consider adding both extensions at or after k. Then the numerical set begins $\{0, 2, 4, \ldots, k, \ldots\}$. Everything in the numerical set between 0 and the first extension must be even, so k is even. However, k must be odd by the original assumption, a contradiction. So it is impossible to produce a good numerical set by adding both extensions after reaching k.

So exactly one extension must occur prior to reaching k.

Consider adding an extension at k and an extension between 0 and k. The numerical set is $S = \{0, 2, 4, \ldots, n, n+1, \ldots, k, k+3, \ldots, g-1, g+1 \rightarrow\}$. $2+(g-1) = g+1 \in S$. Therefore, $(g-1) \in A(S)$. This is only allowed if $k = g-1$, but $g-1 = 2n+1-1 = 2n$, which is even, and k must be odd by the original assumption, a contradiction. So it is impossible to produce a good numerical set by adding an extension at k and an extension between 0 and k.

Consider adding an extension at 0 and an extension after k. The numerical set is $S = \{0, 1, 3, 5, \ldots, k, k+2, \ldots, n, n+3, \ldots, g-1, g+1 \rightarrow\}$. $1, k \in S$, but $(k+1) \notin S$, so $k \notin A(S)$. Therefore, S is not a good numerical set for T. So it is impossible to produce a good numerical set by adding an extension at 0 and an extension after k.

Consider adding an extension between 0 and k, and an extension after k. The numerical set is $S = \{0, 2, 4, \ldots, n, n+1, \ldots, k, k+2, \ldots, m, m+3, \ldots, g-1, g+1 \rightarrow\}$. $2 + (g-1) = g+1 \in S$, so $(g-1) \in A(S)$. This is only allowed if $k = g-1$, but

$g-1 = 2n+1-1 = 2n$, which is even, and k must be odd by the original assumption, a contradiction. So it is impossible to produce a good numerical set by adding an extension between 0 and k and an extension after k.

Therefore, adding one row and one column extension cannot produce a good numerical set. □

Lemma 4.8. *Let T be a numerical semigroup with Frobenius number $g = 2n$ where $n \in \mathbb{Z}$ and a single small nyb k such that k is even. Then there are no possible good numerical sets for T with more than n hooks of length 1 and if k is even, then there are no possible good numerical sets for T with more than $n-1$ hooks of length 1.*

Proof. Let T be a numerical semigroup with Frobenius number $g = 2n$ where $n \in \mathbb{Z}$ and a single small nyb k. First consider the pure yggle with length and height of n, so that it has n hooks of length one. Assume for now that k is even. Then the Frobenius number for this numerical set is $2n-1$, and by adding a column extension WLOG will then give the new set a Frobenius number $2n$ while still having n hooks of length one. The new set will be $S = \{0, 2, 4, \ldots, 2m, 2m+1, 2m+3, \ldots, g-1, g+1, \rightarrow\}$. If $m > 0$, then $g-1 \in A(S)$ since $0 + g - 1 = g - 1 \in S$, and $2 + g - 1 = g + 1 \in S$ and every integer greater than $g+1 \in S$ also. But $g - 1 \neq k$ since k is even, thus S is not a good numerical set for T. If $m = 0$, then $S = \{0, 1, 3, 5, \ldots, g-1, g+1, \rightarrow$. But then $k \notin S$ since it only contains odd numbers between 0 and g, so S is not a good numerical set for T. Now, consider the pure yggle with length and height greater than or equal to $n+1$ and let k be even or odd. Then the Frobenius number of the corresponding set is at least $2n+1 > g$, so this set is also not a good numerical set for T. Thus, there are no possible good numerical sets for T with more than n hooks of length one and if k is even, then there are no possible good numerical sets that have $n-1$ hooks of length 1.. □

4.2 No Small Nyb

Next, consider numerical semigroups with no small nyb (no elements less than the Frobenius number). The hooks of length 1 take a different pattern than those in Young diagrams with a small nyb.

4.2.1 Rectangles

Removing the small nyb allows Young diagrams to have exactly one hook of length 1, and therefore a rectangular shape.

Lemma 4.9. *A Young diagram with one hook of length* 1 *is a good numerical set for a numerical semigroup with no small nybs.*

Proof. A Young diagram with one hook of length 1 must be rectangular. Let it be an $a \times b$ rectangle. Then the rows of the Young diagram contain the hooks $\{1, \ldots, a\}, \{2, \ldots, a+1\}, \ldots, \{b, \ldots, g\}$. The union of all these sets with then be $\{1, \ldots, g\}$, thus the numerical semigroup the Young diagram maps to has all gaps between 0 and the Frobenius number meaning it has no small nybs. $\square$

Lemma 4.10. *The number of good numerical sets with one hook of length* 1 *a numerical semigroup with no small nyb and Frobenius number g is g.*

Proof. Lemma 4.9 shows that every rectangular Young diagram is a good numerical set for a numerical semigroup with no small nybs. An $a \times b$ rectangle has Frobenius number $a+b-1$. Let $a+b-1 = g$, then $a = g+1-b$ with $a \geq 1, b \geq 1$. Thus, there are g options for a which will determine b which means that there are g rectangular numerical sets with Frobenius number g. Therefore there are g good numerical sets for the numerical semigroup with no small nybs and Frobenius number g. $\square$

4.2.2 Hooks of Length 1

Other Young diagram structures are also possible in the no small nyb case.

Lemma 4.11. *Let T be a numerical semigroup with Frobenius number g and no small nyb. The maximum number of hooks of length 1 in a Young diagram corresponding to a good numerical set for T is* $\left\lceil \frac{g-1}{2} \right\rceil$.

Proof. Case 1: g is even. Then $S = \{0, 1, 3, 5, \ldots, g-1, g+1 \rightarrow\}$ is a good numerical set for T, since $\forall a \in S$ such that $a \neq 0$, $(g-a) \in S$, and $a + (g-a) = g \notin S$, so

$a, (g-a) \notin A(S)$. The conjugate of S is $\{0, 2, 4, \ldots, g-2, g+1 \rightarrow\}$. This is also a good numerical set for T. Both S and its conjugate have $\frac{g}{2}$ hooks of length 1.

Since g is even, $g = 2n$ for some $n \in \mathbb{Z}$, and the Young diagram with the maximal amount of hooks of length 1 must have length and height $n \times (n+1)$.

Case 2: g is odd. Then $S = \{0, 1, 3, \ldots, g-2, g+1 \rightarrow\}$ is a good numerical set for T, since $\forall a \in S$ such that $a \neq 0$, $a+1 \notin S$, so $a \notin A(S)$. S is its own conjugate, and S has $\frac{g-1}{2}$ hooks of length 1.

$R = \{0, 2, 4, \ldots, g+1 \rightarrow\} \setminus \{g-1\}$ is also a good numerical set for T, since $\forall a \in S$ such that $a \neq 0$, $g-1-a \in S$. $(g-1-a)+a = g-1 \notin S$, so $a, (g-1-a) \notin A(S)$. The conjugate of R is $\{0, 1, 2, 4, 6, \ldots, g-1, g+1 \rightarrow\}$. This is also a good numerical set for T. Both R and its conjugate have $\frac{g-1}{2}$ hooks of length 1.

Let $g = 2n-1$ for some $n \in \mathbb{Z}$. Then the Pure Yggle has the largest number of hooks of length 1. In the Pure Yggle, the corresponding numerical set contains all the even numbers in $\{0, 1, \ldots, 2n-1\}$. The number of even numbers in this set is n, and so is the number of odd numbers. Therefore, the largest row and largest column of the Pure Yggle each have length n. However, a Young diagram of this form corresponds to numerical set $S = \{0, 2, 4, \ldots, g-1, g+1 \rightarrow\}$. $2+(g-1) = g+1 \in S$, so $g+1 \in A(S)$, so S is not a good numerical set. So a Young diagram with an odd Frobenius number cannot have more than $\frac{g-1}{2}$ hooks of length 1.

So the maximum number of hooks of length 1 in a Young diagram corresponding to a good numerical set for semigroup T is $\left\lceil \frac{g-1}{2} \right\rceil$.

$\square$

Lemma 4.12. *Let T be a numerical semigroup with no small nyb and Frobenius number g such that g is even. Then there are two good numerical sets with the maximum number of hooks of length 1.*

Proof. $\{0, 1, 3, 5, \ldots, g-1, g+1 \rightarrow\}$ and its conjugate are both good numerical sets for T with $\frac{g}{2}$ hooks of length 1, the maximum number. This is equivalent to a column extension of the Pure Yggle at 0, and in the conjugate to a row extension of the Pure Yggle at g. Adding this extension anywhere else produces a bad numerical set for T:

$S = \{0, 2, 4, \ldots, n, n+1, \ldots, g-1, g+1 \rightarrow\}$. This set is bad because $\forall s \in S$ such that $s \neq 0$, $s + (g-1) \geq g+1$, and therefore $g - 1 \in A(S)$. So there are exactly two good numerical sets for T with the maximum possible number of hooks of length 1. $\square$

4.2.3 Counts of Specific Shapes

The number of L-shapes is also different in the no small nyb case.

Theorem 4.6. *For numerical semigroups with Frobenius number g and no small nybs, the amount of good numerical sets that are L-shaped is $\dfrac{g(g-1)(g-3)}{8}$ when g is odd and $\dfrac{g(g-2)^2}{8}$ when g is even.*

Proof. Consider an L-shape with the down arm being a $a \times b$ rectangle and the right arm being a $c \times d$ rectangle. Then the down arm contains the hook lengths $\{1, \ldots, a+b-1\}$, the right arm contains the hook lengths $\{1, \ldots, c+d-1\}$, and the body contains the hook lengths $\{b+c+1, \ldots, a+b+c+d-1\}$. A numerical semigroup with no small nybs has all gaps up until the Frobenius number, therefore in order for an L-shape to be a good numerical set either $a+b \geq b+c+1$ or $c+d \geq b+c+1$ and $g = a+b+c+d-1$ as this ensures that every number less than and equal to g is a gap in the corresponding numerical semigroup.

Without loss of generality assume that the L-shape satisfies $a+b \geq b+c+1$. Then $a = c+1+x_1$ where $x_1 \geq 0$. This then gives that $x_1 + b + 2c + d = g$. Let $b = x_2+1, c = x_3+1, d = x_4+1$ where $x_2, x_3, x_4 \geq 0$, this ensures that a, b, c, d are all positive integers. Thus, we have a good L-shape when $x_1+x_2+2x_3+x_4 = g-4$. Let $x_3 = i$, then our equation becomes $x_1+x_2+x_4 = g-4-2i$ where $x_1, x_2, x_4 \geq 0$ and $0 \leq i \leq \left\lfloor \dfrac{g-4}{2} \right\rfloor$. For a given i there are $\binom{g-2-2i}{g-4-2i}$ solutions, thus the total number of solutions is

$$\sum_{i=0}^{\lfloor (g-4)/2 \rfloor} \binom{g-2-2i}{g-4-2i}.$$

If we relabel a to d and c to b we find the number of cases when $c+d \geq b+c+1$. However, this double counts the cases when both are true. Assume that both $a+b \geq b+c+1$ and $c+d \geq b+c+1$ are satisfied. Then we get that $a = c+1+x_1$ and $d = b+1+x_4$ where $x_1, x_4 \geq 0$. Then let $b = x_2+1$ and $c = x_3+1$ where $x_2, x_3 \geq 0$. Then $a+b+c+d-1 = g$ becomes $x_1+2x_2+2x_3+x_4 = g-5$. Let $x_2+x_3 = j$, then our equation becomes $x_1+x_4 = g-5-2j$ where $x_1, x_4 \geq 0$ and $0 \leq j \leq \left\lfloor \frac{g-5}{2} \right\rfloor$. For a given j there are $\binom{g-4-2j}{g-5-2j}$ solutions to $x_1+x_4 = g-5-2j$ and $\binom{j+1}{j}$ solutions to $x_2+x_3 = j$. Thus, the total number of solutions is

$$\sum_{j=0}^{\lfloor (g-5)/2 \rfloor} \binom{g-4-2j}{g-5-2j}\binom{j+1}{j}.$$

Therefore, the total number of good L-shapes for a given g is

$$2\sum_{i=0}^{\lfloor (g-4)/2 \rfloor} \binom{g-2-2i}{g-4-2i} - \sum_{j=0}^{\lfloor (g-5)/2 \rfloor} \binom{g-4-2j}{g-5-2j}\binom{j+1}{j}. \tag{4}$$

Suppose g is odd then $\left\lfloor \frac{g-4}{2} \right\rfloor = \frac{g-5}{2}$ and $\left\lfloor \frac{g-5}{2} \right\rfloor = \frac{g-5}{2}$. Equation 4 then simplifies to $\frac{g(g-1)(g-3)}{8}$. Suppose g is even then $\left\lfloor \frac{g-4}{2} \right\rfloor = \frac{g-4}{2}$ and $\left\lfloor \frac{g-5}{2} \right\rfloor = \frac{g-6}{2}$. Equation 4 then simplifies to $\frac{g(g-2)^2}{8}$. □

An extension of the previous formula to examine three hooks of length one:

Theorem 4.7. *For numerical semigroups with Frobenius number g and no small nybs the amount of good numerical sets with three hooks of length 1 is equal to the number of groups of integers $a, b, c, d, e, f \geq 1$ with $a+b+c+d+e+f-1 = g$ that satisfies at least one of $a \geq b+1$, $a+d \geq c+e+1$, $e \geq d+1$, $b \geq c+1$, $c+f \geq b+d+1$, or $f \geq e+1$, also satisfies at least one of $a+b+d \geq c+1$, $c+e+f \geq d+1$, $a+d \geq max(b+d+1, c+e+1)$, $b+e \geq max(b+d+1, c+e+1)$, or $c+f \geq max(b+d+1, c+e+1)$ and satisfies at least one of the pairs $a \geq c+1$ or $f \geq d+1$.*

Proof. Consider a Young diagram with three hooks of length 1. Let a be the hook length of the hook in the bottom left corner, let b be the hook length of the hook that is in the same row as the second hook of length 1 and one column to the right of the first hook of length 1, let c be the hook length of the hook that is in the same row as the third hook of length 1 and one column to the right of the second hook of length 1, let d be the hook length of the hook that is one row below the second hook of length 1 and is in the same column as the first hook of length 1, let e be the hook length of the hook that is one row below the third hook of length 1 and is in the same column as the second hook of length 1, and let f be the hook length of the hook in the top right corner. Then the Young diagram can be split into six rectangles: a $a \times b$, $a \times e$, $a \times f$, $b \times e$, $b \times f$, and $c \times f$ rectangle. The $a \times d$ rectangle will contain the hook lengths $\{1, \ldots, a+d-1\}$, the $b \times e$ rectangle will contain the hook lengths $\{1, \ldots, b+e-1\}$, and the $c \times f$ rectangle will contain the hook lengths $\{1, \ldots, c+f-1\}$. The $a \times e$ rectangle will contain the hook lengths $\{b+d+1, \ldots, a+b+d+e-1\}$ and the $b \times f$ rectangle will contain the hook lengths $\{c+e+1, \ldots, b+c+e+f-1\}$. Finally, the $a \times f$ rectangle will contain the hook lengths $\{b+c+d+e+1, \ldots, a+b+c+d+e+f-1\}$. In order for the numerical set to be good it must contain every hook length from 1 to g. The largest hook length is $a+b+c+d+e+f-1$, thus it must be the Frobenius number. In order for every other hook length first one of the sets $\{1, \ldots, a+d-1\}$, $\{1, \ldots, b+e-1\}$, $\{1, \ldots, c+f-1\}$ must overlap or have no gap between either $\{b+d+1, \ldots, a+b+d+e-1\}$ or $\{c+e+1, \ldots, b+c+e+f-1\}$. Thus, one of the following must be satisfied: $a+d \geq b+d+1$, $a+d \geq c+e+1$, $b+e \geq b+d+1$, $b+e \geq c+e+1$, $c+f \geq b+d+1$, or $c+f \geq c+e+1$ which can be simplified to $a \geq b+1$, $a+d \geq c+e+1$, $e \geq d+1$, $b \geq c+1$, $c+f \geq b+d+1$, or $f \geq e+1$. Then we must ensure that there is no hook length in between the sets $\{b+d+1, \ldots, a+b+d+e-1\}$ and $\{c+e+1, \ldots, b+c+e+f-1\}$ left behind, thus either these two sets overlap or have no gaps between them, requiring $a+b+d+e \geq c+e+1$ or $b+c+e+f \geq b+d+1$, or one of the sets $\{1, \ldots, a+d-1\}$, $\{1, \ldots, b+e-1\}$, or $\{1, \ldots, c+f-1\}$ must have no gaps between both of the sets $\{b+d+1, \ldots, a+b+d+e-1\}$ and $\{c+e+1, \ldots, b+c+e+f-1\}$ meaning that $a+d \geq \max(b+d+1, c+e+1)$, $b+e \geq \max(b+d+1, c+e+1)$, or $c + f \geq \max(b + d + 1, c + e + 1)$. Thus, one of $a + b + d \geq c + 1$, $c + e + f \geq d + 1$, $a + d \geq \max(b + d + 1, c + e + 1)$, $b + e \geq \max(b + d + 1, c + e + 1)$, or $c + f \geq \max(b + d + 1, c + e + 1)$ must be satisfied. Finally one of the sets $\{b + d + 1, \ldots, a + b + d + e - 1\}$ and $\{c + e + 1, \ldots, b + c + e + f - 1\}$ must either overlap or have no gaps between the set $\{b+c+d+e+1, \ldots, a+b+c+d+e+f-1\}$ requiring that $a + b + d + e \geq b + c + d + e + 1$ or $b + c + e + f \geq b + c + d + e + 1$ which simplifies to $a \geq c + 1$ or $f \geq d + 1$. Thus, every hook length from 1 to g is in the Young diagram as long as at least one of $a \geq b + 1$, $a + d \geq c + e + 1$, $e \geq d + 1$, $b \geq c + 1$, $c + f \geq b + d + 1$, or $f \geq e + 1$, at least one of $a + b + d \geq c + 1$, $c + e + f \geq d + 1$, $a + d \geq \max(b + d + 1, c + e + 1)$, $b + e \geq \max(b + d + 1, c + e + 1)$, or $c + f \geq \max(b + d + 1, c + e + 1)$ is also satisfied and at least one $a \geq c + 1$ or $f \geq d + 1$ is satisfied. Therefore, when this condition is satisfied the numerical set is good. $\square$

5 Equivalence Classes

After several data was generated on the number of good numerical sets that map to a single small atom numerical semigroup, we investigated creating a good numerical set from these semigroups based on adding certain elements in a particular equivalence class modulo the multiplicity. With a single small atom m and a Frobenius number g, it can be easily seen that $g \not\equiv 0 \pmod{m}$, otherwise g would be a multiple of m and therefore not the Frobenius number. Furthermore, the semigroup itself is the numerical set that adds 0 equivalence classes modulo m.

5.1 Adding One Equivalence Class

Next, we calculated how many ways a single equivalence class could be added to a single small atom numerical semigroup. If T has a single small atom m and Frobenius number g such that $g = nm+x$ for some integers n, x and $x < m$, then T is of the form $\{0, m, 2m, \ldots, nm, nm+x+1, \rightarrow\}$. Then the numerical set S that is constructed by adding a single equivalence class to T is of the form $\{0, a, m, a+m, 2m, \ldots, (n-1)m+a, nm, nm+a, nm+x+1, \rightarrow\}$ if $a < x$ or of the form $\{0, a, m, a+m, 2m, \ldots, (n-1)m+a, nm, nm+x+1, \rightarrow\}$ if $x < a$. It is easily shown that $a \not\equiv x \pmod{m}$, otherwise $g \in S$, and from Lemma 1.1 S is not a good numerical set for T.

Lemma 5.1. *Let T be a numerical set with a single small atom m and Frobenius number g such that $g = nm + x$ for some integer n and $x < m$. Then there are $\left\lfloor \dfrac{m+x-2}{2} \right\rfloor - \varepsilon_0$ ways to add a single equivalence class modulo m to T, where*

$$\varepsilon_0 = \begin{cases} 1 \text{ if } m \equiv 0 \pmod{2} \text{ and } \dfrac{m}{2} \neq x, \\ 0 \text{ otherwise.} \end{cases}$$

Proof. The numerical set created by adding a single equivalence class a is $S = \{0, a, m, m+a, \ldots, g-x-m+a, nm, nm+a, g+1, \rightarrow\}$. Then $xm + S \subseteq S$ for any $x \in \mathbb{N}_0$. Then for $0 < y < n-1$, $ym + 2a \not\equiv 0$ or a, when $a \neq \dfrac{m}{2}$ meaning that

$ym + a + S \not\subseteq S$ as long as $a \neq \frac{m}{2}$. If $a \leq \frac{m+x+1}{2}$, then $g - x - m + 2a < g + 1$ or $(n-1)m + 2a < g + 1$ meaning that as long as $a \neq \frac{m}{2}$ then $g - x - m + 2a \notin S$. Thus, if $a < \frac{m+x+1}{2}$ and $a \neq \frac{m}{2}$ then $g - x - m + a + S \not\subseteq S$. Thus, as long as $a < \frac{m+x+1}{2}$, $a \neq \frac{m}{2}$, and $a \neq x$ (as then $g \in S$) S is a good numerical set. There are $\left\lfloor \frac{m+x}{2} \right\rfloor$ possible values for a such that $a < \frac{m+x+1}{2}$. However, this also counts the cases when $a = \frac{m}{2}$ and $a = x$. If m is odd then $\frac{m}{2} \notin \mathbb{N}_0$ therefore $a = \frac{m}{2}$ cannot happen, so just the case when $a = x$ needs to be removed, so the number of possible values for a is $\left\lfloor \frac{m+x-2}{2} \right\rfloor$. If $x = \frac{m}{2}$ then $a = \frac{m}{2}$ and $a = x$ are the same case cannot, so the number of possible values for a is $\left\lfloor \frac{m+x-2}{2} \right\rfloor$. If m is even and $x \neq \frac{m}{2}$ then both cases can occur meaning that there are $\left\lfloor \frac{m+x-4}{2} \right\rfloor$ possible values for a. Thus there are $\left\lfloor \frac{m+x-2}{2} \right\rfloor - \varepsilon_0$ total values for a. $\square$

Lemma 5.1 gives us the total number of ways to add a single equivalence class modulo m. This formula tells us the total number of numerical sets that are of this form. The next step is to consider the good numerical sets that can be created by adding two equivalence classes modulo m. In these cases we will let these classes be a and b, with $a < b$ without loss of generality.

5.2 Adding Two Equivalence Classes

Adding in two equivalence classes allows several new different cases when creating a good numerical set for a single small atom numerical semigroup. For example, consider the numerical semigroup $T = \{0, 9, 18, 27, 35, \rightarrow\}$ and the numerical set $S = \{0, 1, 3, 9, 10, 12, 18, 19, 21, 27, 28, 30, 35, \rightarrow\}$. S is a good numerical set for T that contains the 1 and 3 equivalence classes. Now consider the numerical set $S_1 = \{0, 3, 9, 10, 12, 18, 19, 21, 27, 28, 30, 35, \rightarrow\}$. S_1 is also a good numerical set for T that contains elements in the 1 and 3 equivalence classes, even though $1 \notin S_1$. Similarly,

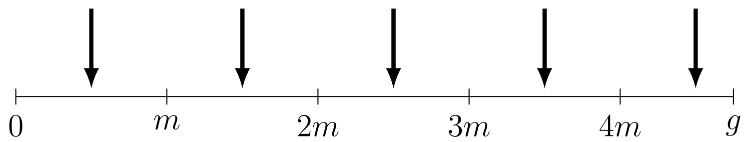


Figure 10: Optional Elements

$S_2 = \{0, 1, 9, 10, 12, 18, 19, 21, 27, 28, 30, 35, \rightarrow\}$ is a good numerical set for T that contains the 1 and 3 equivalence classes, but $3 \notin S_2$. The inclusion of multiple equivalence classes now allows us to choose the starting location for the equivalence classes, which we will define as *optional* elements.

Definition 5.1 (Optional)**.** *Consider a numerical semigroup T with multiplicity m. An equivalence class a is* optional *if a good numerical set for T can include some or all of the equivalence class, but does not need to include the entire class, namely, does not need to include a itself.*

Figure 10 above shows a representation of optional elements. Each of the arrows points to the same equivalence class modulo m, and when adding that equivalence class to form a good numerical set, the smallest element in that equivalence class can be added in at each of the arrows. Once the element is added in, it will also be added in at all instances of the arrows after that element, so that $m \in A(S)$, but the element does not necessarily need to be placed at the first arrow.

When adding the a and b equivalence classes to a semigroup, there are four cases: neither a or b is optional, a is optional and b is not optional, a is not optional and b is optional, and both a and b are optional. In the case where both a and b are optional, at least one of the equivalence classes must start before m, otherwise the numerical set will not be a good numerical set for our semigroup. To demonstrate, when adding an element to create a numerical set, all multiples of the multiplicity must be added to that element, and those sums must also be added to the set. Thus, there will be some maximal element of that equivalence class that is less than the Frobenius number. But when adding the multiplicity to that element, the sum is then greater than the Frobenius number, and if there are no elements in the set less than the multiplicity, that element is then in the atomic monoid, making the set a bad numerical set.

There are several instances in counting where certain cases only occur when certain conditions are met. We will define these conditions with piecewise functions as follows:

$$\varepsilon_2 = \begin{cases} 1 \text{ if } m \equiv 0 \pmod 4 \text{ and } x < \dfrac{m}{4}, \\ 0 \text{ otherwise.} \end{cases}$$

$$\varepsilon_3 = \begin{cases} 1 \text{ if } x < \min\left(\left\lceil \dfrac{m+x+1}{4} \right\rceil, \left\lceil \dfrac{m-3}{3} \right\rceil\right), \\ 0 \text{ otherwise} \end{cases}$$

$$\varepsilon_4 = \begin{cases} 1 \text{ if } x < \left\lceil \dfrac{m+x-2}{3} \right\rceil, \\ 0 \text{ otherwise} \end{cases}$$

$$\varepsilon_5 = \begin{cases} 1 \text{ if } \max\left(\left\lceil \dfrac{m+x+1}{4} \right\rceil, \left\lfloor \dfrac{m-3}{3} \right\rfloor\right) \leq x \leq \left\lceil \dfrac{m+x-2}{3} \right\rceil, \\ 0 \text{ otherwise} \end{cases}$$

$$\varepsilon_6 = \begin{cases} 1 \text{ if } m \equiv 0 \pmod 2 \text{ and } x < \dfrac{m}{2}, \\ 0 \text{ otherwise} \end{cases}$$

$$\varepsilon_7 = \begin{cases} 1 \text{ if } x < \left\lfloor \dfrac{m-2}{2} \right\rfloor, \\ 0 \text{ otherwise} \end{cases}$$

$$\varepsilon_8 = \begin{cases} 1 \text{ if } \left\lfloor \dfrac{m+2}{2} \right\rfloor \leq \dfrac{2m}{3} \leq \min\left(m-x-1, \left\lceil \dfrac{m+x-1}{2} \right\rceil\right) \text{ and } m \equiv 0 \pmod 3), \\ 0 \text{ otherwise} \end{cases}$$

$$\varepsilon_9 = \begin{cases} 1 \text{ if } x < \dfrac{m-1}{3}, \\ 0 \text{ otherwise.} \end{cases}$$

Table 6 shows a complete formula for the total number of ways to add two equivalence classes to a numerical semigroup with multiplicity m and Frobenius number congruent to $x \mod m$.

Theorem 5.1. *For a numerical semigroup with a single small atom, m, and Frobenius number $g = mn + x$ the total number of ways to create a good numerical set from adding two equivalence classes modulo m is demonstrated by Table 6 as the sum of the each element in the Scalar column multiplied by the corresponding element in the Formula column.*

Proof. Lemma 5.14 states that for each pair of equivalence classes modulo m, a and b, there are a certain amount of good numerical sets that can be created depending on what kind of pair a and b are. This is represented in the Scalar column. Lemmas 5.2 through 5.13 prove how many pairs there are of each kind represented by the Formula column. The total amount of good numerical sets is the sum of how many good numerical sets each kind of pair can create multiplied by how many of that kind there are for each kind of pair. Thus, the sum of the each element in the Scalar column multiplied by the corresponding element in the Formula column from Table 6 is the total number of good numerical sets that can be created by adding two equivalence classes modulo m. $\square$

Table 6: Number of Ways to Add 2 Equivalence Classes, $g = nm + x$

Cases	Scalar	Formula
$x > b$, n	1	$\min\left(\left\lceil \frac{x+1}{3} \right\rceil, \left\lceil \frac{m}{2} \right\rceil\right) - \left\lceil \frac{x+1}{4} \right\rceil$
$x > b$, a	$n+1$	$\min\left(\left\lceil \frac{x+1}{4} \right\rceil, \left\lceil \frac{m}{2} \right\rceil\right) - 1$
$x > b$, b	$n+1$	$\left\lceil \frac{x-1}{2} \right\rceil \left(x - \left\lceil \frac{x-1}{2} \right\rceil - \frac{1}{2} \left\lceil \frac{x+1}{2} \right\rceil\right) - \min\left(\left\lceil \frac{x+1}{3} \right\rceil, \left\lceil \frac{m}{2} \right\rceil\right) + \left\lceil \frac{x+1}{4} \right\rceil$
$x > b$, both	$2n+1$	$\left\lfloor \frac{x}{4} \right\rfloor \left(\left\lfloor \frac{x-4}{2} \right\rfloor + 2 \left\lfloor \frac{x \pmod 4}{2} \right\rfloor\right)$
$a < x < b$, n	1	$\max\left(0, \min\left(\left\lceil \frac{m-x-1}{2} \right\rceil - \varepsilon_1, \left\lceil \frac{m+x+1}{2} \right\rceil - \left\lceil \frac{m-x-1}{2} \right\rceil - 1\right)\right)$
$a < x < b$, b	n	$\max\left(\left\lceil \frac{x-3}{2} \right\rceil (m-x-1), \left\lceil \frac{x-1}{2} \right\rceil (m-x-2)\right)$
$x < a$, n	1	$\varepsilon_2 + \varepsilon_3 * \max\left(0, \left\lceil \frac{m-3}{3} \right\rceil - \max\left(\left\lceil \frac{m+x+1}{4} \right\rceil, x+1\right) + 1\right) +$ $\varepsilon_4 * \max\left(0, \left\lceil \frac{m+x-2}{3} \right\rceil - \varepsilon_5 x + (\varepsilon_5 - 1) \max\left(\left\lceil \frac{m+x-3}{4} \right\rceil, \left\lfloor \frac{m}{3} \right\rfloor\right)\right)$ $+\varepsilon_6 \left(\frac{m}{2} + x + 1 - \left\lceil \frac{m+x+1}{2} \right\rceil\right) + \varepsilon_7 * \max\left(0, \min\left(m-x-1, \left\lceil \frac{m+x-1}{2} \right\rceil - \left\lfloor \frac{m}{2} \right\rfloor - \varepsilon_8\right)\right)$
$x < a$, a	n	$\left(\left\lceil \frac{m+x+1}{2} \right\rceil - \frac{m}{2} - 1\right) \varepsilon_6 + \left\lceil \frac{m+x+1}{4} \right\rceil - 1 - x - \varepsilon_2$
$x < a$, b	n	$\varepsilon_9 \left(x \left(m - x + 1 - \left\lceil \frac{m+x+1}{2} \right\rceil\right) + m \left(\frac{2x-m-1}{2}\right) + \frac{(m-x+1)(m-x)}{2}\right) + \varepsilon_6 \left(\frac{m}{2} - x - 1 - \varepsilon_2\right) +$ $(\varepsilon_9 + (-1)^{\varepsilon_9}) \left(m \left(m - \left\lceil \frac{m+x+1}{2} \right\rceil\right) - \frac{m(m-1)}{2} + \frac{1}{2} \left\lceil \frac{m+x-1}{2} \right\rceil \left\lceil \frac{m+x+1}{2} \right\rceil\right) - \varepsilon_6 \left\lfloor \frac{x+1}{2} \right\rfloor -$ $\max\left(\left\lceil \frac{m+x-2}{3} \right\rceil - \max\left(\left\lfloor \frac{m}{3} \right\rfloor, x\right), 0\right) + \varepsilon_9 \left(\frac{1}{2} \left\lfloor \frac{m-3x-1}{2} \right\rfloor \left\lfloor \frac{m-3x-3}{2} \right\rfloor - \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{m+x+1}{4} \right\rceil\right)$
$x < a$, both	$2n-1$	$\frac{1}{2} \left\lceil \frac{m+x-1}{2} \right\rceil \left\lceil \frac{m+x-3}{2} \right\rceil - \left\lceil \frac{m+x-1}{2} \right\rceil x + \frac{x(x+1)}{2} - \max\left(\left\lfloor \frac{m+x}{4} \right\rfloor - x, 0\right) -$ $\varepsilon_6 \frac{m-2x-2}{2} - \max\left(0, \min\left(\left\lfloor \frac{m-1}{2} \right\rfloor - x, \left\lfloor \frac{m-1}{2} \right\rfloor - \left\lfloor \frac{m-x+1}{2} \right\rfloor + 1\right)\right) - \left\lfloor \frac{x}{2} \right\rfloor \varepsilon_6 + \varepsilon_2$

Lemma 5.2. *Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Then when adding two equivalence classes mod m to T, a and b, $a < b$, then when $x > b$, neither a or b are optional when $2b \geq x + 1$ and $2a = b$ and $3a < x + 1 \leq 4a$ giving a total of* $\min\left(\left\lceil \frac{x+1}{3} \right\rceil, \left\lceil \frac{m}{2} \right\rceil\right) - \left\lceil \frac{x+1}{4} \right\rceil$ *pairs of a and b.*

Proof. Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Consider the numerical set $S = \{0, a, b, m, a + m, b + m, 2m \ldots, nm, nm + a, nm + b, mx + 1 + 1 \rightarrow\}$. Since $x > b$, $nm + a, nm + b \in S$ and $nm + a, nm + b < nm + x + 1$. Now if $nm + 2b \notin S$, b is the element that breaks $nm + b$, meaning that $nm + a + b \in S$, but if $nm + a + b \in S$, $nm+a+b \geq nm+x+1$, then $nm+2b \geq nm+x+1$ also meaning $nm+b \in S$, a contradiction. Thus for neither to be optional, $nm + 2a, nm + 2b \in S$ and $nm + a + b \notin S$. Now since $nm + 2b \in S$, $nm + 2b \geq nm + x + 1$, meaning $2b \geq x + 1$. Now if $nm + 2a \geq nm + x + 1$, then $nm + a + b \geq nm + x + 1$ meaning $nm + a + b \in S$, which is a contradiction. Thus $nm + 2a$ can only equal $nm + b$. Then $2a = b$. Then $nm + a + b < nm + x + 1$ since $nm + a + b \notin S$. Then $a + b < x + 1$. Since $2a = b$, $3a < x + 1$ and since $2b \geq x + 1$, then $x + 1 \leq 4a$, meaning $3a < x + 1 \leq 4a$.

Then $3a < x + 1 \leq 4a$ can be rewritten as $\frac{x+1}{4} \leq a < \frac{x+1}{3}$. But for $2a = b$, it must be true that $a < \frac{m}{2}$, otherwise $b > m$. Then a is constrained more by the minimum of $\frac{x+1}{3}$ and $\frac{m}{2}$. Either way, the lower bound on a does not change, meaning there is a total of $\min\left(\left\lceil \frac{x+1}{3} \right\rceil, \left\lceil \frac{m}{2} \right\rceil\right) - \left\lceil \frac{x+1}{4} \right\rceil$ total combinations. $\square$

Lemma 5.3. *Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Then when adding two equivalence classes mod m to T, a and b, $a < b$, with $a < x < b$, if neither a, b are optional, then $2a < x + 1 \leq a + b = m$ and $2b \neq a + m$ and $2b < m + x + 1$.*

Proof. Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Let S be a good numerical set for T such that S is a superset of T and contains 2 equivalence classes a, b with $a < x < b$. Then $S = \{0, a, b, m, a+m, b+m, \ldots (n-1)m+a, (n-1)m+b, nm, nm+a, nm+x+1 \rightarrow\}$. Let neither a nor b be optional.

Case 1: $nm + 2a, (n-1)m + 2b \in S$, $(n-1)m + a + b, nm + a + b \notin S$. Then $nm + 2a \geq nm + x + 1$, so $2a \geq x + 1$. Also, $nm + a < nm + a + b < nm + x + 1$, so $a + b < x + 1$. But then $2a < a + b < x + 1$, and this is a contradiction because $2a \geq x + 1$.

Case 2: $nm + 2a, (n-1)m + 2b \notin S$, $(n-1)m + a + b, nm + a + b \in S$. Then $nm + a + b \geq nm + x + 1$, so $a + b \geq x + 1$. Also, $nm + 2a < nm + x + 1$. So $2a < x + 1$.

Case 2.1: $(n-1)m + a + b = nm$. Then $a + b = m$.

Case 2.1.1: $(n-1)m + 2b < nm$. Then $2b < m$, but then $(n-1)m + a + b < nm$, which is a contradiction.

Case 2.1.2: $nm < (n-1)m + 2b < nm + a$. Then $m < 2b < a + m$.

Case 2.1.3: $nm + a < (n-1)m + 2b < nm + x + 1$. Then $a + m < 2b < m + x + 1$.

Case 2.2: $(n-1)m + a + b = nm + a$. Then $b = m$. This is a contradiction.

Case 2.3: $(n-1)m+a+b \geq nm+x+1$. This is a contradiction because $(n-1)m+2b > (n-1)m + a + b$, and if the statement was true, then $(n-1)m + 2b > nm + x + 1$ and then $(n-1)m + 2b \in S$, which was stated to be false.

So if neither a, b are optional, then $2a < x+1 \leq a+b$ and $a+b = m$ and $2b \neq a+m$ and $2b < m + x + 1$.

$x < b$ and $2b < m + x + 1$, so $x < b \leq \left\lceil \dfrac{m + x - 1}{2} \right\rceil$ and $a + b = m$ and $a < \dfrac{x+1}{2}$ gives that $b > \dfrac{2m - x - 1}{2}$. If $x > \dfrac{2m - x - 1}{2}$ then there are $\left\lceil \dfrac{m + x + 1}{2} \right\rceil - x - 1$ choices for b which simplifies to $\left\lceil \dfrac{m - x - 1}{2} \right\rceil$ and if $x \leq \dfrac{2m - x - 1}{2}$ there are $\left\lceil \dfrac{m + x + 1}{2} \right\rceil - \left\lceil \dfrac{m - x - 1}{2} \right\rceil - 1$ choices for b. Since a is determined once b is chosen this will be the total number of pairs. However, some of these pairs are problematic. If $m \equiv 0 \pmod 3$ and $\dfrac{m}{4} < x < \frac{2}{3}m$, then one of these pairs is $a = \frac{1}{3}m, b = \frac{2}{3}m$ as it requires $x < \frac{2}{3}m < \dfrac{m + x - 1}{2}$, and the corresponding numerical set is $\{a, 2a, 3a, 4a, \ldots, g+1 \rightarrow\}$, which is a bad numerical set for T. So there are $\max\left(0, \min\left(\left\lceil\frac{m-x-1}{2}\right\rceil - \varepsilon_1, \left\lceil\frac{m+x+1}{2}\right\rceil - \left\lceil\frac{m-x-1}{2}\right\rceil - 1\right)\right)$

□

Lemma 5.4. *Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Then when adding two equivalence classes mod m to T, a and b, $a < b$, with $x < a$, if neither a nor b is optional, then either $2b = m$ and $2a = b$, or $a + b < m + x + 1 \leq 2b$ and $a + b \neq m$ and $2a = b$ or $2a = m$, or $a + b = m$ and $2b < m + x + 1$ and $2a \neq b$.*

Proof. Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Let S be a good numerical set for T such that S is a superset of T and contains 2 equivalence classes a, b with $x < a$. Then $S = \{0, a, b, m, a+m, b+m, \ldots (n-1)m + a, (n-1)m + b, nm, nm + x + 1 \rightarrow\}$. Let neither a nor b be optional.

Case 1: $(n-1)m + 2a, (n-1)m + 2b \in S, (n-1)m + a + b \notin S$.

Case 1.1: $(n-1)m + 2b = nm$. Then $2b = m$. Also, $(n-1)m + 2a = (n-1)m + b$. Then $2a = b$. Also, $(n-1)m + a + b < nm$, so $a + b < m$.

Case 1.2: $(n-1)m + 2b \geq nm + x + 1$. Then $2b \geq m + x + 1$.

Case 1.2.1: $(n-1)m + 2a = (n-1)m + b$. Then $2a = b$.

Case 1.2.1.1: $(n-1)m + a + b < nm$. Then $a + b < m$.

Case 1.2.1.2: $nm < (n-1)m + a + b < nm + x + 1$. Then $m < a + b < m + x + 1$.

Case 1.2.2: $(n-1)m + 2a = nm$. Then $2a = m$. Also, $nm < (n-1)m + a + b < nm + x + 1$. Then $m < a + b < m + x + 1$.

Case 2: $(n-1)m + 2a, (n-1)m + 2b \notin S, (n-1)m + a + b \in S$.

Case 2.1: $(n-1)m + a + b = nm$. Then $a + b = m$. Also, $nm < (n-1)m + 2b < nm + x + 1$, then $m < 2b < m + x + 1$.

Case 2.1.1: $(n-1)m + 2a < (n-1)m + b$. Then $2a < b$.

Case 2.1.2: $(n-1)m + b < (n-1)m + 2a < nm$. Then $b < 2a < m$.

Case 2.1.3: $nm < (n-1)m + 2a < nm + x + 1$. This produces a contradiction, since it requires that $(n-1)m + a + b = nm < (n-1)m + 2a$, and therefore $a + b < 2a$.

Case 2.2: $(n-1)m + a + b \geq nm + x + 1$. This produces a contradiction because

$a < b$, so the inequality implies $(n-1)m + 2b > nm + x + 1$, so $(n-1)m + 2b \in S$, and this is stated to be false in the Case 2 statement.
So if neither a nor b is optional, either $2b = m$ and $2a = b$, or $a + b < m + x + 1 \leq 2b$ and $a + b \neq m$ and $2a = b$ or $2a = m$, or $a + b = m$ and $2b < m + x + 1$ and $2a \neq b$.

The cases:

A: $2a = b, 2b = m$.

B: $2b \geq m + x + 1, 2a = b, a + b < m$.

C: $2b \geq m + x + 1, 2a = b, m < a + b < m + x + 1$.

D: $2b \geq m + x + 1, 2a = m, m < a + b < m + x + 1$.

E: $a + b = m, m < 2b < m + x + 1, 2a \neq b, 2a < m$.

A: If $m \equiv 0 \pmod 4$, then $4a = m$ is a possible case. So if $x < \dfrac{m}{4}$ and $m \equiv 0 \pmod 4$, add 1, which occurs when $\varepsilon_2 = 1$.
B: $a + b = 3a < m$, so $a < \dfrac{m}{3}$, and $a \leq \left\lceil \dfrac{m-3}{3} \right\rceil$. $b \geq \left\lceil \dfrac{m+x+1}{2} \right\rceil$ and $2a = b$, so $a \geq \left\lceil \dfrac{m+x+1}{4} \right\rceil$. Also, $a \geq x + 1$. Therefore, $\max\left(\left\lceil \dfrac{m+x+1}{4} \right\rceil, x+1\right) \leq a \leq \left\lceil \dfrac{m-3}{3} \right\rceil$. So there are $\left\lceil \dfrac{m-3}{3} \right\rceil - \max\left(\left\lceil \dfrac{m+x+1}{4} \right\rceil, x+1\right) + 1$ pairs of this form. However, if this number is negative or if $x < \min\left(\left\lceil \dfrac{m+x+1}{4} \right\rceil, \left\lceil \dfrac{m-3}{3} \right\rceil\right)$, there are actually 0 pairs of this form. So there are
$\left(\varepsilon_3 * \max\left(0, \left\lceil \dfrac{m-3}{3} \right\rceil - \max\left(\left\lceil \dfrac{m+x+1}{4} \right\rceil, x+1\right) + 1\right)\right)$ pairs of this form.

C: $a + b = 3a$, so $m < 3a < m + x + 1$, and therefore $\left\lfloor \dfrac{m+3}{3} \right\rfloor \leq a \leq \left\lceil \dfrac{m+x-2}{3} \right\rceil$. Also, $b \geq \left\lceil \dfrac{m+x+1}{2} \right\rceil$ and $2a = b$, so $a \geq \left\lceil \dfrac{m+x+1}{4} \right\rceil$. So $\max\left(\left\lceil \dfrac{m+x+1}{4} \right\rceil, \left\lfloor \dfrac{m+3}{3} \right\rfloor\right) \leq a \leq \left\lceil \dfrac{m+x-2}{3} \right\rceil$. So there are $\left\lceil \dfrac{m+x-2}{3} \right\rceil - \max\left(\left\lceil \dfrac{m+x+1}{4} \right\rceil, \left\lfloor \dfrac{m+3}{3} \right\rfloor\right)$ pairs of this form. However, if this number is negative or $x \geq \left\lceil \dfrac{m+x-2}{3} \right\rceil$, there are actually 0 pairs of this form. Then add

$\varepsilon_{17} * \max\left(0, \left\lceil \frac{m+x-2}{4} \right\rceil - \max\left(\left\lceil \frac{m+x+1}{4} \right\rceil, \left\lfloor \frac{m+3}{3} \right\rfloor\right) + 1\right)$. Additionally, this formula sometimes catches some elements that are less than or equal to x, which is not allowed since $x < a$. So given the cases in which $\max\left(\left\lceil \frac{m+x+1}{4} \right\rceil, \left\lfloor \frac{m+3}{3} \right\rfloor\right) \leq x \leq \left\lceil \frac{m+x-2}{3} \right\rceil$, subtract all elements in this range and less than or equal to x, so subtract $\varepsilon_5 \left(0, x - \max\left(\left\lceil \frac{m+x+1}{4} \right\rceil, \left\lfloor \frac{m+3}{3} \right\rfloor\right) + 1\right)$ pairs. So there are:

$$\varepsilon_4 * \max\left(0, \left\lceil \frac{m+x-2}{3} \right\rceil - \max\left(\left\lceil \frac{m+x+1}{4} \right\rceil, \left\lfloor \frac{m+3}{3} \right\rfloor\right) + 1 - \varepsilon_5 \left(x - \max\left(\left\lceil \frac{m+x+1}{4} \right\rceil, \left\lfloor \frac{m+3}{3} \right\rfloor\right) + 1\right)\right)$$

good pairs of this form.

D: If m is odd, this case never occurs. Otherwise, $a = \frac{m}{2}$, so $x < \frac{m}{2}$. $b \geq \left\lceil \frac{m+x+1}{2} \right\rceil$. Also $a + b < m + x + 1$, so $b < \frac{m}{2} + x + 1$, and therefore $b \leq \frac{m}{2} + x$. So $\left\lceil \frac{m+x+1}{2} \right\rceil \leq b \leq \frac{m}{2} + x$. So there are $\left(\varepsilon_6 * \left(\frac{m}{2} + x + 1 - \left\lceil \frac{m+x+1}{2} \right\rceil\right)\right)$ pairs of this form.

E: $m < 2b < m + x + 1$, so $\frac{m}{2} < b < \frac{m+x+1}{2}$, and therefore $\left\lfloor \frac{m+2}{2} \right\rfloor \leq b \leq \left\lceil \frac{m+x-1}{2} \right\rceil$. Also, $a + b = m$ and $x + 1 \leq a$, so $b \leq m - x - 1$. So $\left\lfloor \frac{m+2}{2} \right\rfloor \leq b \leq \min\left(m - x - 1, \left\lceil \frac{m+x-1}{2} \right\rceil\right)$. So there are $\min\left(m - x - 1, \left\lceil \frac{m+x-1}{2} \right\rceil\right) - \left\lfloor \frac{m+2}{2} \right\rfloor + 1$ pairs of this form. If this number is negative, however, the number of pairs is 0. Also, this bound includes cases in which $2a = b$ whenever $\left\lfloor \frac{m+2}{2} \right\rfloor \leq \frac{2m}{3} \leq \min\left(m - x - 1, \left\lceil \frac{m+x-1}{2} \right\rceil\right)$ and $m \equiv 0 \pmod 3$, so in these cases, 1 must be subtracted. Additionally, $a \leq \left\lceil \frac{m-2}{2} \right\rceil$, so if $x > \left\lceil \frac{m-2}{2} \right\rceil$, this entire case cannot occur. So there are $\left(\varepsilon_7 * \max\left(0, \min\left(m-x-1, \left\lceil \frac{m+x-1}{2} \right\rceil\right) - \left\lfloor \frac{m+2}{2} \right\rfloor + 1 - \varepsilon_8\right)\right)$ pairs of this form.

The sum of all these cases is:

$$\varepsilon_2 + \varepsilon_3 * \max\left(0, \left\lceil \frac{m-3}{3} \right\rceil - \max\left(\left\lceil \frac{m+x+1}{4} \right\rceil, x+1\right) + 1\right) +$$
$$\varepsilon_4 * \max\left(0, \left\lceil \frac{m+x-2}{3} \right\rceil - \max\left(\left\lceil \frac{m+x+1}{4} \right\rceil, \left\lfloor \frac{m+3}{3} \right\rfloor\right) + 1 - \right.$$
$$\left.\varepsilon_5\left(x - \max\left(\left\lceil \frac{m+x+1}{4} \right\rceil, \left\lfloor \frac{m+3}{3} \right\rfloor\right) + 1\right)\right) +$$
$$\varepsilon_6 * \left(\frac{m}{2} + x + 1 - \left\lceil \frac{m+x+1}{2} \right\rceil\right) +$$
$$\varepsilon_7 * \max\left(0, \min\left(m-x-1, \left\lceil \frac{m+x-1}{2} \right\rceil - \left\lfloor \frac{m+2}{2} \right\rfloor + 1 - \varepsilon_8\right)\right)$$

□

Next we will consider the cases where a is optional.

Lemma 5.5. *Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Then when adding two equivalence classes mod m to T, a and b, $a < b$, then when $x > b$, only a is optional when $2a = b$ and $2b < x + 1$. This occurs for a total of $\min\left(\left\lceil \frac{x+1}{4} \right\rceil, \left\lceil \frac{m}{2} \right\rceil\right) - 1$ combinations of a and b.*

Proof. Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm+x$ for some integer n. Consider the numerical set $S = \{0, a, b, m, a+m, b+m, 2m, \ldots, nm+a, nm+b, nm+x+1 \rightarrow\}$. If a is optional, then S is still a good numerical set for T when a is removed. Then b must be the element that breaks $nm+a, nm+b$. Thus $nm+a+b, nm+2b \notin S$, and $nm+2a \in S$, otherwise

b may not be necessary. Now if $nm+2a \geq nm+x+1$, $nm+a+b \geq nm+x+1$, so $nm+a+b \in S$, which is a contradiction. Thus $nm+2a$ can only equal $nm+b$. Then $2a = b$. Now since $nm+2b \notin S$, $nm+2b < nm+x+1$, meaning $2b < x+1$. Then $nm+a+b$ will also be less than $x+1$ since $a < b$.
Since $2b = a$ and $2b < x+1$ it must be true that $4a < x+1$. But for $2a$ to equal b, it must also be true that $a < \dfrac{m}{2}$. Then a is more restricted by the minimum of these numbers, so the total number of pairs of a and b is $\min\left(\left\lceil \dfrac{x+1}{4} \right\rceil, \left\lceil \dfrac{m}{2} \right\rceil\right) - 1$. □

Lemma 5.6. *Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Then when adding two equivalence classes mod m to T, a and b, $a < b$, with $a < x < b$, a is never optional.*

Proof. Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Let S be a good numerical set for T such that S is a superset of T and contains 2 equivalence classes a, b with $a < x < b$. Then $S = \{0, a, b, m, a+m, b+m, \ldots (n-1)m+a, (n-1)m+b, nm, nm+a, nm+x+1 \rightarrow\}$.
Let a be optional. Then $nm+a+b, (n-1)m+2b \notin S$, and $nm+2a \in S$.
Since $nm+2a \in S$, $nm+2a > nm+x+1$, so $2a > x+1$. Also, since $nm+a+b \notin S$, $nm+a+b < nm+x+1$, so $a+b < x+1$. This implies that $a+b < x+1 < 2a$, but $a < b$, so this is a contradiction. So there are no pairs a, b with $a < x < b$ such that a is optional and b is not. □

Lemma 5.7. *Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Then when adding two equivalence classes mod m to T, a and b, $a < b$, with $x < a$, if only a is optional, then either $2a = b$ and $2b \neq m$ and $2b < m+x+1$ and $a+b \neq m$, or $2a = m$ and $2b < m+x+1$. The total number of pairs of a and b that satisfy these conditions is* $\left(\left\lceil \dfrac{m+x+1}{2} \right\rceil - \dfrac{m}{2} - 1\right)\varepsilon_6 + \left\lceil \dfrac{m+x+1}{4} \right\rceil - 1 - x - \varepsilon_2$.

Proof. Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Let S be a good numerical set for T such that S is a superset of T and contains 2 equivalence classes a, b with $x < a$. Then

$S = \{0, a, b, m, a+m, b+m, \dots (n-1)m+a, (n-1)m+b, nm, nm+x+1 \rightarrow\}$.
Let only a be optional. Then $(n-1)m+a+b, (n-1)m+2b \notin S, (n-1)m+2a \in S$.
Case 1: $(n-1)m+2a = (n-1)m+b$. Then $2a = b$.
Case 1.1: $(n-1)m+2b < nm$. Then $2b < m$. Also, $(n-1)m+a+b < nm$. Then $a+b < m$.
Case 1.2: $nm < (n-1)m+2b < nm+x+1$. Then $m < 2b < m+x+1$.
Case 1.2.1: $(n-1)m+a+b < nm$. Then $a+b < m$.
Case 1.2.2: $nm < (n-1)m+a+b < nm+x+1$. Then $m < a+b < m+x+1$.
Case 2: $(n-1)m+2a = nm$. Then $2a = m$. Also, $nm < (n-1)m+2b < nm+x+1$. Then $m < 2b < m+x+1$. Also, $nm < (n-1)m+a+b < nm+x+1$. Then $m < a+b < m+x+1$.
Case 3: $(n-1)m+2a \geq nm+x+1$. This produces a contradiction because $a < b$, so the inequality implies $(n-1)m+2b > nm+x+1$, so $(n-1)m+2b \in S$, and this is stated to be false.
So if only a is optional, then either $2a = b$ and $2b \neq m$ and $2b < m+x+1$ and $a+b \neq m$, or $2a = m$ and $2b < m+x+1$.
To count the total number of pairs that occur, first consider the case where $2b < m+x+1$ and $2a = m$. Then $a = \frac{m}{2}$ and $b < \frac{m+x+1}{2}$. Since there is a fixed value for a, then only the values for b need to be counted. Then $\frac{m}{2} < b < \frac{m+x+1}{2}$. Then there are $\left\lceil \frac{m+x+1}{2} \right\rceil - \frac{m}{2} - 1$ values for b. But for this case to occur, it also must be true that $m \equiv 0 \pmod 2$ and $x < \frac{m}{2}$. Then we multiply this number by ε_6. So the total number of pairs for case 1 is $\left(\left\lceil \frac{m+x+1}{2} \right\rceil - \frac{m}{2} - 1 \right) \varepsilon_6$.
Now the total number of pairs for the case where $2b < m+x+1$, $2a = b$, $a+b \neq m$, and $2b \neq m$ must be added. First consider when $2b < m+x+1$ and $2a = b$. Then $b < \frac{m+x+1}{2}$ and since $2a = b$, $a < \frac{m+x+1}{4}$. Since $x < a$ also, then the total number of possible values for a and b is $\left\lceil \frac{m+x+1}{4} \right\rceil$. Now the case where $a+b = m$ must be subtracted. If $a+b = m$ and $2a = b$, then $3a = m$. Then this case occurs when $m \equiv 0 \pmod 3$ and $\frac{m}{3} < \frac{m+x+1}{4}$ which simplifies to $x > \frac{m-3}{3}$. But since

$x < a$ also, then $x < \dfrac{m}{3}$, which cannot occur. Now the case where $2b = m$ must be subtracted. Then if $2a = b$ and $2b = m$, $4a = m$. Now $\dfrac{m}{4} < \dfrac{m+x+1}{4}$ always occurs, so the only thing that must be subtracted is 1 when $m \equiv 0 \pmod 4$ and $p < \dfrac{m}{4}$. In other words, subtract ε_2.

Then the total number of pairs is $\left(\left\lceil \dfrac{m+x+1}{2} \right\rceil - \dfrac{m}{2} - 1\right)\varepsilon_6 + \left\lceil \dfrac{m+x+1}{4} \right\rceil - 1 - x - \varepsilon_2$. □

Next we will consider the cases where b is optional.

Lemma 5.8. *Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Then when adding two equivalence classes mod m to T, a and b, $a < b$, with $x > b$, if b is optional, then $a+b < x+1, 2b \geq x+1$, and $2a \neq b$. This gives a total of* $\left\lceil \dfrac{x-1}{2} \right\rceil \left(x - \left\lceil \dfrac{x-1}{2} \right\rceil - \dfrac{1}{2}\left\lceil \dfrac{x+1}{2} \right\rceil\right) - \min\left(\left\lceil \dfrac{x+1}{3} \right\rceil, \left\lceil \dfrac{m}{2} \right\rceil\right) + \left\lceil \dfrac{x+1}{4} \right\rceil$ *possible pairs of a and b.*

Proof. Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Let S be a good numerical set for T such that S is a superset of T and contains 2 equivalence classes a, b with $a < b$ and $x > b$. Then $S = \{0, a, b, m, a+m, b+m, \ldots, nm, nm+a, nm+b, nm+x+1 \rightarrow\}$. Let b be optional. Then $nm + a + b, nm + 2a \notin S$, $nm + 2b \in S$.

Since $nm + b \in S$, $nm + 2b \geq nm + x + 1$, so $2b \geq x + 1$. Since $nm + a + b \notin S$, $nm + b < nm + a + b < nm + x + 1$, so $a + b < x + 1$.

Case 1: $nm + 2a < nm + b$. Then $2a < b$.

Case 2: $nm + b < nm + 2a < nm + x + 1$. Then $b < 2a < x + 1$. We already have $a < b$ and $a + b < x + 1$, so $2a < a + b < x + 1$.

In either case, $2a \neq b$. So if b is optional, then $a + b < x + 1, 2b \geq x + 1$, and $2a \neq b$.

To count the total number of pairs of a and b where b is optional, the conditions $a+b < x+1 \leq 2b$ and $2a \neq b$ must be satisfied. First, consider when $a+b < x+1 \leq 2b$ is true. These can be rewritten to state $b < x + 1 - a$ and $b \geq \dfrac{x+1}{2}$. For each of these $b's$, we can have any $a < b$, so then the total number of pairs is

$$\sum_{a=1}^{\left\lceil \frac{x-1}{2} \right\rceil} x - a - \left\lceil \frac{x-1}{2} \right\rceil$$

which simplifies to $\left\lceil \frac{x-1}{2} \right\rceil \left(x - \left\lceil \frac{x-1}{2} \right\rceil - \frac{1}{2} \left\lceil \frac{x+1}{2} \right\rceil \right)$. Then the cases where $2a = b$ must be subtracted to satisfy all of the conditions. If $2a = b$, then the inequalities from above can be rewritten as $\frac{x+1}{2} \leq 2a < x + 1 - a$, which results in $\frac{x+1}{4} \leq a$ and $a < \frac{x+1}{3}$. But for $2a = b$, we also must have that $a < \frac{m}{2}$, otherwise $b > m$. Then the total number of cases that must be subtracted is $\min\left(\left\lceil \frac{x+1}{3} \right\rceil, \left\lceil \frac{m}{2} \right\rceil \right) + \left\lceil \frac{x+1}{4} \right\rceil$. Combining these results yields a total number of $\left\lceil \frac{x-1}{2} \right\rceil \left(x - \left\lceil \frac{x-1}{2} \right\rceil - \frac{1}{2} \left\lceil \frac{x+1}{2} \right\rceil \right) - \min\left(\left\lceil \frac{x+1}{3} \right\rceil, \left\lceil \frac{m}{2} \right\rceil \right) + \left\lceil \frac{x+1}{4} \right\rceil$ possible pairs of a and b. $\square$

Lemma 5.9. *Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Then when adding two equivalence classes mod m to T, a and b, $a < b$, with $a < x < b$, if b is optional, then $2a < x + 1$ and $a + b \neq m$.*

Proof. Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Let S be a good numerical set for T such that S is a superset of T and contains 2 equivalence classes a, b with $a < x < b$. Then $S = \{0, a, b, m, a+m, b+m, \dots (n-1)m+a, (n-1)m+b, nm, nm+a, nm+x+1 \rightarrow\}$. Let b be optional. Then $(n-1)m + a + b, nm + 2a \notin S$, and $(n-1)m + 2b \in S$. Since $nm + 2a \notin S, nm + 2a < nm + x + 1$, so $2a < x + 1$. But also since $x < b$, $nm + a + b > nm + a + x \geq nm + x + 1$, meaning $nm + a + b \in S$, thus, a must be the element to break $nm + a$. For b to be optional, then $(n-1)m + b + a \notin S$ must be true, or else b could not be optional.

Case 1: $nm - m + a + b < nm$. Then $a + b < m$.

Case 2: $nm < nm - m + a + b < nm + a$. Then $m < a + b < a + m$.

Case 3: $nm + a < nm - m + a + b < nm + x + 1$. Then $a + m < a + b < m + x + 1$. But this case is a contradiction since $a + m < a + b$ implies $m < b$.
Thus, for b to be optional, $2a < x + 1$ and $a + b \neq m$.

The total number of pairs will be how many satisfy $x < b < m$ and $a < \dfrac{x+1}{2}$ minus the ones where $a+b = m$. Under the conditions $x < b < m$ and $a < \dfrac{x+1}{2}$ there are always $m-x-1$ options for b and $\left\lceil \dfrac{x-1}{2} \right\rceil$ options for a. Then as long as $m-x > a$ there is a case where $a+b = m$. When $m - x > a$ for every a, b will always have one less option giving total number of pairs of $\left\lceil \dfrac{x-1}{2} \right\rceil (m - x - 2)$. When there exists some a where $m-x \leq a$ then for each a satisfying $1 \leq a \leq m-x-1$ has one less case. This gives a total number of pairs of $\left\lceil \dfrac{x-1}{2} \right\rceil (m-x-1)-(m-x-1)$ which simplifies to $\left\lceil \dfrac{x-3}{2} \right\rceil (m-x-1)$. If there exists an a where $m-x \leq a$, then $m-x < \left\lceil \dfrac{x-1}{2} \right\rceil$ meaning that $\left\lceil \dfrac{x-3}{2} \right\rceil (m - x - 1) \geq \left\lceil \dfrac{x-1}{2} \right\rceil (m - x - 2)$. Otherwise, $m - x \geq \left\lceil \dfrac{x-1}{2} \right\rceil$ which means that $\left\lceil \dfrac{x-3}{2} \right\rceil (m-x-1) < \left\lceil \dfrac{x-1}{2} \right\rceil (m-x-2)$. Thus, the total number of pairs of a and b is $\max\left(\left\lceil \dfrac{x-3}{2} \right\rceil (m - x - 1), \left\lceil \dfrac{x-1}{2} \right\rceil (m - x - 2)\right)$. $\square$

Lemma 5.10. *Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm+x$ for some integer n. Then when adding two equivalence classes mod m to T, a and b, $a < b$, with $x < a$, if only b is optional, then $2a \neq b$, $a + b \neq m$, and either $2b = m$, or $a + b < m + x + 1 \leq 2b$ and $2a \neq m$.*

Proof. Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Let S be a good numerical set for T such that S is a superset of T and contains 2 equivalence classes a, b with $x < a$. Then $S = \{0, a, b, m, a + m, b + m, \ldots (n - 1)m + a, (n - 1)m + b, nm, nm + x + 1 \rightarrow\}$.
Let only b be optional. Then $(n-1)m+a+b, (n-1)m+2a \notin S, (n-1)m+2b \in S$.
Case 1: $(n - 1)m + 2b = nm$. Then $2b = m$. Also, $(n - 1)m + a + b < nm$, so $a + b < m$.

Case 1.1: $(n-1)m+2a<(n-1)m+b$. Then $2a<b$.
Case 1.2: $(n-1)m+b<(n-1)m+2a<nm$. Then $b<2a<m$.
Case 2: $(n-1)m+2b\geq nm+x+1$. Then $2b\geq m+x+1$.
Case 2.1: $(n-1)m+a+b<nm$. Then $a+b<m$.
Case 2.1.1: $(n-1)m+2a<(n-1)m+b$. Then $2a<b$.
Case 2.1.2: $(n-1)m+b<(n-1)m+2a<nm$. Then $b<2a<m$.
Case 2.2: $nm<(n-1)m+a+b<nm+x+1$. Then $m<a+b<m+x+1$.
Case 2.2.1: $(n-1)m+2a<(n-1)m+b$. Then $2a<b$.
Case 2.2.2: $(n-1)m+b<(n-1)m+2a<nm$. Then $b<2a<m$.
Case 2.2.3: $nm<(n-1)m+2a<nm+x+1$. Then $m<2a<m+x+1$.
So if only b is optional, then $2a\neq b$, $a+b\neq m$, and either $2b=m$, or $a+b<m+x+1\leq 2b$ and $2a\neq m$.
The total number of pairs of a and b under these conditions will be the how many satisfy $x<a<b$, $2a\neq b$, and $2b=m$ plus how many satisfy $x<a<b$, $2a\neq b$, $2b\geq m+x+1$, and $a+b<m$ plus how many satisfy $x<a<b$, $2a\neq b$, $m<a+b<m+x+1$, and $2a\neq m$.
Let $x<a<b$, $2a\neq b$, and $2b=m$ be true. Then $b=\frac{m}{2}$, thus requiring that $m\equiv\pmod 2$ and $x<\frac{m}{2}$. Then a will be restricted by $x<a<\frac{m}{2}$, giving $\frac{m}{2}-x-1$ choices for a. However, the choice of $a=\frac{m}{4}$ must be removed as long as it is possible. Since $a\in\mathbb{Z}$ and $x<a$ this case only comes up when $m\equiv 0\pmod 4$ and $x<\frac{m}{4}$. Therefore, for this case the total number of pairs is $\varepsilon_6\left(\frac{m}{2}-x-1-\varepsilon_2\right)$.
Let $x<a<b$, $2a\neq b$, $2b\geq m+x+1$, and $a+b<m$ be true. $2b\geq m+x+1$ gives that $b\geq\left\lceil\frac{m+x+1}{2}\right\rceil$ and $a+b<m$ gives $b<m-a<m-x-1$ as $x+1\leq a$. Therefore, $\left\lceil\frac{m+x+1}{2}\right\rceil\leq b<m-x-1$. Then $x<a$ and $a+b<m$ gives that $x<a<m-b$. Thus, the total amount of choices for a pair of a and b is

$$\sum_{b=\left\lceil\frac{m+x+1}{2}\right\rceil}^{m-x-1} m-b-x-1.$$

The first term is $m - \left\lceil \dfrac{m+x+1}{2} \right\rceil - x - 1$ and the subsequent terms are one less, thus by reindexing this sum can be written as

$$\sum_{i=1}^{m-\left\lceil \frac{m+x+1}{2} \right\rceil - x - 1} i$$

which can be further simplified to

$$\sum_{i=1}^{\left\lfloor \frac{m-3x-3}{2} \right\rfloor} i.$$

This sum simplifies into $\frac{1}{2}\left\lfloor \dfrac{m-3x-1}{2} \right\rfloor \left\lfloor \dfrac{m-3x-3}{2} \right\rfloor$. Note that since $x < a < m-b$ and $b \geq \dfrac{m+x+1}{2}$ there are only options as long as $x < m - \dfrac{m+x+1}{2}$ which simplifies to $x < \dfrac{m-1}{3}$. Thus, the total amount of options for this case is $\dfrac{\varepsilon_9}{2}\left(\left\lfloor \dfrac{m-3x-1}{2} \right\rfloor \left\lfloor \dfrac{m-3x-3}{2} \right\rfloor\right)$.

Next we subtract the cases where $2a = b$ which occurs when $\dfrac{m+x+1}{4} \leq a < \dfrac{m}{3}$ as $\dfrac{a}{2} = b$, $2b \leq m+x+1$, and $a+b < m$ (so $3a < m$). Thus, we must subtract $\left\lceil \dfrac{m}{3} \right\rceil - \left\lceil \dfrac{m+x+1}{4} \right\rceil$ cases. This makes the total number of pairs being

$$\varepsilon_9\left(\frac{1}{2}\left(\left\lceil \frac{m-3x-1}{2} \right\rceil \left\lceil \frac{m-3x-3}{2} \right\rceil\right) - \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{m+x+1}{4} \right\rceil\right)$$

Let $x < a < b$, $2b \geq m+x+1$, $m < a+b < m+x+1$, $2a \neq b$, and $2a \neq m$ be true. Then $x < a$, $m-b < a$, and $a < m+x+1-b$. When $x < m-b$, a is restricted by $m-b < a < m+x+1-b$ resulting in x choices for a. When $x \geq m-b$, a is restricted by $x < a < m+x+1-b$ resulting in $m-b$ choices for a. Then b

is restricted by $\left\lceil\frac{m+x+1}{2}\right\rceil \leq b \leq m-1$. Thus, when there exists a $b < m-x$ meaning that $\frac{m+x+1}{2} < m-x$ which simplifies to $x < \frac{m-1}{3}$ the total number of pairs is

$$\sum_{b=\left\lceil\frac{m+x+1}{2}\right\rceil}^{m-x} x + \sum_{b=m-x+1}^{m-1} m-b$$

which simplifies to

$$x\left(m-x+1-\left\lceil\frac{m+x+1}{2}\right\rceil\right)+m(x-1)-\frac{m(m-1)}{2}+\frac{(m-x+1)(m-x)}{2}.$$

If $\frac{m+x+1}{2} \geq m-x$ then there does not exist a $b < m-x$ meaning that the total number of pairs is

$$\sum_{b=\left\lceil\frac{m+x+1}{2}\right\rceil}^{m-1} m-b$$

which simplifies to

$$m\left(m-\left\lceil\frac{m+x+1}{2}\right\rceil\right)-\frac{m(m-1)}{2}+\frac{1}{2}\left\lceil\frac{m+x-1}{2}\right\rceil\left\lceil\frac{m+x+1}{2}\right\rceil.$$

These also count the cases when $2a = m$ or $2a = b$, so the next step is subtract those cases. Luckily only one can happen at a time. When $2a = m$ it must also be true that $x < \frac{m}{2}$ and $m \equiv 0 \pmod 2$. Then b is restricted by $\left\lceil\frac{m+x+1}{2}\right\rceil \leq b < \frac{m}{2}+x+1$ resulting in a total of $\varepsilon_6\left\lfloor\frac{x+1}{2}\right\rfloor$ options where $\varepsilon_6 = 1$ when $x < \frac{m}{2}$ and $m \equiv 0 \pmod 2$ and $\varepsilon_6 = 0$ otherwise.

When $a = \frac{b}{2}$, $x < a < m+x+1-b$ becomes $x < \frac{b}{2} < m+x+1-b$ which can be simplified to $2x < b < \frac{2m+2x+2}{3}$ or $x < a < \frac{m+x+1}{3}$ and $m-b < a < m+x+1-b$ becomes $m-b < \frac{b}{2} < m+x+1-b$ which can be simplified to

$\frac{2m}{3} < b < \frac{2m+2x+2}{3}$ or $\frac{m}{3} < a < \frac{m+x+1}{3}$. Thus, the total choices for a is $\max\left(\left\lceil\frac{m+x-2}{3}\right\rceil - \max\left(\left\lfloor\frac{m}{3}\right\rfloor, x, \left\lfloor\frac{m+x-3}{4}\right\rfloor\right), 0\right)$ as the larger lower bound is the one restricting a.

Thus, the total number of pairs of a and b where $x < a < b$, $2b \geq m+x+1$, $m < a+b < m+x+1$, $2a \neq b$, and $2a \neq m$ are true is

$$\begin{aligned}
&\varepsilon_9\left(x\left(m-x+1-\left\lceil\frac{m+x+1}{2}\right\rceil\right)+m(x-1)-\frac{m(m-1)}{2}+\frac{(m-x+1)(m-x)}{2}\right)\\
&+(\varepsilon_9+(-1)^{\varepsilon_9})\left(m\left(m-\left\lceil\frac{m+x+1}{2}\right\rceil\right)-\frac{m(m-1)}{2}+\frac{1}{2}\left\lceil\frac{m+x-1}{2}\right\rceil\left\lceil\frac{m+x+1}{2}\right\rceil\right)\\
&-\varepsilon_6\left\lfloor\frac{x+1}{2}\right\rfloor-\max\left(\left\lceil\frac{m+x-2}{3}\right\rceil-\max\left(\left\lfloor\frac{m}{3}\right\rfloor, x, \left\lfloor\frac{m+x-3}{4}\right\rfloor\right), 0\right).
\end{aligned}$$

Finally, this gives us that the total number of pairs of a and b where only b is optional and $x < a$ is

$$\begin{aligned}
&\varepsilon_6\left(\frac{m}{2}-x-1-\varepsilon_2\right)+\varepsilon_9\left(\frac{1}{2}\left\lfloor\frac{m-3x-1}{2}\right\rfloor\left\lfloor\frac{m-3x-3}{2}\right\rfloor-\left\lceil\frac{m}{3}\right\rceil+\left\lceil\frac{m+x+1}{4}\right\rceil\right)\\
&+\varepsilon_9\left(x\left(m-x+1-\left\lceil\frac{m+x+1}{2}\right\rceil\right)+m(x-1)-\frac{m(m-1)}{2}+\frac{(m-x+1)(m-x)}{2}\right)\\
&+(\varepsilon_9+(-1)^{\varepsilon_9})\left(m\left(m-\left\lceil\frac{m+x+1}{2}\right\rceil\right)-\frac{m(m-1)}{2}+\frac{1}{2}\left\lceil\frac{m+x-1}{2}\right\rceil\left\lceil\frac{m+x+1}{2}\right\rceil\right)\\
&-\varepsilon_6\left\lfloor\frac{x+1}{2}\right\rfloor-\max\left(\left\lceil\frac{m+x-2}{3}\right\rceil-\max\left(\left\lfloor\frac{m}{3}\right\rfloor, x\right), 0\right)
\end{aligned}$$

$\square$

Finally, we will consider the cases where both a and b are optional.

Lemma 5.11. *Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm+x$ for some integer n. Then when adding two equivalence classes mod m to T, a and b, $a < b$, with $x > b$, if both a, b are optional, then $2b < x+1$, and $2a \neq b$. The total number of possible pairs of a and b for these conditions are $\frac{x(x-4)}{8}$ if $x \equiv 0 \pmod 4$, $\frac{(x-1)(x-5)}{8}$ if $x \equiv 1 \pmod 4$, $\frac{(x-2)^2}{8}$*

if $x \equiv 2 \pmod 4$*, and* $\dfrac{(x-3)^2}{8}$ *if* $x \equiv 3 \pmod 4$*.*

Proof. Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Let S be a good numerical set for T such that S is a superset of T and contains 2 equivalence classes a, b with $a < b$ and $x > b$. Then $S = \{0, a, b, m, a+m, b+m, \ldots, nm, nm+a, nm+b, nm+x+1 \rightarrow\}$. Let a, b be optional. Then $nm+a+b, nm+2a, nm+2b \notin S$.

Since $nm+b \notin S$, $nm+2b < nm+x+1$, so $2b < x+1$. Since $nm+a+b \notin S$, $nm+b < nm+a+b < nm+x+1$, so $a+b < x+1$.

Case 1: $nm+2a < nm+b$. Then $2a < b$.

Case 2: $nm+b < nm+2a < nm+x+1$. Then $b < 2a < x+1$. We already have $a < b$ and $a+b < x+1$, so $2a < a+b < x+1$.

In either case, $2a \neq b$. So if b is optional, then $a+b < x+1, 2b < x+1$, and $2a \neq b$. Now, if $4x$ is odd, the possible values for b are $2, 3, 4, \ldots, \dfrac{x-1}{2}$. For each value of b, there are $b-1$ possible values for a. Thus there are a total of $\dfrac{(x-3)(x-1)}{8}$ combinations. Then each of the instances where $2a = b$ must be removed. This will happen exactly once for each b that is even. Then if $x \equiv 1 \pmod 4$, there are $\dfrac{x-1}{4}$ even $b's$, and if $x \equiv 3 \pmod 4$, there are $\dfrac{x-3}{4}$ even $b's$. Subtracting these numbers yields $\dfrac{(x-1)(x-5)}{8}$ combinations when $x \equiv 1 \pmod 4$ and $\dfrac{(x-3)^2}{8}$ combinations when $x \equiv 3 \pmod 4$. Now consider when x is even. Then the possible values for b are $2, 3, 4, \ldots, \dfrac{x}{2}$. Similar to above, there are $\dfrac{x(x-2)}{8}$ combinations of a and b. The cases where $2a = b$ must also be removed, and if $x \equiv 0 \pmod 4$, there are $\dfrac{x}{4}$ cases where b is even and if $x \equiv 2 \pmod 4$, there are $\dfrac{x-2}{4}$ instances where b is even. Then there are a total of $\dfrac{(x-2)^2}{8}$ combinations when $x \equiv 2 \pmod 4$ and $\dfrac{x(x-4)}{8}$ combinations when $x \equiv 0 \pmod 4$. □

Lemma 5.12. *Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm+x$ for some integer n. Then when adding two equivalence classes mod m to T, a and b, $a < b$, with $a < x < b$, then there are no pairs a, b with* *both a and b are optional.*

Proof. Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm+x$ for some integer n. Let S be a good numerical set for T such that S is a superset of T and contains 2 equivalence classes a, b with $a < x < b$. Then $S = \{0, a, b, m, a+m, b+m, \ldots (n-1)m+a, (n-1)m+b, nm, nm+a, nm+x+1 \rightarrow\}$. Let both a and b be optional. Then $(n-1)m+a+b, (n-1)m+2b, nm+2a, nm+a+b \notin S$. $nm + a + b < nm + x + 1$, so $a + b < x + 1$. This is a contradiction because $a \geq 1$ and $a < x < b$, so $x + 1 \leq x + a < b + a$, and therefore $x + 1 < a + b$.
So there are no pairs a, b with $a < x < b$ such that a and b are both optional. $\square$

Lemma 5.13. *Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm+x$ for some integer n. Then when adding two equivalence classes mod m to T, a and b, $a < b$, with $x < a$, if both a and b are optional, then $2a \neq b$, $2b \neq m$, $a + b \neq m$, $2a \neq m$, and $2b < m + x + 1$.*

Proof. Let T be a numerical semigroup with a single small atom m and Frobenius number $g = nm + x$ for some integer n. Let S be a good numerical set for T such that S is a superset of T and contains 2 equivalence classes a, b with $x < a$. Then $S = \{0, a, b, m, a + m, b + m, \ldots (n-1)m + a, (n-1)m + b, nm, nm + x + 1 \rightarrow\}$.
Let both a and b be optional. Then $(n-1)m+a+b, (n-1)m+2a, (n-1)m+2b \notin S$.
Case 1: $(n-1)m + 2b < nm$. Then $2b < m$. Also, $(n-1)m + a + b < nm$. Then $2a < b$.
Case 1.1: $(n-1)m + 2a < (n-1)m + b$. Then $2a < b$.
Case 1.2: $(n-1)m + b < (n-1)m + 2a < nm$. Then $b < 2a < m$.
Case 2: $nm < (n-1)m + 2b < nm + x + 1$. Then $m < 2b < m + x + 1$.
Case 2.1: $(n-1)m + a + b < nm$. Then $a + b < m$.
Case 2.1.1: $(n-1)m + 2a < (n-1)m + b$. Then $2a < b$.
Case 2.1.2: $(n-1)m + b < (n-1)m + 2a < nm$. Then $b < 2a < m$.
Case 2.2: $nm < (n-1)m + a + b < nm + x + 1$. Then $m < a + b < m + x + 1$.
Case 2.2.1: $(n-1)m + 2a < (n-1)m + b$. Then $2a < b$.
Case 2.2.2: $(n-1)m + b < (n-1)m + 2a < nm$. Then $b < 2a < m$.
Case 2.2.3: $nm < (n-1)m + 2a < nm + x + 1$. Then $m < 2a < m + x + 1$.

So if both a and b are optional, then $2a \neq b$, $2b \neq m$, $a + b \neq m$, $2a \neq m$, and $2b < m + x + 1$.

First, all of the cases where $2b < m + x + 1$ are true. If $2b < m + x + 1$, then $b \leq \left\lceil \frac{m+x-1}{2} \right\rceil$. Then b can equal $1, 2, 3, \ldots, \left\lceil \frac{m+x-1}{2} \right\rceil$. But since $a < b$, the first a integers must be subtracted. Then the total number of combinations is

$$\sum_{a=x+1}^{\left\lceil \frac{m+x-3}{2} \right\rceil} \left\lceil \frac{m+x-1}{2} \right\rceil - a.$$

The sum starts with $a = x + 1$ since $a > x$. Then this sum simplifies to

$$\frac{\left\lceil \frac{m+x-1}{2} \right\rceil \left\lceil \frac{m+x-3}{2} \right\rceil}{2} - \left\lceil \frac{m+x-1}{2} \right\rceil x + \frac{x(x+1)}{2}.$$

Next we must subtract the cases where $2a = b$, $2b = m$, $a + b = m$, or $2a = m$.

When $2a = b$, since we have that $b \leq \left\lceil \frac{m+x-1}{2} \right\rceil$, $x < a \leq \left\lfloor \frac{m+x}{4} \right\rfloor$. Thus, there are $\left\lfloor \frac{m+x}{4} \right\rfloor - x$ options for a as long as $x < \left\lfloor \frac{m+x}{4} \right\rfloor$ otherwise there is no such a that exists. When $2b = m$, m must be even and $b = \frac{m}{2}$. Then $x < a < \frac{m}{2}$ which gives $\frac{m-2x-2}{2}$ options for a when $b = \frac{m}{2}$. When $a + b = m$, b is restricted by $b \leq \left\lceil \frac{m+x+1}{2} \right\rceil$. Since $a + b = m$, $m \leq \left\lceil \frac{m+x+1}{2} \right\rceil + a$ therefore, $a \geq m - \left\lceil \frac{m+x+1}{2} \right\rceil$. a is also under the condition $x < a \leq \left\lfloor \frac{m-1}{2} \right\rfloor$. Thus, the total number of possible pairs is $\max\left(0, \min\left(\left\lfloor \frac{m-1}{2} \right\rfloor - x, \left\lfloor \frac{m-1}{2} \right\rfloor - \left\lfloor \frac{m-x+1}{2} \right\rfloor + 1\right)\right)$. When $2a = m$, m is even and $a < b \leq \left\lceil \frac{m+x-1}{2} \right\rceil$. Therefore, for each $\frac{m}{2} < b \leq \left\lceil \frac{m+x-1}{2} \right\rceil$ can picked, giving a total of $\left\lfloor \frac{x}{2} \right\rfloor$ choices as long as $x < \frac{m}{2}$. Finally, we must add back on the cases when two of $2a = b$, $2b = m$, $a + b = m$, or $2a = m$ are met, which can only be $2a = b$ and $2b = m$. This means that $a = \frac{m}{4}$ and $b = \frac{m}{2}$ which requires that $m \equiv 0 \pmod 4$ and $x < \frac{m}{2}$.

Thus, there number of pairs of a and b is

$$\frac{\left\lceil \frac{m+x-1}{2} \right\rceil \left\lceil \frac{m+x-3}{2} \right\rceil}{2} - \left\lceil \frac{m+x-1}{2} \right\rceil x + \frac{x(x+1)}{2} - \max\left(\left\lfloor \frac{m+x}{4} \right\rfloor - x, 0\right)$$
$$- \varepsilon_6 \frac{m-2x-2}{2} - \max\left(0, \min\left(\left\lfloor \frac{m-1}{2} \right\rfloor - x, \left\lfloor \frac{m-1}{2} \right\rfloor - \left\lfloor \frac{m-x+1}{2} \right\rfloor + 1\right)\right)$$
$$- \left\lfloor \frac{x}{2} \right\rfloor \varepsilon_6 + \varepsilon_2.$$

$\square$

Using each of Lemmas 5.2 through 5.13 gives the total number of possible pairs of a and b under certain conditions. Depending on these conditions, each of these totals must be multiplied by a different coefficient.

Lemma 5.14. *Let T be a numerical semigroup with a single small atom m and Frobenius number $g = mn + x$, where n is an integer. Then when adding two equivalence classes of m, a and b, $a < b$ to T to create a numerical set that maps to T, then if neither a or b are optional, there is only one way to create that form of numerical set, if a is optional and $x < a$, there are n ways to create that form of numerical set, if a is optional and $a < x$, there are $n + 1$ ways to create that numerical set, if b is optional and $x < b$, there are n ways to create that form of numerical set, if b is optional and $b < x$, there are $n + 1$ ways to create that form of numerical set, if both a and b are optional and $x < a$, there are $2n - 1$ ways to create that form of numerical set, if both a and b are optional and $a < x < b$, there are $2n$ ways to create that form of numerical set, and if both a and b are optional and $b < x$, then there are $2n + 1$ ways to create that form of numerical set.*

Proof. Let T be a numerical semigroup with a single small atom m and Frobenius number $g = mn + x$, where n is an integer. Consider the numerical set S which is

a superset of T but with the elements of $a \pmod m$ and $b \pmod m$ added to it. If neither a or b are optional, then there is only one way to create S since both a, b.
Now consider when a is optional. Then each of $a, a+m, \ldots, (n-1)m+a$ can be starting elements for the a class. If $x < a$, then these are the only possible starting elements, but if $a < x$, then $nm+a$ is also a possible starting element for the a class. Then when $x < a$, there are n possible starting locations for the a class, and when $x < a$, there are $n+1$ starting locations for the a class.
Now consider when b is optional. Similar to above, there are n possible starting locations for the b class when $x < b$ and there are $n+1$ starting locations for the b class when $b < x$.
Now consider when both a and b are optional. First, consider when $x < a$. At least one of a or b classes needs to start before m, otherwise S will not map to T. So there are 2 choices to pick a starting element. Then there are $n-1$ choices to pick the starting location for the other element that are not in the position before m. Thus there is a total of $2(n-1) = 2n-2$. However, the case where both a and b are starting before m is not counted, so one more must be added to this count, making the total $2n-1$. Now, when $a < x < b$, one more additional case exists where b is only element before m and the a equivalence class starts after the nm element, making the total $2n$. Similarly, one additional case is gained when $b < x$, making that total $2n+1$. $\square$

Given each of the sums, multiplying each part by its corresponding coefficient from Lemma 5.14, the final equation can be created, which is displayed in Table 6. This equation is the complete equation for the number of ways to add two equivalence classes to a numerical semigroup modulo its multiplicity to create a good numerical set for that numerical semigroup. To find the total number of ways to add 0 or 1 equivalence classes modulo m, then add $1 + \left\lfloor \dfrac{m+x-2}{2} \right\rfloor - \varepsilon_0$ to the equation in Table 6. In the semigroups that have a small enough multiplicity such that no more than 2 equivalence classes can be added to make a numerical set, then this formula completely counts the number of good numerical sets for that multiplicity. Otherwise, this formula serves as a lower bound for the count of good numerical sets for the numerical semigroup with a single small atom.

6 Appendix

g	k	Total Number of Good Numerical Sets	Good % of Numerical Sets with same g	Good % of Feasible Sets from the pair poset
3	2	1	50%	100%
4	3	2	50%	100%
5	4	2	25%	50%
6	5	6	37.5%	75%
7	6	7	21.88%	43.75%
8	7	18	28.13%	56.25%
9	8	28	21.88%	43.75%
10	9	60	23.44%	46.88%
11	10	108	21.09%	42.19%
12	11	228	22.27%	44.53%
13	12	423	20.65%	41.31%
14	13	868	21.19%	40.51%
15	14	1659	20.25%	40.51%
16	15	3392	20.70%	41.41%
17	16	6557	20.01%	40.02%
18	17	13290	20.28%	40.56%
19	18	25983	19.82%	39.65%
20	19	52500	20.03%	40.05%

Table 7: The amount of numerical sets that map to a numerical semigroup with a single small nyb. Small nyb = Frobenius - 1.

g	k	Total Number of Good Numerical Sets	Good % of Numerical Sets with same g	Good % of Feasible Sets from the pair poset
5	3	2	25%	66.67%
6	4	4	25%	66.67%
7	5	6	18.75%	50.00%
8	6	10	15.63%	41.67%
9	7	22	17.19%	45.83%
10	8	40	15.63%	41.67%
11	9	78	15.23%	40.63%
12	10	148	14.45%	38.54%
13	11	290	14.16%	37.76%
14	12	564	13.77%	36.72%
15	13	1116	13.62%	36.72%
16	14	2188	13.35%	35.61%
17	15	4364	13.32%	35.51%
18	16	8616	13.15%	35.06%
19	17	17158	13.09%	34.91%
20	18	33992	12.97%	34.58%

Table 8: The amount of numerical sets that map to a numerical semigroup with a single small nyb. Small nyb = Frobenius number - 2.

g	k	Total Number of Good Numerical Sets	Good % of Numerical Sets with same g	Good % of Feasible Sets from the pair poset
7	4	3	9.375%	33.33%
8	5	6	9.375%	33.33%
9	6	9	7.03%	25.00%
10	7	18	7.03%	25.00%
11	8	31	6.05%	21.53%
12	9	68	6.64%	23.61%
13	10	125	6.10%	21.70%
14	11	262	6.40%	22.74%
15	12	484	5.91%	21.01%
16	13	994	6.07%	21.57%
17	14	1886	5.76%	20.46%
18	15	3844	5.87%	20.86%
19	16	7465	5.70%	20.25%
20	17	15096	5.76%	20.48%

Table 9: The amount of numerical sets that map to a numerical semigroup with a single small nyb. The small nyb = Frobenius number - 3.

g	k	Total Number of Good Numerical Sets	Good % of Numerical Sets with same g	Good % of Feasible Sets from the pair poset
9	5	8	6.250%	29.63%
10	6	16	6.250%	29.63%
11	7	26	5.078%	24.07%
12	8	52	5.078%	24.07%
13	9	96	4.688%	22.22%
14	10	182	4.443%	21.06%
15	11	370	4.517%	21.41%
16	12	718	4.382%	20.78%
17	13	1442	4.401%	20.86%
18	14	2836	4.327%	20.52%
19	15	5590	4.265%	20.22%
20	16	11048	4.214%	19.98%

Table 10: The amount of numerical sets that map to a numerical semigroup with a single small nyb. Small nyb = Frobenius number - 4.

g	k	Total Number of Good Numerical Sets	Good % of Numerical Sets with same g	Good % of Feasible Sets from the pair poset
11	6	18	3.516%	22.22%
12	7	36	3.516%	22.22%
13	8	61	2.979%	18.83%
14	9	122	2.979%	18.83%
15	10	229	2.795%	17.67%
16	11	458	2.795%	17.67%
17	12	857	2.615%	16.53%
18	13	1766	2.695%	17.03%
19	14	3399	2.593%	16.39%
20	15	6920	2.640%	16.69%

Table 11: The amount of numerical sets that map to a numerical semigroup with a single small nyb. Small nyb = Frobenius number - 5.

g	k	Total Number of Good Numerical Sets	Good % of Numerical Sets with same g	Good % of Feasible Sets from the pair poset
13	7	50	2.441%	20.58%
14	8	100	2.441%	20.58%
15	9	176	2.148%	18.11%
16	10	352	2.148%	18.11%
17	11	664	2.026%	17.08%
18	12	1328	2.026%	17.08%
19	13	2578	1.967%	16.58%
20	14	5068	1.933%	16.29%

Table 12: The amount of numerical sets that map to a numerical semigroup with a single small nyb. Small nyb = Frobenius number - 6.

g	k	Total Number of Good Numerical Sets	Good % of Numerical Sets with same g	Good % of Feasible Sets from the pair poset
15	8	135	1.648%	18.52%
16	9	270	1.648%	18.52%
17	10	485	1.48%	16.63%
18	11	970	1.48%	16.63%
19	12	1854	1.414%	15.90%
20	13	3708	1.414%	15.90%
21	14	7202	1.374%	15.44%
22	15	14404	1.374%	15.44%
23	16	28031	1.337%	15.02%
24	17	56640	1.350%	15.17%

Table 13: The amount of numerical sets that map to a numerical semigroup with a single small nyb. Small nyb = Frobenius number - 7.

g	k	Total Number of Good Numerical Sets	Good % of Numerical Sets with same g	Good % of Feasible Sets from the pair poset
17	9	385	1.175%	17.60%
18	10	770	1.175%	17.60%
19	11	1417	1.081%	16.20%
20	12	2834	1.081%	16.20%
21	13	5432	1.036%	15.52%
22	14	10864	1.036%	15.52%
23	15	21192	1.011%	15.14%
24	16	42384	1.011%	15.14%
25	17	83465	0.9950%	14.91%
26	18	165930	0.9890%	14.82%

Table 14: The amount of numerical sets that map to a numerical semigroup with a single small nyb. Small nyb = Frobenius number - 8.

g	k	Total Number of Good Numerical Sets	Good % of Numerical Sets with same g	Good % of Feasible Sets from the pair poset
19	10	1065	0.8125%	16.23%
20	11	2130	0.8125%	16.23%
21	12	3962	0.7557%	15.10%
22	13	7924	0.7557%	15.10%
23	14	15289	0.7290%	14.56%
24	15	30578	0.7290%	14.56%
25	16	59887	0.7139%	14.26%
26	17	119774	0.7139%	14.26%

Table 15: The amount of numerical sets that map to a numerical semigroup with a single small nyb. Small nyb = Frobenius number - 9.

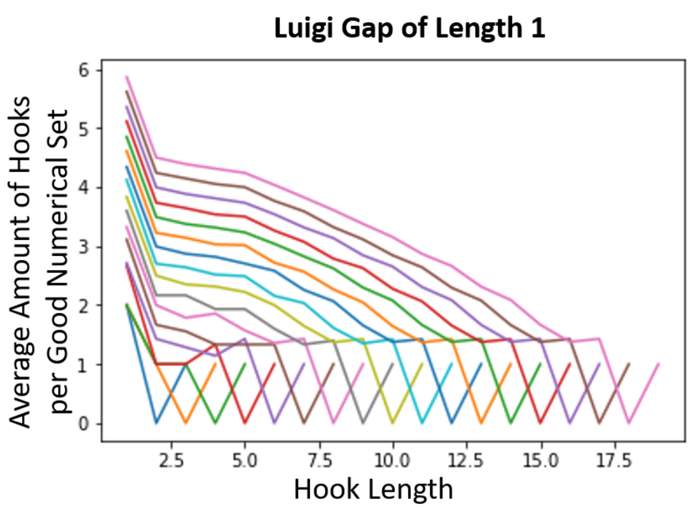


Figure 11: Luigi dinosaur for $\ell = 1$, the average number of hooks of each length per good numerical set for numerical semigroups with a single small nyb and luigi gap of 1. Each line corresponds to a different numerical semigroup with the small nyb being the value at which there are no hooks of that length.

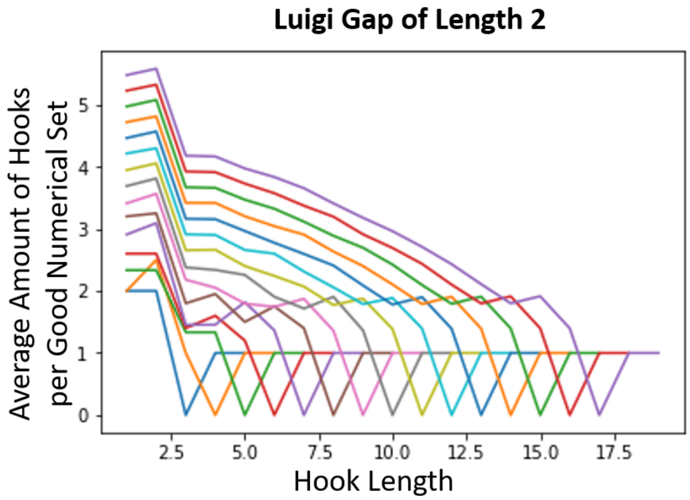


Figure 12: Luigi dinosaur for $\ell = 2$, the average number of hooks of each length per good numerical set for numerical semigroups with a single small nyb and luigi gap of 2. Each line corresponds to a different numerical semigroup with the small nyb being the value at which there are no hooks of that length.

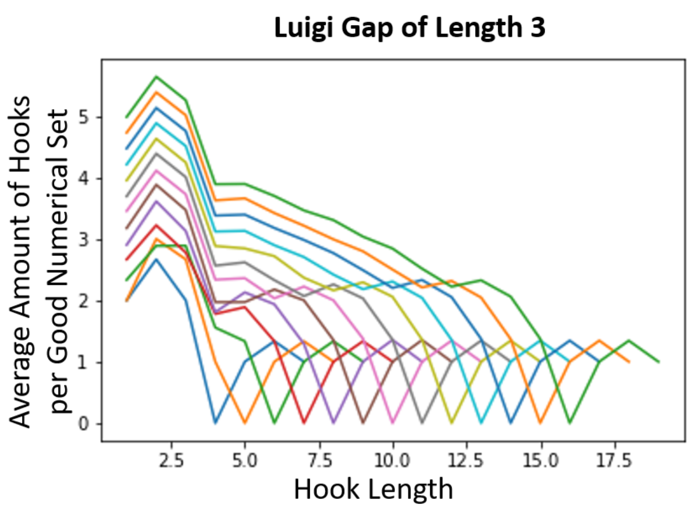


Figure 13: Luigi dinosaur for $\ell = 3$, the average number of hooks of each length per good numerical set for numerical semigroups with a single small nyb and luigi gap of 3. Each line corresponds to a different numerical semigroup with the small nyb being the value at which there are no hooks of that length.

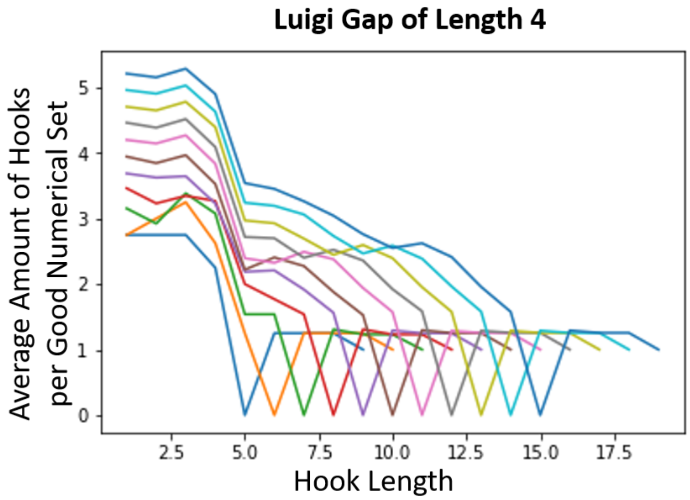


Figure 14: Luigi dinosaur for $\ell = 4$, the average number of hooks of each length per good numerical set for numerical semigroups with a single small nyb and luigi gap of 4. Each line corresponds to a different numerical semigroup with the small nyb being the value at which there are no hooks of that length.

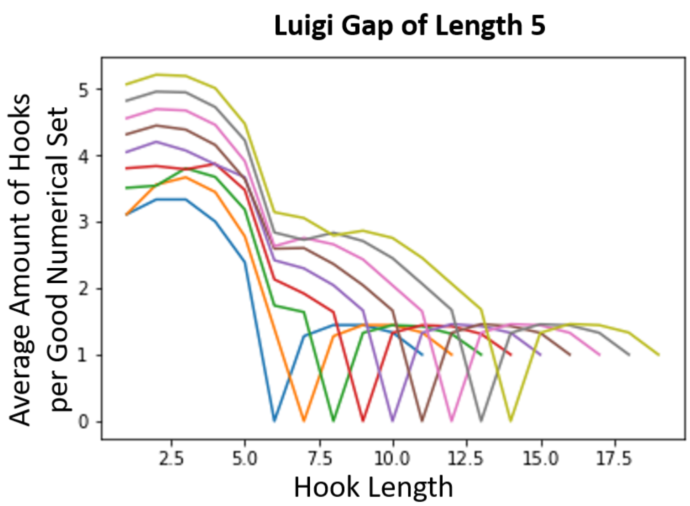


Figure 15: Luigi dinosaur for $\ell = 5$, the average number of hooks of each length per good numerical set for numerical semigroups with a single small nyb and luigi gap of 5. Each line corresponds to a different numerical semigroup with the small nyb being the value at which there are no hooks of that length.

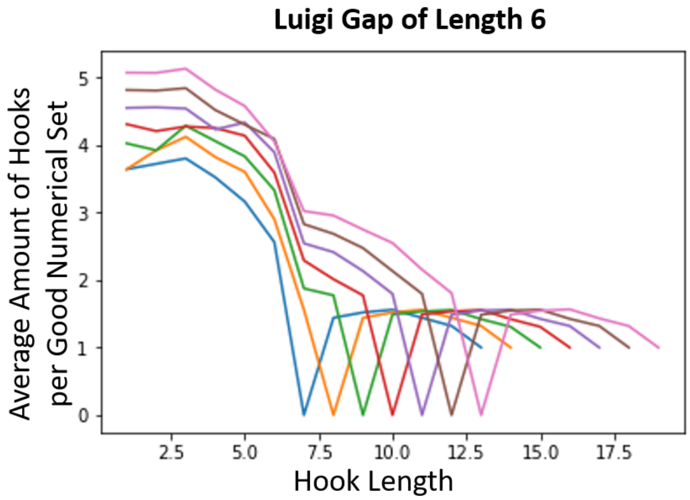


Figure 16: Luigi dinosaur for $\ell = 6$, the average number of hooks of each length per good numerical set for numerical semigroups with a single small nyb and luigi gap of 6. Each line corresponds to a different numerical semigroup with the small nyb being the value at which there are no hooks of that length.

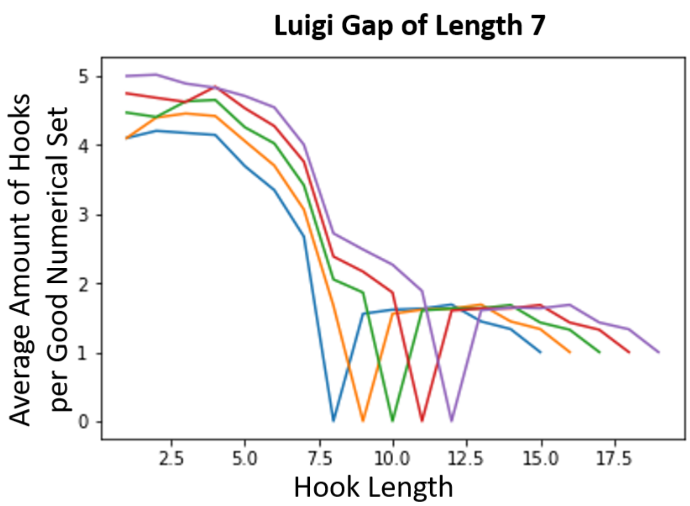


Figure 17: Luigi dinosaur for $\ell = 7$, the average number of hooks of each length per good numerical set for numerical semigroups with a single small nyb and luigi gap of 7. Each line corresponds to a different numerical semigroup with the small nyb being the value at which there are no hooks of that length.

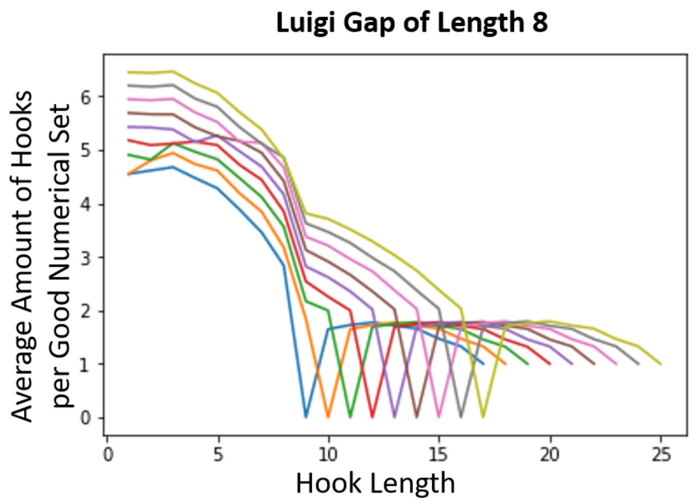


Figure 18: Luigi dinosaur for $\ell = 8$, the average number of hooks of each length per good numerical set for numerical semigroups with a single small nyb and luigi gap of 8. Each line corresponds to a different numerical semigroup with the small nyb being the value at which there are no hooks of that length.

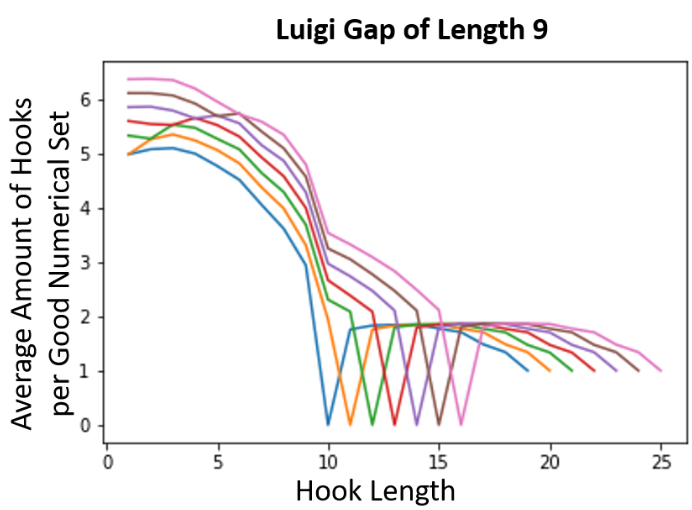


Figure 19: Luigi dinosaur for $\ell = 9$, the average number of hooks of each length per good numerical set for numerical semigroups with a single small nyb and luigi gap of 9. Each line corresponds to a different numerical semigroup with the small nyb being the value at which there are no hooks of that length.

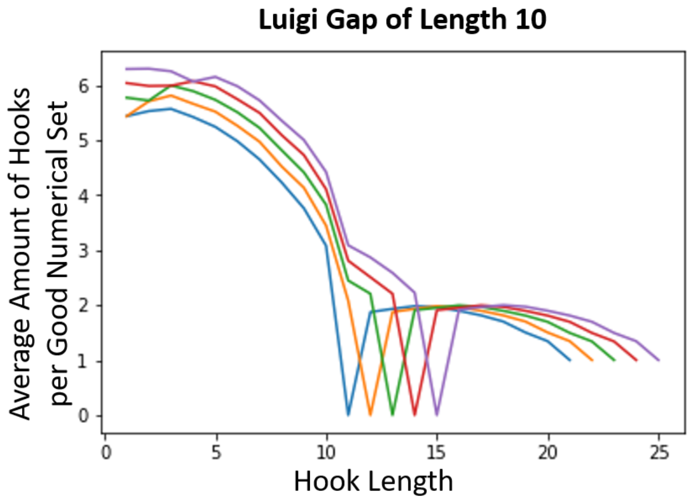


Figure 20: Luigi dinosaur for $\ell = 10$, the average number of hooks of each length per good numerical set for numerical semigroups with a single small nyb and luigi gap of 10. Each line corresponds to a different numerical semigroup with the small nyb being the value at which there are no hooks of that length.

7 Acknowledgements

Thanks to San Diego State University for generously hosting this research. Thanks to April Chen, Liam Lawson, and Deepesh Singhal for the void poset.

References

[1] J. Marzuloa and A. Miller. Counting numerical sets with no small atoms. *Journal of Combinatorial Theory, Series A*, 2010.