QUESTION 0.4

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Contents

1. Preliminaries

Definition 1. (Numerical Semigroup) We say $S \subseteq \mathbb{Z}_{\geq 0}$ is a numerical semigroup (or monoid) if:

(1) S is closed under addition
(2) S has a finite complement
(3) $0 \in S$

Definition 2. (Frobenius Number) We say the Frobenius number of a semigroup S is the largest element in the complement of S, i.e.

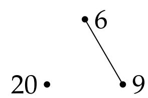
$$F(S) = \max(\mathbb{Z}_{\geq 0} \backslash S)$$

Definition 3. (Simplicial Complex) A *simplicial complex* is $\Delta \subseteq \mathcal{P}([n])$ such that if $A \in \Delta$ and $B \subseteq A$, then $B \in \Delta$.

Definition 4. (Δ_m) Let $S = \langle n_1, ..., n_k \rangle$ and fix $n \in S$. Then

$$\Delta_n = \{F \subseteq \{n_1, \ldots, n_k\} : n - \sum_{n_i \in F} n_i \in S\}$$

Example 5. Let $S = \langle 6, 9, 20 \rangle$. Then $\Delta_{60} = \{20, \{6, 9\}, 6, 9, \emptyset\}$



Definition 6. (Euler Characteristic) For a simplicial complex Δ, we say $f_{-1} = 1$, and f_i is the number of i-dimensional faces in Δ for all $i \geq 0$. We define the *Euler characteristic* of Δ to be

$$\chi(\Delta) = \sum_{i=-1}^{\infty} (-1)^{i+1} f_i$$

Definition 7. (Homology) For a simplicial complex Δ, we say $H_i(\Delta)$ is the number of i-dimensional holes of Δ.

Definition 8. (Hilbert Series) We say the *Hilbert series* of a numerical semigroup N is given by

$$\mathcal{H}(N) = \sum_{n \in N} t^n$$

It is known that

$$\mathcal{H}(N; t) = \frac{\mathcal{K}(N; t)}{(1 - t^{n_1}) \cdots (1 - t^{n_k})}$$

where $\mathcal{K}(N; t) = \sum_{n \in N} \chi(\Delta_n) t^n$. If, for some $n \in N$, $\chi(\Delta_n) \neq 0$, we say that n is *shaded*.

2. On the Subject of Skeletal Monoids

Throughout, let $a_1, \ldots, a_k$ be pairwise relatively prime, $P = a_1 \cdots a_k$, and $n_i = P/a_i$ for $1 \leq i \leq k$.

Definition 9. (Skeletal Monoids) We say $S = \langle n_1, \cdots, n_k \rangle$ is *skeletal*.

Theorem 10. *(Modernistic) Let $S = \langle \frac{P}{a_1}, \cdots, \frac{P}{a_k} \rangle$ be skeletal. Then, for $0 \leq n \leq k - 1$, Δ_{nP} is a uniform graph of dimension n on k vertices.*

Proof. Fix n. For any subset T of generators of size $m \leq n$, we have a factorization of nP using exactly the generators in T by multiplying each generator $n_j = P/a_j$ by $m_j a_j$, where m_j is a natural number and so that the m_j sum to n (which is always possible so long as $m \leq n$). Hence the uniform graph of dimension n on k verties is a subgraph of Δ_{nP}. Then, it suffices to show that there are no factorizations of nP using strictly more than n generators. Suppose such a factorization existed, of the form

$$\sum_{i=1}^{k} c_i n_i = nP \tag{11}$$

where at least $n+1$ of the c_i are nonzero. Since a_i divides P and each generator but n_i, we must have that $a_i \mid c_i$ for each $1 \leq i \leq k$. As at least $n+1$ of the c_i are nonzero, we must have that $c_i \geq a_i$ for at least $n+1$ coefficients, and the remaining coefficients are at least nonnegative. Substituting these observations into the left hand side of (11) gives us

$$(n+1)P \leq nP$$

a contradiction. □

Corollary 12. *Let $S = \langle \frac{P}{a_1}, \cdots, \frac{P}{a_k} \rangle$ be skeletal. Then, Δ_P is the totally disconnected k-graph.*

• This is the statement Vadim asked us to look at originally. - Sam

Proof. Set $n = 1$ in the preceding Theorem. □

Proposition 13. *Let $S = \langle \frac{P}{a_1}, \cdots, \frac{P}{a_k} \rangle$ be skeletal. Then, if Δ_Q is a uniform graph of dimension n on k vertices with $n < k$, $Q = nP$.*

Proof. Since Δ_Q contains every subset of size n, we have a factorization

$$\sum_{i=2}^{n+1} c_i n_i = Q$$

where each $c_i \neq 0$. Since $a_1 \mid n_i$ for all $i \neq 1$, $a_1 \mid Q$. By a similar argument we have that $a_i \mid Q$ for all i, so $P \mid Q$, i.e. $Q = mP$ for some m. If $m \geq k$, then we have a completely filled in graph, as we have a factorization using all the generators by multiplying each generator $n_j = P/a_j$ by $m_j a_j$, where m_j is a natural number so that the m_j sum to m (which is always possible so long as $k \leq m$). So $0 \leq m \leq k-1$, and the characterization from Theorem 10 assures us $m = n$ and $Q = nP$. □

Corollary 14. *Let $S = \langle \frac{P}{a_1}, \cdots, \frac{P}{a_k} \rangle$ be skeletal. Then*

$$F(S) = (k-1)P - \sum_{i=1}^{k} n_i$$

Proof. For any numerical semigroup S, we have that $\Delta_{F(S)+\sum_{i=1}^{k} n_i}$ is the uniform graph of dimension $k-1$ on n vertices, since subtracting any proper subset of the generators from this quantity remains larger than $F(S)$. The preceding Theorem then guarantees that $F(S) + \sum_{i=1}^{k} n_i = (k-1)P$. □

Lemma 15. *Let $S = \langle \frac{P}{a_1}, \cdots, \frac{P}{a_k} \rangle$ be skeletal. For all $Q \in S$, Q is of the form*

$$Q = mP + \sum_{i=1}^{k} c_i n_i \tag{16}$$

such that $c_i < a_i$ for all i. Further, c_i and m are unique.

Proof. Suppose Q has two factorizations, $Q = \sum_{i=1}^{k} c_i' n_i = \sum_{i=1}^{k} b_i' n_i$. Letting $c_i = c_i' \pmod{a_i}$, $b_i = b_i' \pmod{a_i}$, $m_1 = \sum_{i=1}^{k} \lfloor \frac{c_i'}{a_i} \rfloor$ and $m_2 = \sum_{i=1}^{k} \lfloor \frac{b_i'}{a_i} \rfloor$, we have $m_1 P + \sum_{i=1}^{k} c_i n_i = m_2 P + \sum_{i=1}^{k} b_i n_i$. By cancellation properties, we would have $\sum_{i=1}^{k} (b_i - c_i) n_i = (m_1 - m_2) P$. For each a_i, $a_i | P$, but $a_i \nmid n_i$, hence $a_i | (b_i - c_i)$. But by construction $b_i, c_i < a_i$, so $|b_i - c_i| < a_i$. Hence $a_i \nmid (b_i - c_i)$. Therefore $b_i - c_i = 0$, and so $m_1 - m_2 = 0$. Therefore we can uniquely write Q as $Q = mP + \sum_{i=1}^{k} c_i n_i$ □

Lemma 17. *Let $S = \langle \frac{P}{a_1}, \cdots, \frac{P}{a_k} \rangle$ be skeletal. Then the Δ_{mP} is uniform of dimension $m - 1$ on k vertices.*

Proof. Since for all $i, a_i n_i = P$, there is a factorization for mP using any m distinct generators. Hence Δ_{mP} has all $m - 1$ dimensional faces. □

Lemma 18. *For $n \in \mathbb{N}$ and $n < k$,*

$$\sum_{m=0}^{n} \binom{k}{m} (-1)^m = \binom{k-1}{n} (-1)^n$$

Proof. For $n = 1$, we have

$$\sum_{m=0}^{n} \binom{k}{m} (-1)^m = \binom{k}{0} - \binom{k}{1} = 1 - k = \binom{k-1}{n} (-1)^n$$

Proceed by induction; assuming the statement holds for n, note

$$\begin{aligned}
\sum_{m=0}^{n+1} \binom{k}{m} (-1)^m &= \sum_{m=0}^{n} \binom{k}{m} (-1)^m + \binom{k}{n+1} (-1)^{n+1} \\
&= (-1)^n \frac{(k-1)!}{(k-1-n)!n!} + (-1)^{n+1} \frac{k!}{(k-n-1)!(n+1)!} \\
&= (-1)^{n+1} \frac{-(k-1)!(n+1) + k!}{(k-1-n)!(n+1)!} = (-1)^{n+1} \frac{(k-1)!(k-n-1)}{(k-n-1)!(n+1)!} \\
&= (-1)^{n+1} \frac{(k-1)!}{(k-n-2)!(n+1)!} = \binom{k-1}{n+1} (-1)^{n+1}
\end{aligned}$$

□

Theorem 19. *Let $S = \langle \frac{P}{a_1}, \cdots, \frac{P}{a_k} \rangle$ be skeletal. and $Q \in S$. Then,*

(1) If $P \nmid Q$, $\chi(\Delta_Q) = 0$.

(2)

• TODO: Prove the second equality.

$$\mathcal{K}(S; t) = \sum_{n=0}^{k-1} \sum_{m=0}^{n} \binom{k}{m} t^{Pn} (-1)^m = \sum_{n=0}^{k-1} \binom{k-1}{n} (-1)^n t^{Pn} = (1 - t^P)^{k-1}$$

Proof. By Lemma 15, Q can be uniquely written as $Q = mP + \sum_{i=1}^{k} b_i n_i$. Denote the number of nonzero b_i as β. Looking at the simplicial complex of Q restricted to n_i such that $b_i = 0$, this is exactly the simplicial complex of mP' in the restricted semigroup generated by those n_i, and P' is the product of them, and by Lemma 17, is uniform of dimension $m-1$ on $k-\beta$ vertices. The Euler characteristic of this restricted graph is given by

$$A = \sum_{i=0}^{k-\beta} \binom{k-\beta}{i} (-1)^i$$

For each facet in this restricted graph, we may attach any subset of the β other vertices, increasing the cardinality of each facet by the number attached. Extending the graph to include all k vertices, we in the restricted graph, we may from each face in the restricted graph, we add a face for each new vertex, and pair of vertices and so on. Hence the Euler characteristic of the full graph is given by

• TODO: Make wording more clear

$$\sum_{i=0}^{\beta} \binom{\beta}{i} (-1)^i A$$

By factoring and using the binomial theorem, we have

$$\chi(\Delta_Q) = A \sum_{i=0}^{\beta} \binom{\beta}{i} (-1)^i = A(1 + (-1))^\beta = A(0)^\beta = 0$$

Therefore if $\beta \neq 0$, then $\chi(\Delta_Q) = 0$, hence (1).
If $\beta = 0$, the $P|Q$ and $\chi(\Delta_Q) = A = \sum_{i=0}^{k} \binom{k}{i} (-1)^i$. Thus the only nonzero coefficients of $\mathcal{K}(S;t)$ are from $Q = mP$, and

$$\mathcal{K}(S;t) = \sum_{n=0}^{k-1} \sum_{m=0}^{n} \binom{k}{m} t^n (-1)^m$$

The second equality follows from Lemma 18 and the last equality follows from the Binomial Theorem. □

Lemma 20. *Let $S = \langle n_1, \cdots, n_k \rangle$. If $lcm(n_1, \cdots, n_k)/n_i$ are pairwise relatively prime, then S is skeletal.*

Proof. Write $a_i = \mathrm{lcm}(n_1, \cdots, n_k)/n_i$. Then, $n_i = \mathrm{lcm}(n_1, \cdots, n_k)/a_i$, and so $\mathrm{lcm}(n_1, \cdots, n_k)$ must be exactly $a_1 \cdots a_k$. □

Theorem 21. *Let $S = \langle n_1, \cdots, n_k \rangle$, and suppose there exists an $m \in S$ so that Δ_m is k singletons. Then, S is a skeletal monoid. Further, $m = lcm(n_1, \ldots, n_k)$.*

Proof. Since Δ_m is k singletons, Δ_{m-n_i} must contain only the single vertex n_i, since any other vertices imply a factorization of m using multiple generators. So $n_i \mid m - n_i$ for all n_i, and in particular $n_i \mid m$ for all m; thus $b \cdot \mathrm{lcm}(n_1, \ldots, n_k) = m$ for some natural number b. If $b > 1$, then we have a factorization of $m = \mathrm{lcm}(n_1, \ldots, n_k) + (b-1)\mathrm{lcm}(n_1, \ldots, n_k)$ using multiple generators,
so $m = \mathrm{lcm}(n_1, \ldots, n_k) = a_i n_i$. Now, since Δ_m is k singletons, then

$$\mathcal{Z}_S(m) = \{(a_1, 0, \cdots, 0), \cdots, (0, \cdots, 0, a_k)\}$$

Suppose towards contradiction $\gcd(a_i, a_j) = c > 1$ for some i, j. Then $a_i = ca'_i$ and $a_j = ca'_j$. Therefore $m = ca'_i n_i = ca'_j n_j \Rightarrow a_i = a_j$. So we have $m = ca'_i n_i = (c-1)a'_i n_i + a'_i n_i = (c-a)a'_i n_i + a'_j n_j$ and $(0, \cdots, (c-1)a'_i, \cdots, a'_j, \cdots, 0) \in \mathcal{Z}_S(m)$, a contradiction. Hence for all i, j, $i \neq j$, $\gcd(a_i, a_j) = 1$. □

3. On the Subject of Gluing

The following lemma will save us much grief:

Lemma 22. *(Rosales, Sanchez, [2]) Let S_1 and S_2 be numerical semigroups minimally generated by $\{n_1, \ldots, n_r\}$ and $\{n_{r+1}, \ldots, n_e\}$, respectively. Let $\lambda \in S_1 \backslash \{n_1, \ldots, n_r\}$ and let $\mu \in S_2 \backslash \{n_{r+1}, \ldots, n_e\}$ with gcd$(\lambda, \mu) = 1$. Then,*

$$S = \langle \mu n_1, \cdots, \mu n_r, \lambda n_{r+1}, \cdots, \lambda n_e \rangle$$

is minimally generated. We say the above is a gluing.

Proposition 23. *For any natural number N, let*

$$S = \langle 4N, 4N+1, \cdots, 5N-1, 5N+1, 5N+2, \cdots, 6N \rangle$$

Then, Δ_{10N} is the simplicial complex with N disconnected line segments.

Proof. Since the greatest common divisor of two consecutive numbers is 1, S is indeed a numerical semigroup, and is minimally generated since twice the smallest generator $4N$ is larger than $6N$, the largest generator. Now, any factorization of $10N$ must involve exactly two generators, since $10N$ is not a scalar multiple of any generator, greater than the largest generator, and smaller than 3 times the smallest generator. Clearly, these factorizations are given by pairing the generator $AN + B$ with $(10-A)N - B$. □

Lemma 24. *Let $S = \langle n_1, \cdots, n_k \rangle$ and let $m \in S$, with $m \equiv 1$ modulo p for some prime p. Then,*

$$S' = \langle pn_1, \cdots, pn_k, m \rangle$$

is minimally generated.

Proof. Let $m = \sum_{i=1}^{k} a_i n_i$. Suppose the presentation above is not minimal and hence there exists an expression:

$$qm + p \sum_{i \neq j} c_i n_i = q \sum_{i=1}^{k} a_i n_i + p \sum_{i \neq j} c_i n_i = pn_j$$

for some j, and at least some $c_i > 0$. We have that $p \mid q$ and so we may factor out p:

$$q' \sum_{i=1}^{k} a_i n_i + \sum_{i \neq j} c_i n_i = n_j \tag{25}$$

If $a_j = 0$, then we may write n_j in terms of the other generators of S, and so S was not minimally generated. Hence, $a_j \geq 1$, and we have that $q \geq 1$ by the same argument. But now the left of (25) is strictly larger than the right; a contradiction. □

Theorem 26. *(Adding Points) Let $S = \langle n_1, \cdots, n_k \rangle$ and let $m \in S$, with $m \equiv 1$ modulo p for some prime p. Then, Δ_{pm} in*

$$S' = pS + m\langle 1 \rangle = \langle pn_1, \cdots, pn_k, m \rangle$$

is the simplicial complex of m in S with an additional disconnected vertex.

Proof. Certainly, any relations between m and the n_i will hold between pm and pn_i, so Δ_{pm} contains a copy of Δ_m in S. Further, we have the vertex m since pm is a multiple of this last generator. Any lines connecting the vertex m to any other vertex implies a factorization of $pm - m = (p-1)m$ using only multiples of p; since both $p-1$ and m are relatively prime to p this is impossible. □

Corollary 27. *(Attack of a Thousand Raisins) Let $S = \langle n_1, \ldots, n_k \rangle$ and let $m \in S$ and $m > 2$. There exists a simplicial complex with a copy of Δ_m and any number of finitely many disconnected vertices.*

Proof. By the Fundamental Theorem of Arithmetic we may always find, for $m > 2$ a prime p so that $m \equiv 1$ modulo p. Then apply Theorem 26 as many times as desired. □

Theorem 28. *(Glue the World) Let $S = \langle u_1, \cdots, u_k \rangle$, $T = \langle v_1, \cdots, v_\ell \rangle$, and let $m_1 \in S$ and $m_2 \in T$ be relatively prime. Then, $\Delta_{m_1 m_2}$ in the simplicial complex*

$$m_2 S + m_1 T = \langle m_2 u_1, \cdots, m_2 u_k, m v_1, \cdots, m_1 v_k \rangle$$

is the disconnected union of Δ_{m_1} in S and Δ_{m_2} in T.

Proof. First we must show that $m_2 S + m_1 T$ is indeed minimally generated. Let $m_1 = \sum_{i=1}^{k} c_i u_i$ and $m_2 = \sum_{i=1}^{\ell} d_i v_i$, and, without loss of generality, suppose towards a contradiction we have a factorization

$$m_2 \sum_{i \neq j} a_i u_i + m_1 \sum_{i=1}^{\ell} b_i v_i = m_2 u_j$$

for some coefficients a_i, b_i, at least some of which are positive, and some u_j. We must have $m_2 \mid \sum_{i=1}^{\ell} b_i u_i$ and we may factor out m_2:

$$\sum_{i \neq j} a_i u_i + C \sum_{i=1}^{k} c_i u_i = u_j \tag{29}$$

for some constant $C \geq 1$. If $c_j = 0$ then we have that $\langle u_1, \cdots, u_k \rangle$ is not minimally presented; hence $c_j \geq 1$, and the left hand side of (29) is strictly larger than the right.

Now, since any relation between m_1 and the u_i will hold between $m_2 m_1$ and $m_2 u_1$, and any relation between m_2 and the v_i will hold between $m_2 m_1$ and $m_1 v_i$, $\Delta_{m_1 m_2}$ contains both a copy of Δ_{m_1} in S and Δ_{m_2} in T. It now suffices to show that these two components are not connected. Suppose

$$m_1 m_2 = m_2 \sum_{i=1}^{k} a_i u_i + m_1 \sum_{i=1}^{\ell} b_i v_i$$

with at least one of the $b_i \neq 0$. Using the same trick as before, we may factor out m_2 and write

$$m_1 = \sum_{i=1}^{k} a_i u_i + m_1 C'$$

with $C \geq 1$. So $C = 1$ and $a_i = 0$ for all i. By an analogous argument starting with at least one of the $a_i \neq 0$, we have that no factorization of $m_1 m_2$ exists using both generators from $m_2 S$ and $m_1 T$. □

Example 30. In $S_1 = \langle 3, 4, 5 \rangle$. Δ_{13} is the unfilled triangle. To add a disconnected line to this complex we may take Δ_{15} in $S_2 = \langle 7, 8 \rangle$ and glue the semigroups together:

$$S_3 = 15S_1 + 13S_2 = \langle 45, 60, 75, 91, 104 \rangle$$

looking at $\Delta_{15 \cdot 13} = \Delta 195$. We may add a dot to the complex by taking Δ_2 in $S_4 = \langle 1 \rangle$ giving us:

$$S_5 = 2S_3 + S_4 = \langle 90, 120, 150, 182, 208, 195 \rangle$$

looking at Δ_{390}. Add one last dot using Δ_7 in $S_4 = \langle 1 \rangle$ to get

$$S_6 = 7S_5 + S_4 = \langle 630, 840, 1050, 1274, 1456, 1365, 390 \rangle$$

And $\Delta_{390 \cdot 7} = \Delta_{2730}$ in this last semigroup is the disconnected union of all the components.

4. On the Subject of the Pseudo-Frobenius Numbers

Definition 31. (Pseudo-Frobenius Number) Let S be a numerical semigroup. We say $x \notin S$ is a pseudo-Frobenius number if $x + s \in S$ for all $s \in S \backslash \{0\}$. We denote the set of pseudo-Frobenius numbers as $PF(S)$, and the type of S to be the cardinality of $PF(S)$.

From this definition it is obvious that the Frobenius number of S is also a pseudo-Frobenius number.

Proposition 32. *Let $S = \langle n_1, \cdots, n_k \rangle$. Then, for $m \in S$, Δ_m is the $(k-1)$-skeleton if and only if $m = s + \sum_{i=1}^{k} n_i$ for some $s \in PF(S)$.*

Proof. For each $s \in PF(S)$, we may subtract the sum of any $k-1$ generators from $s + \sum_{i=1}^{k} n_i$ and remain in S by definition. And subtracting all the generators gives us s which is not in S also by definition.

For the other direction, if Δ_m is the $(k-1)$-skeleton, then, since $m - \sum_{i \neq j}^{k} n_i$, are in S for all j, but $m - \sum_{i=1}^{k} n_i \notin S$, we have that $m - \sum_{i=1}^{k} n_i \in PF(S)$. □

Corollary 33. *Let S be a skeletal monoid. Then, the only pseudo-Frobenius number is $F(S)$.*

Proof. This follows from Proposition 32 and Proposition 13. □

5. On the Subject of Stars

Definition 34. (Stars) We say a simplicial complex Δ is a (k,ℓ)-star if there is one vertex v connected to k other vertices, and all ℓ dimensional facets with v as a subset are in Δ, and no other connections exist.

Proposition 35. *Let $S = \langle a_1a_2, a_1a_3, a_2a_3\rangle$ be a skeletal monoid. Then, $\Delta_{a_1a_2a_3+a_2a_3}$ is a 2-star.*

Proof. Clearly at least a_2a_3 is connected to the other vertices. We cannot have only a connection between a_1a_2 and a_1a_3 since this complex would then have nonzero Euler characteristic and contradict Theorem 19. So it suffices to show that we do not have the complete simplicial complex. This would imply there exists $b_i \neq 0$ such that

$$a_1a_2a_3 + a_2a_3 = b_1a_1a_2 + b_2a_1a_3 + b_3a_2a_3$$

We must have that $a_3 \mid b_1 > 0$, so we may factor $b_1 = b_1'a_3$ where $b_1' \geq 1$. Then,

$$(a_1+1)a_2 = b_1'a_1a_2 + b_2a_1 + b_3a_2$$

and we may factor $b_2 = b_2'a_2$ with $b_2' \geq 1$ by the same argument, giving us:

$$a_1 + 1 = b_1'a_1 + b_2'a_1 + b_3 \geq 2a_1 + b_3$$

a contradiction. □

Proposition 36. *Let $S = \langle \frac{P}{a_1}, \cdots, \frac{P}{a_k}\rangle$ be a skeletal monoid. Then, $\Delta_{P+\frac{P}{a_k}}$ is the $(k-1)$-star centered at $\frac{P}{a_k}$.*

Proof. Since P is a multiple of each generator, $P + \frac{P}{a_k} = a_i\frac{P}{a_i} + \frac{P}{a_k}$ for all a_i, so $\frac{P}{a_k}$ is connected to every other vertex.

Now suppose towards a contradiction that

$$P + \frac{P}{a_k} = \sum_i^k b_i\frac{P}{a_i} \tag{37}$$

with $b_j, b_l \neq 0$, for some $j, l \neq k$. a_j divides the left of (37) and hence divides $b_j\frac{P}{a_j}$. So $a_j|b_j$ and we have $b_j \geq a_j$. Similarly, $b_l \geq a_l$. Hence

$$\sum_i^k b_i\frac{P}{a_i} \geq 2P > P + \frac{P}{a_k}$$

a contradiction. □

Theorem 38. *Let $S = \langle \frac{P}{a_1}, \cdots, \frac{P}{a_k}\rangle$ be a skeletal monoid. Then, $\Delta_{nP+\frac{P}{a_k}}$ is the $(k-1,n)$-star centered at $\frac{P}{a_k}$.*

Proof. Certainly we have all the n-dimensional facets with $\frac{P}{a_k}$ as a vertex. Suppose some other factorization existed, of the form:

$$nP + \frac{P}{a_k} = \sum_i^k b_i\frac{P}{a_i}$$

with $n+1$ of the $b_i \neq b_k$ nonzero. Then, since $a_i \mid \frac{P}{a_j}$ for all $i \neq j$, we have that $a_i \mid b_i$ for all i. So $b_i \geq a_i$ for at least $n+1$ coefficients; hence we have

$$\sum_i^k b_i \frac{P}{a_i} \geq (n+1)P > nP + \frac{P}{a_k}$$

a contradiction. □

6. On the subject of Plastered Monoids

Proposition 39. *For $S = \langle n_1, n_2 \rangle$*

$$\mathcal{K}(S;t) = 1 - t^{n_1 n_2}$$

Proof. Since the only simplicial complexes on two vertices with nonzero Euler characteristic are the empty complex and the disconnected complex on two vertices, it suffices to find all of the latter complexes. Given $m \in S$, we know $m - n_1 \in S$. And since $m - n_1 - n_2 \notin S$, we must have that $n_1 \mid m - n_1$ so $n_1 \mid m$. Similarly $n_2 \mid m$. It follows $n_1 n_2 \mid m$, so $m = a n_1 n_2$ for some natural number a. If $a > 1$ then we may factor m as $((a-1)n_1)n_2 + (n_2)n_1$ and the complex would not be disconnected. Hence $m = n_1 n_2$ exactly; its Euler characteristic is -1 and so

$$\mathcal{K}(S;t) = 1 - t^{n_1 n_2}$$

as claimed. □

Corollary 40. *Let $S = \langle n_1, n_2 \rangle$. Then, the only element in $PF(S)$ is $F(S)$.*

Proof. Proposition 32 assures us that all the elements in $PF(S)$ have the simplicial complex of two disconnected vertices; the previous proposition shows that the only such element is $F(S)$. □

Definition 41. Let S be a non-skeletal numerical semigroup and suppose there exists a numerical semigroup T and $t \in T$ so that $S = aT + t\langle 1 \rangle$ with a, t relatively prime. Then we say that S is plastered.

In the case of three generators, let S be the unordered, non-skeletal, minimally generated semigroup $\langle n_1, n_2, n_* \rangle$ such that

$$n_1 = \ell_1 a, \quad n_2 = \ell_2 a, \quad n_* = x\ell_1 + y\ell_2$$

such that ℓ_1, ℓ_2 are relatively prime and $a > 1$. We say that S is plastered.

Proposition 42. *Let $S = \langle n_1, n_2, n_* \rangle$ be plastered. Then, $\Delta_{\ell_1 \ell_2 a}$ is the unique simplicial complex containing exactly the vertices n_1 and n_2.*

Proof. We know that $\Delta_{\ell_1 \ell_2 a}$ contains the vertices n_1 and n_2 and does not include the edge $n_1 n_2$. Suppose toward contradiction that $\Delta_{\ell_1 \ell_2 a}$ also contains the vertex n_*. Then $\ell_1 \ell_2 a = bn_1 + cn_2 + dn_*$ for $b, c \geq 0$ and $d > 0$. Observe that a divides $\ell_1 \ell_2 a$ as well as bn_1, cn_2; thus a must also divide dn_*. However, a does not divide n_*; else S would not be primitive. Thus a must divide d, so $d = ea$ for some $e \in \mathbb{N}$. Thus

$$\ell_1 \ell_2 a = b(\ell_1 a) + c(\ell_2 a) + ea(x\ell_1 + y\ell_2) = (b + ex)(n_1) + (c + ey)(n_2),$$

a contradiction, as it implies the edge n_1n_2. Thus $\Delta_{\ell_1\ell_2 a}$ does not contain the vertex n_*, and therefore is the simplicial complex containing exactly the vertices n_1 and n_2.

Now, suppose we could obtain the simplicial complex Δ_z containing exactly the vertices n_* and n_i, for $i \leq 2$. This can only be obtained if $z = \mathrm{lcm}(n_*, n_i)$. Now suppose $z = \mathrm{lcm}(n_*, n_i)$ and $\gcd(n_*, n_i) = k$. Note that $\gcd(k, a) = 1$, else S would not be primitive. Thus since $k \mid n_i = \ell_i a$, $k \mid \ell_i$; so $\frac{\ell_i}{k} = h$ for some $h \in \mathbb{N}$. Then • Prove "this can only be.."?

$$z = \frac{n_* n_i}{k} = \frac{(x\ell_1 + y\ell_2)(\ell_i a)}{k} = \frac{x\ell_i}{k}\ell_1 a + \frac{y\ell_i}{k}\ell_2 a = xh(n_1) + yh(n_2).$$

Thus Δ_z contains the edge n_1n_2 and cannot be the simplicial complex containing exactly the vertices n_* and n_i, thus $\Delta_{\ell_1\ell_2 a}$ is the unique simplicial complex containing exactly the vertices n_1 and n_2. □

Lemma 43. *(Pajemma Lemma) Let $S = \langle n_1, n_2, n_* \rangle$ be plastered. Then Δ_{an_*} is the unique simplicial complex with three vertices and the edge n_1n_2. Further, Δ_{an_*} is the only simplicial complex containing exactly three vertices and one edge.*

Proof. Since $an_* = xn_1 + yn_2$, n_1 and n_2 are connected. Now suppose we had a factorization $an_* = kn_* + mn_1$, with $k, m > 0$. Through rearranging terms we see $a \mid (a-k)n_*$, a contradiction since $a > a - k$ and $gcd(a, n_*) = 1$. A similar argument holds for n_2; hence n_* is disconnected from the other vertices.

Now suppose there exists an element $z \in S$ such that $z = bn_1 = cn_* + dn_2$ and Δ_z contains exactly every vertex and the edge n_*n_2. Then $b\ell_1 a = c(x\ell_1 + y\ell_2) + d\ell_2 a$. Since a is relatively prime to n_*, we know $a \mid c$; so $c = ea$ for some $e \in \mathbb{N}$. Collecting terms we see $bn_1 = (ex)n_1 + (d + ey)n_2$, a contradiction since Δ_z does not contain the edge n_1n_2. Using a similar argument, we see that for $x \in S$ such that $x = bn_2 = cn_* + dn_1$, Δ_x must contain the edge n_1n_2, a contradiction. Thus neither Δ_z nor Δ_x cannot be a simplicial complex with exactly three vertices and one edge.

Suppose for $q \in S$, $q = bn_* = cn_1 + dn_2$ with $b > 0$, and Δ_q is the simplicial complex with exactly three vertices and the edge n_1n_2. We see $a \mid b$, so $b = ea$ for some $e \in \mathbb{N}$. Through rearranging terms we see $ean_* = ((e-1)a)n_* + xn_1 + yn_2$. Now if $e > 1$, this is a contradiction since Δ_q does not contain the triangle $n_*n_1n_2$. So $e = 1$ and $b = a$; thus $q = an_*$.

□

Definition 44. ($\leq_S$) Let S be a numerical semigroup. Define a partial ordering on the integers $a \leq_S b$ if $b - a \in S$. For $T \subseteq \mathbb{Z}$, let

$$\mathrm{Maximals}_{\leq_S} T = \{t \in T \mid \nexists\, u \in T, t \leq_S u\}$$

Proposition 45. *(Rosales, Sanchez, [2]) Let S be a numerical semigroup and n be a nonzero element of S. Then,*

$$PF(S) = \{w - n \mid w \in Maximals_{\leq_S} Ap(S; n)\}$$

Proposition 46. *Let $S = \langle an_1, an_2, n_* \rangle$ be plastered. Then, the only pseudo-Frobenius number is $F(S)$ and*

$$F(S) = a(n_1n_2 - n_1 - n_2) + (a-1)n_*$$

Proof. Consider the Apéry set of n_* in $\langle n_1, n_2 \rangle$; these elements are the first instances of i modulo n_* in $\langle n_1, n_2 \rangle$ for $0 \le i < n_*$. Since a is relatively prime to n_*, multiplication by a induces a permutation of these elements; hence the first instances of i modulo n_* in S will be exactly $a\mathrm{Ap}(\langle n_1, n_2 \rangle; n_*)$. That is,

$$\mathrm{Ap}(S; n_*) = a\mathrm{Ap}(\langle n_1, n_2 \rangle; n_*)$$

By Corollary 40, the only pseudo-Frobenius number in a two generated monoid is the Frobenius number. Therefore there is only one maximal element in the Apéry set of $\langle n_1, n_2 \rangle$. Since we are scaling our generators and our Apèry set by a, this still holds in $\mathrm{Ap}(S; n_*)$. Hence we know there is only one pseudo-Frobenius number in S which is maximal – the Frobenius number. □

Theorem 47. *(Plastered Polynomial) Let $S = \langle an_1, an_2, n_* \rangle$ be plastered. Then,*

$$\mathcal{K}(S; t) = 1 - t^{an_1n_2} - t^{an_*} + t^{an_1n_2 + an_*}$$

Proof. With three vertices, there are 5 possible simplicial complexes with nonzero Euler characteristic: the empty face, two disconnected vertices, three disconnected vertices, a line and a disconnected vertex, and the unfilled triangle. Three disconnected vertices gives the 1-skeleton, which is impossible since S is plastered and not skeletal. Previous results in this section show that all the other 4 complexes arise exactly once; computing the Euler characteristics gives us

$$\mathcal{K}(S; t) = 1 - t^{an_1n_2} - t^{an_*} + t^{an_1n_2 + an_*}$$

□

7. On the Subject of Geometric Sequences

Definition 48. Let $a, b, p \in \mathbb{N}$ with $a < b, \gcd(a, b) = 1$. Then $\{a^p, a^{p-1}b, \ldots, b^p\}$ is a geometric sequence and we define a *numerical semigroup on a compound sequence* to be $\langle a^p, a^{p-1}b, \ldots, b^p \rangle$, which we will abreviate as NSGS. Throughout this section, we take $a < b$ and the form above. It remains to be justified that this is a numerical semigroup, but it is not difficult to do.

Proposition 49. *Let S be geometric. Then the only pseudo-Frobenius number is $F(S)$.*

Proof. Corollary 40 assures us that the statement holds if S is two-generated. We now induct on the embedding dimension of S. Suppose the statement holds for all geometric monoids $S = \langle n_1, \cdots, n_k \rangle$. Then let $T = a\langle n_1, \cdots, n_k \rangle + n_{k+1}\langle 1 \rangle$ where $a \in \mathbb{N}$ is relatively prime with $n_{k+1} \in S$. Consider the Apéry set of n_{k+1} in S; these elements are the first instances of i modulo n_{k+1} in S for $0 \le i < n_{k+1}$. Since a is relatively prime to n_{k+1}, multiplication by a induces a permutation of these elements; hence the first instances of i modulo n_{k+1} in S will be exactly $a\mathrm{Ap}(S; n_{k+1})$. That is,

$$\mathrm{Ap}(T; n_{k+1}) = a\mathrm{Ap}(S; n_{k+1})$$

Since we know for $S = \langle n_1, \cdots, n_k \rangle$ the only pseudo-Frobenius number is the Frobenius number, there is only one maximal element in the Apéry set of n_{k+1} in

S. Since we are scaling our generators and our Apèry set by a, this still holds in $\mathrm{Ap}(T; n_{k+1})$. Hence we know there is only one pseudo-Frobenius number in T which is maximal – the Frobenius number. □

The following is a generalization of Lemma 7 of [1]:

Lemma 50. *Let S be a NSGS and $m \in S$. Suppose $m = c_k a^k + c_{k-1}a^{k-1}b + \cdots + c_0 b^k = d_k a^k + d_{k-1}a^{k-1}b + \cdots + d_0 b^k$. Let $j \in \{0, \ldots, k\}$. Then,*

(1) *If for all $i < j, c_i = d_i$, then $c_j \equiv d_j \pmod{a}$*
(2) *If for all $i > j, c_i = d_i$, then $c_j \equiv d_j \pmod{b}$*

Proof. We prove (1), the proof for (2) is similar.
Let $j \in \{0, \ldots, k\}$ and suppose for all $i < j$, $c_i = d_i$. Then

$$\sum_{i=0}^{k} c_i a^i b^{k-i} \equiv \sum_{i=0}^{k} d_i a^i b^{k-i} \pmod{a^{j+1}} \iff \sum_{i=j}^{k} c_i a^i b^{k-i} \equiv \sum_{i=j}^{k} c_i a^i b^{k-i} \pmod{a^{j+1}}$$
$$\iff c_j a^j b^{k-j} \equiv d_j a^j b^{k-j} \pmod{a^{j+1}} \iff c_j \equiv d_j \pmod{a}$$

□

The following gives us a membership criterion for any NSGS.

Proposition 51. *Let S be a NSGS and $m \in S$. Then m can be uniquely expressed as $c_k a^k + c_k k - 1a^{k-1}b + \cdots + c_0 b^k$ with $c_1, \ldots, c_k < b$.*

Proof. Let $m \in S$. Then $m = c_k a^k b + c_{k-1}a^{k-1}b + \cdots + c_0 b^k$ for some $c_0, \ldots, c_k \in \mathbb{Z}_{\geq 0}$. If for $i > 0$, $c_i < b$, then m has such an expression. If not, let j be maximal such that $c_j \geq b$. Then using $ba^i b^{k-i} = aa^{i-1}b^{k-1+1}$ the appropriate number of times, the coefficient of $a^i b^{k-i}$ is now less than b. Continuing in this fashion, we can find a factorization such that all coefficients for $a^i b^{k-i}$, $i > 0$ are less than b. Hence if $m \in S$, such an expression exists.
Suppose $m = c_k a^k + c_{k-1}a^{k-1}b + \cdots + c_0 b^k = d_k a^k + d_{k-1}a^{k-1}b + \cdots + d_0 b^k$ with for all $i > 0$, $c_i, d_i < b$. We prove $c_i = d_i$ by strong induction on i.
By Lemma 50, $c_k \equiv d_k \pmod{b}$, hence $c_k = d_k$. Now suppose for all $i > j$ for some j, we have $c_i = d_i$. Then again by Lemma 50, $c_j \equiv d_j \pmod{b}$, hence $c_j = d_j$. Hence for all $i > 0$, $c_i = d_i$. Then we have

$$\begin{aligned}
&c_k a^k + c_{k-1}a^{k-1}b + \cdots + c_0 b^k = d_k a^k + d_{k-1}a^{k-1}b + \cdots + d_0 b^k \\
&\iff c_k a^k + c_{k-1}a^{k-1}b + \cdots + c_0 b^k = c_k a^k + c_{k-1}a^{k-1}b + \cdots + d_0 b^k \\
&\iff c_0 b^k = d_0 b^k \iff c_0 = d_0
\end{aligned}$$

Hence for $m \in S$, we can uniquely express m as $c_k a^k + c_{k-1}a^{k-1}b + \cdots + c_0 b^k$ with $c_1, \ldots, c_k < b$. □

Proposition 52. *Let S be a NSGS. Then $\chi(\Delta_{a^{i+1}b^{k-i}}) = -1$ for $i \in \{0, \ldots, k-1\}$.*

Proof. Observe for $i \in \{0, \cdots, k-1\}$,

$$b \cdot a^{i+1} b^{k-i-1} = a \cdot a^i b^{k-i} = a^{i+1} b^{k-i}$$

and

$$b \cdot a^{i+1}b^{k-i-1} = (b-a)a^{i+1}b^{k-i-1} + (a)a^{i+1}b^{k-i-1}$$

Continuing in this manner, we have

$$a^{i+1}b^{k-i} = a \cdot a^i b^{k-i} = (b-a)a^{i+1}b^{k-i-1} + (b-a)a^{i+2}b^{k-i+2} + \cdots + a^k$$

Hence the vertex corresponding to the generator $a^i b^{k-i}$ is disconnected and all vertices corresponding to $a^\ell b^{k-\ell}$, $\ell > i$ form a simplex. Now suppose towards a contradiction that there is a factorization

$$a^{i+1}b^{k-i} = \sum_{j=0}^{k} c_j a^j b^{k-j}$$

where for some $j < i$, $c_j \neq 0$. Then by Lemma 50, $c_j \equiv 0 \pmod{a}$, so $c_j \geq a$. Then $c_j a^j b^{k-j} \geq a^{j+1}b^{k-j}$. Since $a^j b^{k-j}$ is strictly decreasing with respect to j, we have $c_j a^j b^{k-j} > a^{i+1}b^{k-i}$, a contradiction. Hence no faces that include a larger vertex appear. The totally connected subcomponent has zero Euler characteristic; hence including the last vertex yields $\chi(\Delta_{a^i b^{k-i+1}}) = -1$. □

Using the above proof, we have the following corollary:

Corollary 53. *Let S be a NSGS. Then $\Delta_{a^{i+1}b^{k-i}}$ only contains vertices less than or equal to $a^i b^{k-i}$.*

Proposition 54. *Let S be a NSGS and $\alpha \in \{1, \ldots, a-1, a+1, \ldots, b-1\}$. Then $\chi(\Delta_{\alpha a^i b^{k-i}}) = 0$ for $i \in \{0, \ldots, k\}$.*

Proof. We first prove that $\Delta_{\alpha a^i b^{k-i}}$ contains no vertices larger than $a^i b^{k-i}$.
Suppose $\alpha a^i b^{k-i} = \sum_{j=0}^{k} c_j a^j b^{k-j}$. Let ℓ be minimal such that $c_\ell \neq 0$. Suppose by contradiction $\ell < i$. Since we have a factorization, $\alpha a^i b^{k-i}$, by Lemma 50, $c_j \equiv 0 \pmod{a}$. So $a \mid c_j$ and $c_\ell a^\ell b^{k-\ell} \geq a * a^\ell b^{k-\ell}$. Since $a^j b^{k-j}$ is strictly decreasing in terms of j, $\alpha a^i b^{k-i} < a * a^\ell b^{k-\ell}$, a contradiction. Hence $\ell \geq i$ and $\Delta_{\alpha a^i b^{k-i}}$ doesn't contain any vertices larger than $a^i b^{k-i}$.
Now we prove that for $0 < \alpha < a$, $\Delta_{\alpha a^i b^{k-i}}$ is the vertex $a^i b^{k-i}$.
Let $0 < \alpha < a$, then by Lemma 50, $c_i = \alpha$. Then $c_i a^i b^{k-i} = \alpha a^i b^{k-i}$. Hence we only have the one factorization, $\alpha a^i b^{k-i}$, and $\Delta_{\alpha a^i b^{k-i}}$ is the vertex $a^i b^{k-i}$.
Lastly we prove for $a < \alpha < b$, $\Delta_{\alpha a^i b^{k-i}}$ is the simplex including all vertices $a^j b^{k-j}$, with $j \geq i$.
Let $a < \alpha < b$. We can write $\alpha a^i b^{k-i}$ as

$$\begin{aligned}
\alpha a^i b^{k-i} &= (\alpha - a)a^i b^{k-i} + a * a^i b^{k-i} = (\alpha - a)a^i b^{k-i} + b * a^{i+1}b^{k-i-1} \\
&= (\alpha - a)a^i b^{k-i} + ba^{i+1}b^{k-i-1} \\
&= (\alpha - a)a^i b^{k-i} + (b-a)a^{i+1}b^{k-i-1} + aa^{i+1}b^{k-i-1} \\
&\vdots \\
&= (\alpha - a)a^i b^{k-i} + (b-a)a^{i+1}b^{k-i-1} + \ldots + (b-a)a^{k-1}b^1 + ba^k
\end{aligned}$$

Hence $\Delta_{\alpha a^i b^{k-i}}$ is a simplex including all vertices $a^j b^{k-j}$, $j \geq i$.
In both cases $\chi(\Delta_{\alpha a^i b^{k-i}}) = 0$. □

The following corollary follows directly for $a < \alpha$ and $i = 0$, as such we omit the proof.

Corollary 55. *Let S be a NSGS and $\alpha \in \mathbb{N}$ with $\alpha > a$. Then $\Delta_{\alpha b^k}$ is the simplex including all vertices*

Proposition 56. *Let S be a NSGS and fix $i \in \{0, \ldots, k\}$. Let $m \in S$ such that Δ_m does not contain $a^j b^{k-j}$ with $j \leq i$. Denote $s = a^{i+1}b^{k-i} + m$. Then $\chi(\Delta_s) = \chi(\Delta_m)\chi(\Delta_{a^{i+1}b^{k-i}}) = -\chi(\Delta_m)$.*

Proof. First we prove that faces that contain vertices $a^j b^{k-j}$ with $j < i$ do not appear in Δ_s, then the vertex $a^i b^{k-i}$ is only connected to faces in Δ_m.

Since Δ_m does not contain the vertex $a^j b^{k-j}$ for any $j \leq i$, we have a unique expression $m = \sum_{j>i} c_j a^j b^{k-j}$ by Proposition 51, where $c_j < b$. Since $a \cdot a^i b^{k-i}$ is the unique way of expressing $a^{i+1}b^{k-i}$ as described in Proposition 51, we have that

$$m + a^{i+1}b^{k-i} = \sum_{j>i} c_j a^j b^{k-j} + a \cdot a^i b^{k-i}$$

is the unique way of expressing $m + a^{i+1}b^{k-i}$ with coefficients less than b. Now suppose towards a contradiction there was a factorization of $m + a^{i+1}b^{k-i}$ using larger generators. Such a factorization must have coefficients larger than b by unique factorization, and, for that generator, we may rewrite $ba^j b^{k-j} = a \cdot a^{j-1} b^{k-j+1}$ and repeat as necessary to decrease the coefficient until it is less than b, while raising the coefficient of the next generator. Repeating as such, we arrive at a factorization

$$m + a^{i+1}b^{k-i} = \sum_{j=0}^{k} c_j a^j b^{k-j}$$

where $c_1, \ldots, c_k < b$ with some $c_j \neq 0$ for some $j < i$, violating Proposition 51. Hence in Δ_s there are no faces that contain $a^j b^{k-j}$ with $j < i$.

Now suppose we had a factorization of $m + a^{i+1}b^{k-i}$ using $a^i b^{k-i}$, i.e. a factorization

$$m + a^{i+1}b^{k-i} = \sum_{j>i} c_j a^j b^{k-j} + c_i a^i b^{k-i} \tag{57}$$

with $c_i \neq 0$. By Lemma 50, $a \equiv c_i \pmod{a}$, or $a \mid c_i$, i.e. $c_i = c_i' a$. Then, rearranging terms,

$$m = \sum_{j>i} c_j a^j b^{k-j} + (c_1' a - a) a^i b^{k-i}$$

Since m does not contain $a^i b^{k-i}$ in any factorization, we must have that $c_1 = a$; cancelling terms from (57) gives that $\sum_{j>i} c_j a^j b^{k-j}$ is a factorization of m. Hence in Δ_s the only faces that were not in $\Delta_{a^{i+1}b^{k-i}}$ are the vertex $a^i b^{k-i}$ connected to the faces in Δ_m, in particular, each face in Δ_m.

Δ_m is a subset of the simplex in $\Delta_{a^{i+1}b^{k-i}}$ containing vertices $a^j b^{k-j}$ where $j < i$; this subset, including the empty face, has zero Euler characteristic. The faces in

Δ_m connected to the vertex $a^i b^{k-i}$ are one dimension higher, and hence the sign is flipped in the Euler characteristic. Therefore $\chi(\Delta_s) = 0 - \chi(\Delta_m) = (-1)\chi(\Delta_m) = \chi(\Delta_{a^{i+1}b^{k-i}})\chi(\Delta_m)$. □

Proposition 58. *Let S be a NSGS, fix $i \in \{0, \dots, k\}$, and let $\alpha \in \{1, \dots, a-1, a+1, \dots, b-1\}$. Let $m \in S$ such that Δ_m does not contain $a^j b^{k-j}$ with $j \le i$. Denote $s = \alpha a^i b^{k-i} + m$. Then $\chi(\Delta_s) = \chi(\Delta_m)\chi(\Delta_{a^{i+1}b^{k-i}}) = 0$.*

Proof. First we prove that faces that contain vertices $a^j b^{k-j}$ with $j < i$ do not appear in Δ_s, then the vertex $a^i b^{k-i}$ is only connected to faces in Δ_m.

Since Δ_m does not contain the vertex $a^j b^{k-j}$ for any $j \le i$, we have a unique expression $m = \sum_{j>i} c_j a^j b^{k-j}$ by Proposition 51, where $c_j < b$. Since $\alpha < b$, we have that

$$m + \alpha a^i b^{k-i} = \sum_{j>i} c_j a^j b^{k-j} + \alpha a^i b^{k-i}$$

is the unique way of expressing $m + a^{i+1}b^{k-i}$ with coefficients less than b. Now suppose towards a contradiction there was a factorization of $m + a^{i+1}b^{k-i}$ using larger generators. Such a factorization must have coefficients larger than b by unique factorization, and, for that generator, we may rewrite $ba^j b^{k-j} = a \cdot a^{j-1} b^{k-j+1}$ and repeat as necessary to decrease the coefficient until it is less than b, while raising the coefficient of the next generator. Repeating as such, we arrive at a factorization

$$m + \alpha a^i b^{k-i} = \sum_{j=0}^{k} c_j a^j b^{k-j}$$

where $c_1, \dots, c_k < b$ with some $c_j \neq 0$ for some $j < i$, violating Proposition 51. Hence in Δ_s there are no faces that contain $a^j b^{k-j}$ with $j < i$.

Now suppose $0 < \alpha < a$ and that we had a factorization of $m + \alpha a^i b^{k-i}$ using $a^i b^{k-i}$, i.e. a factorization

$$m + \alpha a^i b^{k-i} = \sum_{j>i} c_j a^j b^{k-j} + c_i a^i b^{k-i} \tag{59}$$

with $c_i \neq 0$. By Lemma 50, $\alpha \equiv c_i \pmod{a}$, or $a \mid (c_i - \alpha)$, i.e. $c_i = ac_i' + \alpha$. Then, rearranging terms,

$$m = \sum_{j>i} c_j a^j b^{k-j} + (c_i' a) a^i b^{k-i}$$

Since m does not contain $a^i b^{k-i}$ in any factorization, we must have that $c_i' = 0$; cancelling terms from (59) gives that $\sum_{j>i} c_j a^j b^{k-j}$ is a factorization of m. Hence in Δ_s the only faces that were not in Δ_m are the vertex $a^i b^{k-i}$ connected to the faces in Δ_m, in particular, each face in Δ_m. The faces in Δ_m connected to the vertex $a^i b^{k-i}$ are one dimension higher, and hence the sign is flipped in the Euler characteristic. Therefore $\chi(\Delta_s) = \chi(\Delta_m) - \chi(\Delta_m) = 0 = 0\chi(\Delta_m) = \chi(\Delta_{\alpha a^i b^{k-i}})\chi(\Delta_m)$.

If $a < \alpha < b$, then Δ_m is a subset of $\Delta_{\alpha a^i b^{k-i}}$ which is a simplex that includes all $a^j b^{k-j}$, $j > i$. We also have not faces that use a larger vertex, so there are no new faces. Hence $\chi(\Delta_s) = \chi(\Delta_{\alpha a^i b^{k-i}}) = 0 = 0\chi(\Delta_m) = \chi(\Delta_{\alpha a^i b^{k-i}})\chi(\Delta_m)$. □

Theorem 60. *Let S be a NSGS and $m \in S$. Let $m = c_k a^k + c_{k-1}a^{k-1}b + \cdots + c_0 b^k$ be the unique factorization described in Proposition 51. Then $\chi(\Delta_m) \neq 0 \iff$ for each $i > 0, c_i = 0$ or $c_i = a$.*

Proof. By Proposition 51, we can write $m = c_k a^k + c_{k-1}a^{k-1}b + \cdots + c_0 b^k$ with $c_j < b$ for $j > 0$ uniquely. By Corollary 53 and Propositions 56 and 58, we can rewrite $\chi(\Delta_m) = \chi(\Delta_{c_k a^k + c_{k-1}a^{k-1}b + \cdots + c_0 b^k}) = \chi(\Delta_{c_k a^k + c_{k-1}a^{k-1}b + \cdots + c_1 ab^{k-1}})\chi(\Delta_{c_0 b^k}) = \cdots = \chi(\Delta_{c_k a^k})\chi(\Delta_{c_{k-1}a^{k-1}b}) \cdots \chi(\Delta_{c_0 b^k})$. Then by Propositions 52 and 54 and Corollary 55, it follows that $\chi(\Delta_m) \neq 0 \iff$ for each $i > 0, c_i = 0$ or $c_i = a$. □

Theorem 61. *Let S be a NSGS. Then*

$$\mathcal{K}(S;t) = \prod_{i=0}^{k-1}(1 - t^{a^{i+1}b^{k-i}})$$

Proof. It follows from Theorem 60 that the only elements with nonzero Euler characteristic will be those that are the sum of some distinct $a^{i+1}b^{k-i}$ with $i \in \{0, \ldots, k-1\}$. Furthermore the Euler charcteristic of those elements is $(-1)^m$ where m is the number of summands. Distributing $\prod_{i=0}^{k-1}(1 - t^{a^{i+1}b^{k-i}})$ gives exactly that. □

Disclaimer: I feel this section can be improved many times over.

Conjecture 62. *We believe this result can be generalized without too much work for* compound sequence *numerical semigroups. Let S be a NSCS (detailed in [1]), then*

$$\mathcal{K}(S;t) = \prod_{i=1}^{k}(1 - t^{b_1 b_2 \cdots b_i a_i a_{i+1} \cdots a_k}) = \prod_{i=1}^{k}\left(1 - t^{(\prod_{j=i}^{k} a_j)(\prod_{j=1}^{i} b_j)}\right)$$

8. On the Subject of Augmented Monoids

Definition 63. Let $S = \langle n_1, \cdots, n_k \rangle$ be a monoid. Then, we say

$$S' = \langle n_1, \cdots, n_k, F(S) \rangle$$

is the *augmented* monoid of S. If $\langle n_1, \cdots, n_k \rangle$ is a skeletal monoid, then we say that S' is an augmented skeletal monoid.

Proposition 64. *Let $S' = \langle n_1, \ldots, n_k, F(S) \rangle$ be an augmented skeletal monoid and let $P = lcm(n_i)$, and $a_i = P/n_i$. Then for $1 \leq m \leq k-2$, Δ_{mP} is the m-skeleton of S.*

Proof. Assume $mP - F(S) = \sum_{i=1}^{k} c_i n_i + c_F F(S)$. Suppose $c_F = 0$. Then it follows

$$mP - (k-1)P + \sum_{i=1}^{k} n_i = \sum_{i=1}^{k} c_i n_i$$

Through rearranging terms we see

$$\sum_{i=1}^{k}(1 - c_i)n_i = (k - 1 - m)P$$

Clearly $a_i \mid (1-c_i)$ and $c_i \geq 0$. Note that if $c_i = 0$, then $a_i \mid 1$. Thus $1 - c_i \leq 0$, a contradiction, so $c_F \geq 1$.

Now in our factorization we have the inequality $mP - F(S) \geq c_F F(S)$, which we may rearrange as:

$$c_F \leq \frac{mP}{F(S)} - 1$$

We know $a_i \geq 2$, so $2\sum_{i=1}^k n_i < kP$. Therefore, we know $2(k-1)P - 2\sum_{i=1}^k n_i > (2k-2)P - kP = (k-2)P$. Since the $m \leq k-2$, we can rewrite this as $mP < 2((k-1)P - 2\sum_{i=1}^k n_i)$ which can be rewritten as $\frac{mP}{F(S)} = \frac{mP}{(k-1)P - \sum_{i=1}^k n_i} < 2$.

It follows that

$$1 \leq c_F \leq \frac{mP}{(k-1)P - \sum_{i=1}^k n_i} - 1 < 1$$

This is a contradiction. Therefore $F(S)$ is not included in any factorization and the simplicial complex is the same as it is in S, i.e. the m skeleton. □

Proposition 65. *Let $S' = \langle n_1, \ldots, n_k, F(S) \rangle$ be an augmented skeletal monoid, $P = lcm(n_i)$, and $a_i = P/n_i$. Then for $1 \leq m \leq k-2$, $F(S) + mP$ is shaded in S'.*

Proof. Observe that

$$\begin{aligned} F(S) + mP &= mP + (k-1)P - \sum n_i \\ &= ma_1 n_1 - \sum n_i + \underbrace{P + \cdots + P}_{(k-1) \text{ times}} \\ &= (ma_1 - 1)n_1 + (a_2 - 1)n_2 + \cdots + (a_k - 1)n_k. \end{aligned}$$

so $\Delta_{F(S)+mP}$ contains the k-simplex with vertices $n_1, \ldots, n_k$. Also, observe that

$$F(S) + mP = F(S) + P + \cdots + P = F(S) + a_j n_j + \cdots + a_\ell n_\ell$$

for m-many arbitrary $n_i \in S$. Thus $\Delta_{F(S)+mP}$ contains every $(m+1)$-simplex containing the vertex $F(S)$.
Now suppose $\Delta_{F(S)+mP}$ contains some $(m+2)$-simplex with the vertex $F(S)$. Then, without loss of generality,

$$F(S) + mP = F(S) + b_1 n_1 + \cdots + b_m n_m + b_{m+1} n_{m+1}$$

with $b_i > 0$, so

$$mP = b_1 n_1 + \cdots + b_m n_m + b_{m+1} n_{m+1}.$$

Then $a_i \mid b_i > 0$ for all i, so the right is at least $(m+1)P$, a contradiction. Thus $\Delta_{F(S)+mP}$ cannot contain any $(m+2)$-simplex with the vertex $F(S)$; this suffices to show that $\Delta_{F(S)+mP}$ contains exactly the k-simplex with vertices $n_1, \ldots, n_k$ and every $(m+1)$-simplex with the vertex $F(S)$.

Now, note that for $0 \leq s \leq m+1$, we have every possible s-simplex; so the number of s-simplexes in $\Delta_{F(S)+mP}$ is $\binom{k+1}{s}$. For $m+2 \leq t \leq k$, we only have the

t-simplexes that do not include $F(S)$, so the number of t-simplexes is $\binom{k}{t}$. Thus

$$\chi\left(\Delta_{F(S)+mP}\right) = \sum_{i=0}^{m+1}(-1)^i\binom{k+1}{i} + \sum_{i=m+2}^{k}(-1)^i\binom{k}{i}.$$

Now suppose $\chi\left(\Delta_{F(S)+mP}\right) = 0$. Using the Binomial Theorem, we have

$$\begin{aligned} 0 &= \sum_{i=0}^{m+1}(-1)^i\binom{k+1}{i} + \sum_{i=m+2}^{k}(-1)^i\binom{k}{i} \\ &= 1 + \sum_{i=1}^{m+1}(-1)^i\binom{k}{i-1} + \sum_{i=1}^{k}(-1)^i\binom{k}{i} \\ &= 1 + \sum_{i=0}^{m}(-1)^{i+1}\binom{k}{i}. \end{aligned}$$

Now by rearranging terms we have $\sum_{i=0}^{m}(-1)^i\binom{k}{i} = 1$. It is known that

$$\sum_{i=0}^{m}(-1)^i\binom{k}{i} = (-1)^m\binom{k-1}{m},$$

so now we have $(-1)^m\binom{k-1}{m} = 1$. This equality only occurs when $m = 0$ or $k-1$, which is impossible.

Thus, $\chi\left(\Delta_{F(S)+mP}\right) \neq 0$, and therefore $F(S) + mP$ is shaded in S'. ☐

Proposition 66. *Let $S' = \langle n_1, \dots, n_k, F(S)\rangle$ be an augmented skeletal monoid, $P =$ lcm(n_i), and $a_i = P/n_i$. Then $\chi(\Delta_{2F(S)}) = -1$.*

Proof. Assume

$$2F(S) = \sum_{i=1}^{k} c_i n_i + c_f F(S)$$

where $c_f \neq 0$. Clearly c_f cannot be greater than 2. If $c_f = 1$, then $F(S) = \sum_{i=1}^{k} c_i n_i$, a contradiction. If $c_f = 2$, then $\sum_{i=1}^{k} c_i n_i = 0$, and thus we have $F(S)$ on its own.

We must consider the case where our skeletal monoid is two-generated and one of our generators is 2 separately. The new augmented monoid would have the form $\langle 2, m, m-2\rangle$. This however is not minimally generated and hence impossible.

Now we may suppose either $k \geq 3$ or 2 is not one of the a_i. We show we have the face composed of $\{n_1, \dots, n_k\}$ by factoring $2F(S) - \sum_{i=1}^{k} n_i$. We see

$$2F(S) - \sum_{i=1}^{k} n_i = (2k-2)P - 3\sum_{i=1}^{k} n_i$$

If $a_j = 2$ for some j, then $P < 3n_j < 2P$. Hence we may group:

$$(P - 3n_1) + (P - 3n_2) + \cdots + (2P - 3n_j) + \cdots + (P - 3n_k) + (k-3)P$$

where each term is nonnegative so long as $k \geq 3$. n_i divides the ith term in this expression, so we have a factorization $\sum_{i=1}^{k} c_i n_i \in S$ of the above.

If $a_i \neq 2$ for all i, then we need only to group each generator with one P:

$$(P - 3n_1) + (P - 3n_2) + \cdots + (P - 3n_k) + (k-2)P$$

where each term is nonnegative so long as $k \geq 2$. The same argument above gives a factorization. □

Corollary 67. *Let $S' = \langle n_1, \ldots, n_k, F(S) \rangle$ be an augmented skeletal monoid, and $a > 2$. Then $\chi(\Delta_{aF(S)}) = 0$.*

Proof. By the above, $\Delta_{2F(S)}$ completely connects the original generators and has $F(S)$ as an isolated vertex. Adding on any more copies of $F(S)$ will connect this vertex to the rest of the complex, resulting in a totally connected complex with Euler characteristic 0. □

Proposition 68. *Let $S' = \langle n_1, \cdots, n_k, F(S) \rangle$ be an augmented skeletal monoid. Then, for any subset $X \in 2^{\{n_1, \cdots, n_k\}}$ with $1 \leq |X| \leq k-1$,*

$$\chi(\Delta_{F(S) + \sum_{x \in X} x}) = (-1)^{|X|}$$

Proof. Clearly, the vertices $\{F(S)\} \cup X$ are completely connected; this subcomponent, including the empty face, has Euler characteristic 0. Using Corollary 14, we may rewrite:

$$F(S) + \sum_{x \in X} x = (k-1)P - \sum_{i=1}^{k} n_i + \sum_{x \in X} x = (k-1)P - \sum_{y \in X^c} y \tag{69}$$

where P is as before $\mathrm{lcm}(n_i)$. There are no connections between $F(S)$ and any generators in $y = n_j \in X^c$, since any such connection implies a factorization:

$$F(S) + \sum_{x \in X} x - F(S) - y = \sum_{x \in X} x - y = \sum_{i=1}^{k} c_i n_i + c_f F(S)$$

First suppose $c_f = 0$; that is, $\sum_{x \in X} x - y \in S$, and we have

$$\sum_{x \in X} x = \sum_{i \neq j} c_i n_i + (c_j + 1) n_j$$

Since a_j must divide $c_j + 1 > 0$, we have that

$$\sum_{x \in X} x = \sum_{i \neq j} c_i n_i + mP$$

for some m. Now, at least one of the c_i corresponding to a generator in x must be 0, as else the right will be strictly bigger than the left; let this generator be n_ℓ. Now, a_ℓ divides every term on the right and every term on the left but n_ℓ, a contradiction. Hence suppose $c_f > 0$ and again write

$$\sum_{x \in X} x = \sum_{i \neq j} c_i n_i + (c_j + 1) n_j + (k-1) c_f P - \sum_{i=1}^{k} n_i = \sum_{i \neq j} (c_i - 1) n_i + c_j n_j + (k-1) c_f P$$

Hence,

$$\sum_{i\neq j} n_i \geq \sum_{i\neq j}(c_i-1)n_i + c_j n_j + (k-1)c_f P \geq \sum_{i\neq j}(c_i-1)n_i + (k-1)P$$

where last inequality assumes the smallest values for c_f and c_j, i.e. that $c_f = 1$ and $c_j = 0$. Then,

$$\sum_{i\neq j}(2-c_i)n_i \geq (k-1)P$$

Since the c_i are all nonnegative and $2n_i \leq P$ with strict inequality for all but at most one generator, the left quantity is at most $\sum_{i\neq j} 2n_i < (k-1)P$, another contradiction. Finally, returning to (69) and rearranging terms, we get the expression

$$F(S) + \sum_{x\in X} x = \sum_{y\in X^c}(P-y) + (k-1-|X^c|)P = \sum_{y\in X^c}(P-y) + (|X|-1)P$$

There are factorizations of this expression involving all $y \in X^c$, since $y \mid P-y$. For the remaining $(|X|-1)$ copies of P, we can use up to $|X|-1$ other generators in the factorization, letting us completely connect the vertices in X^c to any subset of $|X|-1$ or less generators in X. Counting these faces up (and excluding the empty face, which has already been counted), we get:

$$\chi(\Delta_{F(S)+\sum_{x\in X} x}) = \sum_{i=0}^{|X|-1}(-1)^{i+1}\binom{|X|}{i}$$

We note that

$$(-1)^{|X|} - \sum_{i=0}^{|X|-1}(-1)^i\binom{|X|}{i} = -\sum_{i=0}^{|X|}(-1)^i\binom{|X|}{i} = 0$$

which implies the result. □

Proposition 70. *Let $S' = \langle n_1, \cdots, n_k, F(S)\rangle$ be an augmented skeletal monoid. Then, for any subset $X \in 2^{\{n_1,\cdots,n_k\}}$ with $1 \leq |X| \leq k-1$,*

$$\chi(\Delta_{2F(S)+\sum_{x\in X} x}) = (-1)^{|X|+1}$$

Proof. Since $\Delta_{2F(S)} \subseteq \Delta_{2F(S)+\sum_{x\in X} x}$, we have by Proposition 66 that our complex contains $\{n_1, \ldots, n_k\}$. By Proposition 68 and similar argument we have the simplex containing all $y \in X^c$, and each face in that simplex connected to each face up to dimension $|X|-1$ in the simplex containing the vertices from X. The additional copy of $F(S)$ allows us to connect $F(S)$ to all such faces; we now claim there are no other faces in $\Delta_{2F(S)+\sum_{x\in X} x}$.

Suppose towards a contradiction that

$$2F(S) + \sum_{x\in X} x = \sum_{y\in X^c} c_y y + \sum_{x\in X} c_x x + F(S)$$

where at least $|X|$ of the c_x coefficients are nonzero. Then,

$$F(S) + \sum_{x\in X} = \sum_{y\in X^c} c_y y + \sum_{x\in X} c_x x$$

which gives a factorization of $F(S)+\sum_{x\in X}$ using more than $|X|-1$ faces from X, a contradiction of the previous proposition.

Now, since we have the simplex containing all n_i (which has Euler characteristic 0), and the only other faces come from the connecting the generators in X^c to up to $|X|-1$ faces in X as well as to $F(S)$, we are exactly adding one dimension to each face from the previous proposition, thus reversing the sign of the Euler characteristic. That is,

$$\chi(\Delta_{2F(S)+\sum_{x\in X} x}) = -(-1)^{|X|} = (-1)^{|X|+1}$$

as claimed. □

Conjecture 71. *Let $S' = \langle n_1, \ldots, n_k, F(S)\rangle$ be an augmented skeletal monoid with $P = lcm(n_i)$. Then, S' has the following categories of shaded elements:*

- *mP, where $1 \le m \le k-2$,*
- *$F(S)+mP$, where $1 \le m \le k-2$,*
- *$F(S)+\sum_{x\in X} X$, where $X \in 2^{\{n_1,\ldots,n_k\}}$ and $1 \le |X| \le k-1$,*
- *$2F(S)+\sum_{x\in X} X$, where $X \in 2^{\{n_1,\ldots,n_k\}}$ and $1 \le |X| \le k-1$*

and the elements 0 and $2F(S)$. In particular, S has $2^{k+1}+2k-6$ shaded elements.

Conjecture 72. *Let $S = \langle n_1, n_2, n_3\rangle$ be a numerical semigroup.*

(1) If S is plastered and not skeletal, then S has one pseudo-Frobenius number and 4 shaded elements.
(2) If S is skeletal, then S has one pseudo-Frobenius number and 3 shaded elements.
(3) If S is neither, then S has two pseudo-Frobenius numbers and has 6 shaded elements.

Lemma 73. *Let $S' = \langle n_1, n_2, F(S)\rangle$ be an augmented skeletal monoid. Then there do not exist any $m \in S$ so that Δ_m contains only two disconnected vertices.*

Proof. Without loss of generality, we must consider two cases: the case where the vertices are n_1 and n_2, and the case where the vertices are n_1 and $F(S)$.

Consider the first case and let $w = an_1n_2$ for some $a \ge 1$. Then $w - F(S) = n_1(an_2 - n_2 + 1) + n_2 \in S'$. This factorization implies Δ_w has the line connecting n_1 and n_2, a contradiction.

Now, we consider the second case and let $m = an_1F(S)$ for some $a \ge 1$. Then $m - n_1 - F(S) = an_1(n_1n_2 - n_1 - n_2) - n_1 - n_1n_2 + n_1 + n_2$. Through rearranging terms, it follows

$$m - n_1 - F(S) = n_1(an_1n_2 - an_1 - an_2 - n_2) + n_2 \in S'$$

This is a contradiction, because there is no edge connecting the vertices. Therefore we cannot create the simplicial complex containing two vertices from S'. □

Lemma 74. *Let $S' = \langle n_1, n_2, F(S)\rangle$ be an augmented skeletal monoid, and take $\ell \in S'$. Then if Δ_ℓ contains exactly the vertex n_j and the edge $\{n_i, F(S)\}$, $\ell = F(S) + n_i$.*

Proof. Without loss of generality, take $\ell \in S'$ such that $\ell = an_2 = c_1n_1 + c_2F(S)$ and Δ_ℓ contains exactly the vertex n_2 and the edge $\{n_1, F(S)\}$. Then

$$c_1n_1 + c_2F(S) = c_1n_1 + c_2n_1n_2 - c_2n_1 - c_2n_2 = (c_1 - c_2)n_1 + (c_2n_1 - c_2)n_2;$$

but since Δ_ℓ does not include the edge $\{n_1, n_2\}$, it follows that $c_1 - c_2 \leq 0$ and thus $c_2 \geq c_1 \geq 1$.

Now suppose $c_2 \geq 2$. Then

$$\begin{aligned}\ell - n_1 - n_2 &= c_1n_1 + c_2F(S) - n_1 - n_2 \\ &= n_1(c_1 - 1) + c_2F(S) - n_2 \\ &= (c_2 - 1)F(S) + (F(S) - n_2) + n_1(c_1 - 1)\end{aligned}$$

so $\ell - n_1 - n_2 > F(S)$, and Δ_ℓ contains the edge $\{n_1, n_2\}$, a contradiction. Therefore $c_2 = c_1 = 1$, so $\ell = F(S) + n_1$. □

Lemma 75. *Let $S' = \langle n_1, n_2, F(S)\rangle$ be an augmented skeletal monoid, and let $m \in S'$ have the simplicial complex containing $F(S)$ and the edge containing n_1 and n_2.Then $m = \Delta_{2F(S)}$.*

Proof. We have that $m = aF(S) = c_1n_1 + c_2n_2$ for some $a, c_i \geq 1$. Through rewriting terms, we see $an_1n_2 = (a + c_1)n_1 + (a + c_2)n_2$. It follows $n_2 \mid a + c_1 > 0$ and $n_1 \mid a + c_2 > 0$, which implies $a \geq 2$.

Suppose $a \geq 3$. Then $aF(S) - F(S) - n_1 \geq 2F(S) - n_1 = F(S) + (F(S) - n_1) > F(S)$. Therefore $aF(S) - F(S) - n_1 \in S'$, so we have the edge containing $F(S)$ and n_1, a contradiction. Hence $a = 2$, as desired. □

Lemma 76. *Let $S' = \langle n_1, n_2, F(S)\rangle$ be an augmented skeletal monoid, and let $m \in S'$ have the unfilled triangle as its simplicial complex. Then $m = 2F(S) + n_i$ for $i = 1$ or $i = 2$.*

Proof. Observe, without loss of generality, that $m = c_1n_1 + c_2F(S)$ for $c_1, c_2 \geq 1$. Suppose $c_1 \geq 2, c_2 \geq 3$. Then

$$\begin{aligned}m - n_1 - n_2 - F(S) =&(c_1 - 1)n_1 + (c_2 - 3)F(S) + 2F(S) - n_2 \\ =&(c_1 - 1)n_1 + (c_2 - 3)F(S) + (n_2 - 2)n_1 + (n_1 - 3)n_2 \\ =&(c_1 + n_2 - 3)n_1 + (c_2 - 3)F(S) + (n_1 - 3)n_2.\end{aligned}$$

This implies $m - (n_1 + n_2 + F(S)) \in S'$, a contradiction. Thus $c_1 = 1$ and $c_2 \leq 2$. Now suppose $c_2 = 1$. Then $m = F(S) + n_1$, a contradiction by Lemma 74. Thus $c_2 = 2$, and therefore $m = 2F(S) + n_1$. □

Theorem 77. *Let $S' = \langle n_1, n_2, F(S)\rangle$ be an augmented skeletal monoid. Then*

$$\mathcal{H}(S; t) = \frac{1 - t^{F(S)+n_1} - t^{F(S)+n_2} - t^{2F(S)} + t^{2F(S)+n_1} + t^{2F(S)+n_2}}{(1 - t^{n_1})(1 - t^{n_2})(1 - t^{F(S)})}$$

Proof. By Propositions 64, 65, 66, 68, 70, and Lemmas 73, 74, 75, 76, we have that $F(S) + n_i$, $2F(S)$, $2F(S) + n_i$, for $i = 1, 2$, are the only elements $n \in S$ such that

$\chi(\Delta_n) \neq 0$ besides 0. Specifically, $\chi(\Delta_{F(S)+n_1}) = \chi(\Delta_{F(S)+n_2}) = \chi(\Delta_{2F(S)}) = -1$ and $\chi(\Delta_{2F(S)+n_1}) = \chi(\Delta_{2F(S)+n_2}) = \chi(\Delta_0) = 1$. Thus

$$\mathcal{H}(S;t) = \frac{1 - t^{F(S)+n_1} - t^{F(S)+n_2} - t^{2F(S)} + t^{2F(S)+n_1} + t^{2F(S)+n_2}}{(1-t^{n_1})(1-t^{n_2})(1-t^{F(S)})}$$

□

9. On the Subject of Arithmetic Monoids

Definition 78. We say a numerical monoid $S = \langle n_1, n_2, n_3 \rangle$ is *arithmetic* if $S = \langle k, k+a, k+2a \rangle$ such that $\gcd(k,a) = 1$ and k is odd.

Proposition 79. *Let S be an arithmetic monoid. Then*

$$Ap(S;k) = \left\{0, k+a, k+2a, 2k+3a, 2k+4a, \ldots, \left(\frac{k-1}{2}\right)k + (k-1)a\right\}.$$

Proof. Since k and a are relatively prime, $0, a, 2a, \ldots, (k-1)a$ are all distinct modulo k. Thus we know $\mathrm{Ap}(S;k)$ contains all $x \in S$ such that x is the smallest element in S where $x \equiv na \pmod{k}$ for $n = 0, \ldots, k-1$.

Take $x \in S$ such that $x \equiv na \pmod{k}$ for some $n \in \{0, \ldots, k-1\}$. Then $x = mk + na$. Since $x \in S$, we know that $mk + na$ is at least $\frac{n}{2}(k+2a)$. Therefore $m \geq \frac{n}{2}$. If n is even, the smallest x can be is $\frac{n}{2}k + na$. If n is odd, the smallest x can be is $\frac{n+1}{2}k + na$. Thus the smallest elements in S modulo na are of the form $\frac{n}{2}k + na$ for even n, and $\frac{n+1}{2}k + na$ for odd n. Therefore

$$\mathrm{Ap}(S;k) = \left\{0, k+a, k+2a, 2k+3a, 2k+4a, \ldots, \left(\frac{k-1}{2}\right)k + (k-1)a\right\}.$$

□

Corollary 80. *Let S be an arithmetic monoid. The only pseudo-Frobenius numbers of S are*

$$\lambda_1 = \left(\frac{k-1}{2}\right)(k+2a) - (k+a)$$

and

$$\lambda_2 = \left(\frac{k-1}{2}\right)(k+2a) - k.$$

Further, $F(S) = \lambda_2$.

Proof. This follows from Proposition 79 and Proposition 45, as the maximal elements in $\mathrm{Ap}(S;k)$ with respect to $\leq_S$ are $\left(\frac{k-1}{2}\right)(k+2a)$ and $\left(\frac{k-1}{2}\right)(k+2a) - a$. □

Proposition 81. *Let S be an arithmetic monoid. Then $\lambda_i + n_1 + n_3$, $2n_2$ are shaded in S for $i = 1, 2$.*

Proof. Note that by rearranging terms we have

$$\lambda_1 + n_1 + n_3 = \left(\frac{k-3}{2}\right)k + (k-2)a + 2k + 2a = \left(\frac{k+1}{2} + a\right)n_1$$

$$= \left(\frac{k-1}{2}\right) n_3 + n_2.$$

Also note that $\lambda_1 + n_1 + n_3 - (n_1 + n_2) = \lambda_1 + a = \lambda_2 \notin S$, and $\lambda_1 + n_1 + n_3 - (n_1 + n_3) = \lambda_1 \notin S$. Thus $\Delta_{\lambda_1+n_1+n_3}$ contains exactly the vertex n_1 and the edge $\{n_2, n_3\}$.

Similarly,

$$\lambda_2 + n_1 + n_3 = \left(\frac{k-3}{2}\right) k + (k-1)a + 2k + 2a = \left(\frac{k+1}{2}\right) n_3$$
$$= \left(\frac{k-1}{2} + a\right) n_1 + n_2,$$

and $\lambda_2 + n_1 + n_3 - (n_3 + n_2) = \lambda_2 - a = \lambda_1 \notin S$, and $\lambda_2 + n_1 + n_3 - (n_3 + n_1) = \lambda_2 \notin S$. Thus $\Delta_{\lambda_2+n_1+n_3}$ contains exactly the vertex n_3 and the edge $\{n_1, n_2\}$.

Now observe that $2n_2 = n_1 + n_3$, and that $2n_2 - (n_1 + n_2) = a \notin S$ and $2n_2 - (n_2 + n_3) = -a \notin S$, so Δ_{2n_2} contains exactly the vertex n_2 and the edge $\{n_1, n_3\}$.

Now we have $\chi\left(\Delta_{\lambda_1+n_1+n_3}\right) = \chi\left(\Delta_{\lambda_2+n_1+n_3}\right) = \chi\left(\Delta_{n_1+n_3}\right) = -1$. Therefore $\lambda_i + n_1 + n_3$ and $2n_2$ are shaded in S for $i = 1, 2$. □

Corollary 82. *Let S be an arithmetic monoid. Then $\lambda_1 + \sum n_i$ and $\lambda_2 + \sum n_i$ are shaded in S for $1 \leq i \leq 3$. Further, if Δ_m is the unfilled triangle for some $m \in S$, then $m = \lambda_1 + \sum n_i$ or $\lambda_2 + \sum n_i$.*

Proof. This follows from Proposition 32, as S has 3 generators. Thus $\chi\left(\Delta_{\lambda_1+\sum n_i}\right) = \chi\left(\Delta_{\lambda_2+\sum n_i}\right) = 1$ and $\lambda_2 + \sum n_i$ are shaded in S for $1 \leq i \leq 3$. □

Lemma 83. *Let $S = \langle n_1, n_2, n_3\rangle$ be an arithmetic monoid. Then $gcd(n_i, n_j) = 1$.*

Proof. Let $x \mid k + pa$ and $x \mid k + qa$ for some x, where $0 \leq p < q \leq 2$. Then $bx = k + qa$ and $cx = k + pa$, which may be written as $bx = cx + a(q - p)$. We see $1 \leq q - p \leq 2$, so in the case $q - p = 1$, then $x \mid a$. Since $x \mid a$ then $x \mid pa$. If $x \mid pa$ and $x \mid k + pa$, then $x \mid k$, but since k and a are relatively prime, this implies x=1. Now suppose $q - p = 2$. Then $q = 2$ and $p = 0$ so $x \mid k$ and $x \mid 2a$, a contradiction since k is odd by definition. Therefore n_1, n_2, and n_3 are relatively prime for arithmetic monoids. □

Lemma 84. *Let S be an arithmetic monoid. Then there do not exist any $m \in S$ such that Δ_m contains exactly two disconnected vertices.*

Proof. Recall for arithmetic monoids, $F(S) = \left(\frac{k-1}{2}\right)(k + 2a) - k = \frac{k-3}{2}k + (k-1)a$ and n_1, n_2, and n_3 are relatively prime. Note that Δ_m contains exactly the vertices n_i and n_j only when $m = lcm(n_i, n_j)$. Now suppose $m = (k + pa)(k + qa)$ where $0 \leq p < q \leq 2$. Through rearranging terms we see $m - n_i - n_j = (k-2)k + a(p + q)(k-1) + pqa^2 > F(S)$ since this can be rewritten as $\frac{k-1}{2}k + a(k-1)(p + q - 1) + pqa^2 > 0$. This is a contradiction as this implies there is an edge connecting the two vertices. Therefore we cannot create the simplicial complex containing two vertices from S. □

Lemma 85. *Let S be an arithmetic monoid. Then for some $m \in S$, if Δ_m contains exactly the vertex n_2 and the edge $\{n_1, n_3\}$, $m = 2n_2$.*

Proof. Consider $m \in S$ such that Δ_m contains exactly the vertex n_2 and the edge $\{n_1, n_3\}$. Then $m = cn_2$ for some $c \in \mathbb{N}$. If $c = 1$, then $m - n_3 = -a \notin S$, a contradiction. If $c > 2$, then $cn_2 = n_1 + n_2 + n_3 + (c-3)n_2$, a contradiction. Thus $c = 2$ and $m = 2n_2$; therefore Δ_{2n_2} is the only simplicial complex containing exactly the vertex n_2 and the edge $\{n_1, n_3\}$.

□

Lemma 86. *Let S be an arithmetic monoid. Then for some $m \in S$, if Δ_m contains exactly the vertex n_1 and the edge $\{n_2, n_3\}$, $m = \lambda_1 + n_1 + n_3$.*

Proof. Consider $m \in S$ such that Δ_m contains exactly the vertex n_1 and the edge $\{n_2, n_3\}$. Then $m = c_1 n_1 = c_2 n_2 + c_3 n_3$ for some $c_1, c_2, c_3 \in \mathbb{N}$. We know

$$m - n_1 - n_3 = c_2 n_2 + c_3 n_3 - (n_1 + n_3) \notin S.$$

Rearranging terms, we have

$$c_2 n_2 + c_3 n_3 - (n_1 + n_3) = (c_2 + c_3 - 2)k + (c_2 + 2c_3 - 2)a.$$

If $(c_2 + c_3 - 2)k + (c_2 + 2c_3 - 2)a \notin S$, it must be that $c_2 + c_3 - 2 < \frac{c_2 + 2c_3 - 2}{2}$. Thus, by rearranging terms, we have $c_2 < 2$, so $c_2 = 1$. Now we have $c_1 n_1 = n_2 + c_3 n_3$, so we have $c_1 k = (c_3 + 1)k + (2c_3 + 1)a$. This implies $k \mid 2c_3 + 1$, so $c_3 = \frac{dk-1}{2}$ for some $d \in \mathbb{N}$. Since $m - n_1 - n_3 \notin S$, it follows that $m - n_1 - n_3 \leq F(S)$. So we have

$$\left(\frac{dk-1}{2}\right)(k + 2a) - (k + a) \leq \left(\frac{k-1}{2}\right)(k - 2a) - k.$$

This implies that $(d-1)\left(\frac{k}{2}\right)(k + 2a) \leq a$, so since $\frac{k}{2} > 1$ and $k + 2a > a$, it follows that $d - 1 \leq 0$. Thus $d \leq 1$, meaning $d = 1$ since $d \in \mathbb{N}$.

Now we have $m = \left(\frac{k-1}{2}\right)(k + 2a) + (k + a) = \left(\frac{k-1}{2}\right)(k + 2a) - (k + a) + 2(k + a) = \lambda_1 + n_1 + n_3$. Thus if Δ_m contains exactly the vertex n_1 and the edge $\{n_2, n_3\}$, $m = \lambda_1 + n_1 + n_3$. □

Lemma 87. *Let S be an arithmetic monoid. Then for some $m \in S$, if Δ_m contains exactly the vertex n_3 and the edge $\{n_1, n_2\}$, $m = \lambda_2 + n_1 + n_3$.*

Proof. Consider $m \in S$ such that Δ_m contains exactly the vertex n_3 and the edge $\{n_1, n_2\}$. Then $m = c_1 n_3 = c_2 n_1 + c_3 n_2$ for some $c_1, c_2, c_3 \in \mathbb{N}$. We know

$$m - n_1 - n_3 = c_2 n_1 + c_3 n_2 - (n_1 + n_3) = c_2 n_1 + (c_2 - 2) n_2 \notin S.$$

It follows that $c_2 < 2$, so $c_2 = 1$ since $c_2 \in \mathbb{N}$.

Now we have that $m = c_1 n_3 = c_2 n_1 + n_2$; thus $c_1(k + 2a) = c_2 k + k + a$, so $(c_1 - 1)k + (2c_1 - 1)a = c_2 k$. Thus $k \mid 2c_1 - 1$, so $c_3 = \frac{dk+1}{2}$ for some $d \in \mathbb{N}$.

We know that since $m - n_1 - n_3 \notin S$, it follows that $m - n_1 - n_3 \leq F(S)$. So we have

$$\left(\frac{dk+1}{2}\right)(k + 2a) - 2(k + a) \leq \left(\frac{k-1}{2}\right)(k - 2a) - k.$$

Rearranging terms, we have $dk \leq k$, so $d = 1$. Thus $c_3 = \frac{k+1}{2}$, and $m = \left(\frac{k+1}{2}\right)(k + 2a) = \left(\frac{k-1}{2}\right)(k+2a) - k + k + (k+2a) = \lambda_2 + n_1 + n_3$. Therefore if Δ_m contains exactly the vertex n_3 and the edge $\{n_1, n_2\}$, $m = \lambda_2 + n_1 + n_3$. □

Theorem 88 (The Theorem We Came Up With On the Seventh Day of August (TTWCUWOSDA))**.** *Let S be an arithmetic monoid. Then*

$$\mathcal{H}(S;t) = \frac{1 - t^{n_1+n_2} - t^{\lambda_1+n_1+n_3} - t^{\lambda_2+n_1+n_3} + t^{\lambda_1+\sum n_i} + t^{\lambda_2+\sum n_i}}{(1-t^k)(1-t^{k+a})(1-t^{k+2a})}$$

Proof. By Proposition 81, Lemmas 84, 85, 86, 87, and Corollary 82, we have that $\lambda_i + n_1 + n_3$, $n_1 + n_3$, $\lambda_i + \sum n_j$, for $i = 1, 2$ and $1 \leq j \leq 3$, are the only elements $n \in S$ such that $\chi(\Delta_n) \neq 0$ besides 0. Specifically, $\chi(\Delta_{\lambda_1+n_1+n_3}) = \chi(\Delta_{\lambda_2+n_1+n_3}) = \chi(\Delta_{n_1+n_3}) = -1$, $\chi(\Delta_{\lambda_1 \sum n_i})) = \chi(\Delta_{\lambda_2 \sum n_i})) = 1$, and $\chi(\Delta_0) = 1$. Thus

$$\mathcal{H}(S;t) = \frac{1 - t^{n_1+n_2} - t^{\lambda_1+n_1+n_3} - t^{\lambda_2+n_1+n_3} + t^{\lambda_1+\sum n_i} + t^{\lambda_2+\sum n_i}}{(1-t^k)(1-t^{k+a})(1-t^{k+2a})}$$

□

10. On the Subject of Omo Monoids

Definition 89. We say a numerical monoid $S = \langle n_1, n_2, n_3 \rangle$ is *omo* if $S = \langle k, k + a, (m-1)k + ma \rangle$ such that $\gcd(k, a) = 1$ and $k = bm + p$ for $b > 0$ and $0 < p < m$.

Proposition 90. *Let S be an omo monoid. Then*

$$Ap(S;k) = \left\{ \left\lceil \frac{n(m-1)}{m} \right\rceil k + na : 0 \leq a \leq k-1 \right\}$$

Proof. Since k and a are relatively prime, $0, a, 2a, \ldots, (k-1)a$ are all distinct modulo k. Thus we know $\mathrm{Ap}(S;k)$ contains all $x \in S$ such that x is the smallest element in S where $x \equiv na \pmod{k}$ for $n = 0, \ldots, k-1$.

Take $x \in S$ such that $x \equiv na \pmod{k}$ for some $n \in \{0, \ldots, k-1\}$. Then $x = yk + na$. Since $x \in S$, we know that $yk + na \geq \frac{n}{m}((m-1)k + ma)$; however, $yk + na$ may not be a direct multiple of $(m-1)k + ma$; so $y = \left\lceil \frac{n(m-1)}{m} \right\rceil$ is the smallest possible y for all $x = yk + na$. Thus $x = \left\lceil \frac{n(m-1)}{m} \right\rceil k + na$ is the smallest element in S where $x \equiv na \pmod{k}$ for $n = 0, \ldots, k-1$. Therefore

$$\mathrm{Ap}(S;k) = \left\{ \left\lceil \frac{n(m-1)}{m} \right\rceil k + na : 0 \leq a \leq k-1 \right\}.$$

□

Proposition 91. *Let S be an omo monoid. Then the maximals of $Ap(S;k)$ are*

$$(k-1-b)k + (k-1)a \quad \textit{and} \quad (b(m-1))k + (bm-1)a.$$

• maximals notation?

Proof. Let $\alpha = (k-1-b)k + (k-1)a$ and $\beta = (b(m-1))k + (bm-1)a$.

Note that

$$\begin{aligned} \left\lceil \frac{(k-1-p)(m-1)}{m} \right\rceil &= \left\lceil \frac{(bm-1)(m-1)}{m} \right\rceil \\ &= \left\lceil bm - b - \frac{m-1}{m} \right\rceil \\ &= b(m-1) \qquad \text{since } m-1 < m. \end{aligned}$$

Thus $\left\lceil \frac{(k-1-p)(m-1)}{m} \right\rceil k + (k-1-p)a = (b(m-1))k + (bm-1)a = \beta$.

Now take $x \in \text{Ap}(S;k)$ such that $x > \beta$. Then $x = \left\lceil \frac{n(m-1)}{m} \right\rceil k + na$ for $k - p \leq n \leq k-1$. Thus $n = (k-1-p) + d = bm - 1 + d$ for $1 \leq d \leq p$. Then

$$\begin{aligned} x - \beta &= \left\lceil \frac{(bm-1+d)(m-1)}{m} \right\rceil k + (bm-1+d)a - (b(m-1))k - (bm-1)a \\ &= \left\lceil b(m-1) + d - 1 - \frac{d-1}{m} \right\rceil k + (bm-1+d)a - (b(m-1))k - (bm-1)a \\ &= (b(m-1) + d - 1 - b(m-1))k + (bm-1+d-(bm-1))a \\ &= (d+1)k + da. \end{aligned}$$

By our observation in Proposition 90, we know $(d+1)k + da \notin S$, since $d - 1 \geq \frac{d(m-1)}{m}$ if and only if $d \geq m$, which is untrue. Thus $\beta = (b(m-1))k + (bm-1)a$ is a maximal in $\text{Ap}(S;k)$.

Now observe $\left\lceil \frac{(k-1)(m-1)}{m} \right\rceil k + (k-1)a$ is the largest element of $\text{Ap}(S;k)$ and is thus a maximal. Also, we have

$$\begin{aligned} \left\lceil \frac{(k-1)(m-1)}{m} \right\rceil &= \left\lceil \frac{(bm+p-1)(m-1)}{m} \right\rceil \\ &= \left\lceil b(m-1) + p - 1 - \frac{(p-1)}{m} \right\rceil \\ &= k - 1 - b \qquad \text{since } p - 1 < m. \end{aligned}$$

Thus $\left\lceil \frac{(k-1)(m-1)}{m} \right\rceil k + (k-1)a = (k-1-b)k + (k-1)a = \alpha$.

- Reference something saying $\text{Ap}(S;k)$ can't have more than two?

Therefore, α and β are maximals in $\text{Ap}(S;k)$. Since S has three generators, $\text{Ap}(S;k)$ cannot have more than two maximals; thus the only maximals in $\text{Ap}(S;k)$ are α and β. □

Corollary 92. *Let S be an omo monoid. Then the only pseudo-Frobenius numbers of S are*

$$\lambda_1 = (b(m-1))k + (bm-1)a - k$$

and

$$\lambda_2 = (k-1-b)k + (k-1)a - k$$

Further, $F(S) = \lambda_2$.

Proof. This follows from Proposition 45 and Proposition 91. □

Proposition 93. *Let S be an omo monoid. Then mn_2, $\lambda_1 + n_2 + n_3$, $\lambda_2 + n_1 + n_2$, $\lambda_i + \sum n_j$ are shaded in S for $i = 1, 2$ and $1 \leq j \leq 3$.*

Proof. Note that $mn_2 - n_1 - n_3 = 0 \in S$, so the edge between the vertices n_1 and n_3 exists. Now suppose $mn_2 - (n_1 + n_2) = mk + ma - 2k - a \in S$. Through rearranging terms we see $(m-1)k + am - (k+a) = n_3 - n_2 \in S$. This is a contradiction since the semigroup is minimally generated. Thus there does not exist an edge connecting the vertices n_1 and n_2. Now observe $mn_2 - (n_2 + n_3) = -a \notin S$. Therefore Δ_{mn_2} contains exactly the vertex n_2 and the edge n_1, n_3.

Recall that $k = bm + p$ for $0 < p < m$. Now observe that by rearranging terms we have

$$\begin{aligned}\lambda_1 + n_2 + n_3 &= b(m-1)k + (bm-1)a - k + k + a + (m-1)k + ma \\ &= (b+1)n_3 \\ &= (k + a - (b+1))n_1 + (m-p)n_2\end{aligned}$$

Note that $\lambda_1 + n_2 + n_3 - n_2 - n_3 = \lambda_1 \notin S$ and $\lambda_1 + n_2 + n_3 - n_1 - n_3 = \lambda_1 + a = (b(m-1)-1)k + bma \notin S$ since $b(m-1) - 1 < \frac{bm(m-1)}{m}$ for all b, m. Thus $\Delta_{\lambda_1+n_2+n_3}$ contains exactly the vertex n_3 and the edge $\{n_1, n_2\}$.

Similarly, observe that

$$\begin{aligned}\lambda_2 + n_1 + n_2 &= (k-1-b)k + (k-1)a - k + k + k + a \\ &= (k + a - b)n_1 \\ &= (bm + p + a - b)k = pn_2 + bn_3\end{aligned}$$

Note that $\lambda_2 + n_1 + n_2 - n_1 - n_2 = \lambda_2 \notin S$ and $\lambda_2 + n_1 + n_2 - n_1 - n_3 = (k - m - b)k + (k-m)a \notin S$ since $k - m - b < \frac{(k-m)(m-1)}{m}$ for all possible k, m. Thus $\Delta_{\lambda_2+n_1+n_2}$ contains exactly the vertex n_1 and the edge n_2, n_3.

Now we have $\chi\left(\Delta_{mn_2}\right) = \chi\left(\Delta_{\lambda_1+n_2+n_3}\right) = \chi\left(\Delta_{\lambda_2+n_1+n_2}\right) = -1$. Therefore $mn_2, \lambda_1 + n_2 + n_3$, $\lambda_2 + n_1 + n_2$ are shaded in S for $i = 1, 2$.

□

Corollary 94. *Let S be an omo monoid. Then $\lambda_1 + \sum n_j$ and $\lambda_2 + \sum n_j$ are shaded in S for $1 \leq j \leq 3$. Further, if Δ_m is the unfilled triangle for some $m \in S$, then $m = \lambda_1 + \sum n_j$ or $\lambda_2 + \sum n_j$.*

Proof. This follows from Proposition 32, as S has 3 generators; so $\Delta_{\lambda_i+\sum n_j}$ is the unfilled triangle for $i = 1, 2$ and $1 \leq j \leq 3$. Thus $\chi\left(\Delta_{\lambda_1+\sum n_i}\right) = \chi\left(\Delta_{\lambda_2+\sum n_i}\right) = 1$ and $\lambda_i + \sum n_j$ is shaded in S for $i = 1, 2$ and $1 \leq j \leq 3$; and if Δ_m is the unfilled triangle for some $m \in S$, then $m = \lambda_1 + \sum n_j$ or $\lambda_2 + \sum n_j$. □

Lemma 95. *Let S be an omo monoid. Then if Δ_m contains exactly the vertex n_1 and the edge $\{n_2, n_3\}$ for some $m \in S$, $m = \lambda_2 + n_1 + n_2$.*

Proof. Consider $m \in S$ such that Δ_m contains exactly the vertex n_1 and the edge $\{n_2, n_3\}$. Then $m = c_1 n_1 = c_2 n_2 + c_3 n_3$ for some $c_1, c_2, c_3 \in \mathbb{N}$. Since Δ_m does not contain the edge $\{n_1, n_2\}$, it follows that $m - n_1 - n_2 \notin S$. Thus $m - n_1 - n_2 \leq$

$F(S)$, which implies $c_1 n_1 \leq \lambda_2 + n_1 + n_2$, so $c_1 n_1 \leq (k + a - b)n_1$. This implies $c_1 = k + a - b - s$, for some $s \geq 0$. Now observe that

$$\begin{aligned} c_1 n_1 - n_2 - n_3 &= (k + a - b - s)k - (k + a) - ((m-1)k + ma) \\ &= (k - b - s - m)k + (k - m - 1)a \in S \end{aligned}$$

since Δ_m contains the edge $\{n_2, n_3\}$. Thus by our observation in the proof of Proposition 90, $k - b - s - m \geq \frac{(k-m-1)(m-1)}{m}$, which implies $k \geq bm + (sm + 1)$, which is only possible if $s = 0$. Thus $s = 0$ and $c_1 = k + a - b$, which means that $m = \lambda_2 + n_1 + n_2$. Therefore, if Δ_m contains exactly the vertex n_1 and the edge $\{n_2, n_3\}$ for some $m \in S$, $m = \lambda_2 + n_1 + n_2$. □

Lemma 96. *Let S be an omo monoid. Then if Δ_m contains exactly the vertex n_3 and the edge $\{n_1, n_2\}$ for some $m \in S$, $m = \lambda_1 + n_2 + n_3$.*

Proof. Consider $m \in S$ such that Δ_m contains exactly the vertex n_3 and the edge $\{n_1, n_2\}$. Then $m = c_1 n_1 + c_2 n_2 = c_3 n_3$ for some $c_1, c_2, c_3 \in \mathbb{N}$. Suppose $c_3 < b + 1$; then $c_3 = b - d$ for some $0 \leq d \leq b - 1$. Now by rearranging terms we have that $c_3 - n_1 - n_2 = ((b-d)(m-1) - 2)k + ((b-d)m - 1)a \in S$ since Δ_m contains the edge $\{n_1, n_2\}$. This implies that $-1 \geq m$, which is impossible. Thus $c_3 \geq b + 1$.

Now $c_3 = b + 1 + d$ for some $d \geq 0$. Thus $c_3 n_3 = (b+1)n_3 + dn_3 = \lambda_1 + n_2 + n_3 + dn_3$. Then $c_3 n_3 - n_2 - n_3 = \lambda_1 + dn_3 \notin S$ since Δ_m does not contain the edge $\{n_2, n_3\}$. Recall that $F(S) = \lambda_2 = (k - 1 - b)k + (k-1)a - k$ and $\lambda_1 = b(m-1)k + (bm-1)a - k$. Thus $F(S) - \lambda_1 = (p-1)k - pa$. Thus $\lambda_1 + dn_3 \leq F(S) = \lambda_1 + (p-1)k + pa$, which implies $dn_3 \leq (p-1)k + pa$, which is impossible unless $d = 0$. Now we have $c_3 = b + 1$, meaning $m = (b+1)n_3 = \lambda_1 + n_2 + n_3$. Therefore if Δ_m contains exactly the vertex n_3 and the edge $\{n_1, n_2\}$ for some $m \in S$, $m = \lambda_1 + n_2 + n_3$. □

Lemma 97. *Let S be an omo monoid. Then for some $j \in S$, if Δ_j contains exactly the vertex n_2 and the edge $\{n_1, n_3\}$, $j = mn_2$*

Proof. Consider $j \in S$ such that Δ_j contains exactly the vertex n_2 and the edge n_1, n_3. Then $j = \ell n_2$ for some $\ell \in \mathbb{N}$. Now suppose $\ell < m$, then it follows through rearranging terms $\ell k + \ell a - (m-1)k - ma = (k+a)(\ell - m) < 0 \notin S$. Thus $\ell \geq m$. Now suppose $\ell > m$, then through rearranging terms, $\ell k + \ell a - k - k - a - mk + k - ma = (k+a)(c - 1 - m) \geq 0 \in S$. Therefore, $\ell = m$, and if Δ_j contains exactly the vertex n_2 and the edge $\{n_1, n_3\}$, then $j = mn_2$ □

Lemma 98. *Let $S = \langle n_1, n_2, n_3 \rangle$ be an omo monoid. Then $gcd(n_i, n_j) = 1$.*

Proof. We know by Lemma 83, $gcd(k, k+a) = 1$. Now let $x \mid k$ and $x \mid (m-1)k + ma$. It follows $x \mid ma$, but $gcd(k, a) = gcd(k, m) = 1$, so $gcd(k, (m-1)k + ma) = 1$. Now let $x \mid k + a$ and $x \mid (m-1)k + ma$. Through substituting and rearranging terms, it follows $c_2 x = c_1 x(m-1) + a$. Thus, $x \mid a$ and $x \mid k + a$. Since $gcd(k, a) = 1$, it follows $gcd(k + a, (m-1)k + ma) = 1$. Thus n_1, n_2, and n_3 are relatively prime for omo monoids. □

Lemma 99. *Let S be an omo monoid. Then there does not exist any $m \in S$ such that Δ_m is the simplicial complex containing either exactly two or exactly three vertices.*

Proof. By Theorem 21, since S is not skeletal. we know there cannot exist exactly three vertices.

Recall Δ_m can contain exactly the vertices n_i and n_j only when $m = lcm(n_i, n_j)$. We know by Lemma 84, we cannot create the simplicial complex using n_1 and n_2. Now, note that

$$\begin{aligned}(k+a)[(m-1)k+ma] > k[(m-1)k+ma] &> k+a \\ &= F(\langle k, k+a\rangle) + k + k + a \in S\end{aligned}$$

Hence $\mathrm{lcm}(n_1, n_3)$ and $\mathrm{lcm}(n_2, n_3)$ must both have the edge connecting k and $k+a$, and we cannot get two isolated vertices. □

Theorem 100. *Let S be an omo monoid. Then*

$$\mathcal{H}(S;t) = \frac{1 - t^{mn_2} - t^{\lambda_2+n_1+n_2} - t^{\lambda_1+n_2+n_3} + t^{\lambda_1+\sum n_j} + t^{\lambda_2+\sum n_j}}{(1-t^k)(1-t^{k+1})(1-t^{(m-1)k+ma})}$$

Proof. By Proposition 93, Corollary 94, and Lemmas 95, 96, 97, 99, we have that mn_2, $\lambda_2 + n_1 + n_2$, $\lambda_1 + n_2 + n_3$, $\lambda_i + \sum n_j$, for $i = 1, 2$ and $1 \leq j \leq 3$, are the only elements $n \in S$ such that $\chi(\Delta_n) \neq 0$ besides 0. Specifically, $\chi(\Delta_{mn_2}) = \chi(\Delta_{\lambda_2+n_1+n_2}) = \chi(\Delta_{\lambda_1+n_2+n_3}) = -1$, $\chi(\Delta_{\lambda_1 \sum n_i})) = \chi(\Delta_{\lambda_2 \sum n_i})) = 1$, and $\chi(\Delta_0) = 1$. It follows that

$$\mathcal{H}(S;t) = \frac{1 - t^{mn_2} - t^{\lambda_2+n_1+n_2} - t^{\lambda_1+n_2+n_3} + t^{\lambda_1+\sum n_j} + t^{\lambda_2+\sum n_j}}{(1-t^k)(1-t^{k+1})(1-t^{(m-1)k+ma})}$$

□

11. On the Subject of Connected Gluing

In this section, the numerical semigroups are minimally generated by Lemma 22, unless noted otherwise.

Proposition 101. *(Gluing Lines) Let $S = \langle n_1, \cdots, n_k\rangle$, and fix a vertex n_i. Let $m \in S$ with $m - n_i > 0$. Let $p \in \mathbb{N}$ such that* $\gcd(p, m-n_i) = 1$ *Then, Δ_{pm} in the semigroup*

$$T = pS + (m - n_i)\langle 1\rangle$$

is the simplicial complex Δ_m in S with an additional edge connecting $m - n_i$ and pn_i.

Proof. Clearly the faces in Δ_m in S are still in Δ_{pm} since we are just scaling each generator and the element by p. Now, consider any new factorization of pm using $m - n_i$, of the form

$$pn_i + p(m-n_i) = pm = \sum_{i=1}^{k} pc_i n_i + c_0(m - n_i)$$

Since $\gcd(p, m - n_i) = 1$, we have that $p \mid c_0$. Using $c_0 = dp$ with $d \geq 1$ gives us

$$pn_i + p(m-n_i) = \sum_{i=1}^{k} pc_i n_i + dp(m - n_i)$$

Subtracting $p(m - n_i)$ from both sides gives us

$$pn_i = \sum_{i=1}^{k} c_i pn_i + (d-1)p(m - n_i)$$

which implies the semigroup is not minimally generated if $d > 2$ or $c_j \neq 0$ for any $j \neq i$. Hence $d = 1$, and the only way to make the remaining quantity pn_i is using that generator; hence only pn_i is connected to $m - n_i$.

□

We now focus on attaching a vertex to an existing face in a simplicial complex, $X \subseteq \{n_1, \dots, n_k\}$.

Definition 102. Let $S = \langle n_1, \dots, n_k \rangle$ be a numerical semigroup let Δ be a simplicial complex with vertices as a subset of the generators. Then Δ is *accepted* if there exists $m \in S$ with $\Delta_m = \Delta$.

Lemma 103. *Let $S = \langle n_1, \dots, n_k \rangle$ be a numerical semigroup and $X \subseteq \{n_1, \dots, n_k\}$. If the simplex, $2^X = \Delta$, is accepted, then $m = \sum_{x \in X} x$ is one such m where $\Delta_m = \Delta$.*

Proof. Denote $g = \sum_{x \in X} x$. Since 2^X is accepted, for some $m \in S$, $m - g \in S$, but for any generator $n \notin X$, $m - n \notin S$. Then denoting $\{n_{i_1}, \dots, n_{i_j}\} = X$, we have $m = \sum_{\ell=1}^{j} c_\ell n_{i_\ell}$. Δ_g has all faces in the simplex Δ. Now suppose towards a contradiction that Δ_g has a face that Δ does not have. Then for some generator $n \notin X$, $g - n \in S$, so $m - n \in S$, a contradiction. □

Lemma 104. *Let $S = \langle n_1, \dots, n_k \rangle$ be a numerical semigroup, $X \subseteq \{n_1, \dots, n_k\}$, and the simplex 2^X be accepted in S. Let $s \in S$ such that $\Delta_s = 2^X$. Then for any factorization $s = \sum_{i=1}^{k} c_i n_i$, if $n_i \notin X$, then $c_i = 0$.*

Proof. Let $s \in S$ such that $\Delta_s = 2^X$. Suppose towards a contradiction there is a factorization $s = \sum_{i=1}^{k} c_i n_i$ and for some $j, c_j \neq 0$ and $n_j \notin X$. Then $s - n_j \in S$ and n_j is a vertex in Δ_s, a contradiction. □

• Changed Notation again

Proposition 105. *(Inflating Faces) Let $S = \langle n_1, \dots, n_k \rangle$ be a numerical semigroup, $m \in S$, $X \subseteq \{n_1, \dots, n_k\}$, and the simplex 2^X be accepted in S, and denoting*

$$b = m - \sum_{x \in X} x,$$

$b \in S \setminus \{0\}$. Let $p \in \mathbb{N}$ such that $\gcd(p, b) = 1$. Then, Δ_{pm} in the semigroup

$$T = pS + b\langle 1 \rangle$$

is the simplicial complex Δ_m in S with an additional vertex connected to only X.

Proof. Clearly the faces in Δ_m in S are still in Δ_{pm} since we are just scaling each generator and the element by p. Now, consider factorizations of pm using b, of the form

$$\sum_{x \in X} px + p(m - \sum_{x \in X} x) = pm = \sum_{i=1}^{k} c_i pn_i + c_0(m - \sum_{x \in X} x)$$

with $c_0 > 0$. Since $\gcd(p, b) = 1$, $p \mid c_0$, using $c_0 = dp$, with $d \geq 1$ gives us

$$\sum_{x \in X} px + p(m - \sum_{x \in X} x) = \sum_{i=1}^{k} c_i p n_i + dp(m - \sum_{x \in X} x)$$

Subtracting $p(m - \sum_{x \in X} x)$ from both sides and canceling p gives us

$$\sum_{x \in X} x = \sum_{i=1}^{k} c_i n_i + (d-1)(m - \sum_{x \in X} x) \tag{106}$$

Since $b \in S$ and $b > 0$, then

$$m = \sum_{x \in X} x + \sum_{i=1}^{k} d_i n_i$$

with at least one $d_i \neq 0$. Using this in (106), we have

$$\sum_{x \in X} x = \sum_{i=1}^{k} c_i n_i + (d-1)(\sum_{i=1}^{k} d_i n_i) = \sum_{i=1}^{k} (c_i + (d-1)d_i) n_i$$

By Lemma 104 if $n_i \notin X$, then $c_i + (d-1)d_i = 0$. Since $c_i, d_i \geq 0$, we have $n_i \notin X \Rightarrow c_i = 0$. Hence there are no faces that connect b and vertices not in X. □

The above can be seen as taking a simplicial complex and increasing the degree of one face by one, or inflating it, motivating the following definition.

Definition 107. Let Δ be a simplicial complex. We define an inflation of a face f in Δ to be the simplicial complex Δ' where Δ' has all the faces of Δ, as well as the face f connected to a new vertex and every subset of that face.

Definition 108. (THICC) We call a simplicial complex a THICC if the simplicial complex is connected, and if any maximal simplex is connected to at least two other maximal simplexes, removing it causes the simplicial complex to be disconnected.

• Definition could be better

Lemma 109. *Any THICC can be made by repeated inflations on a point.*

Proof. A THICC can be constructed by taking one of its maximal simplexes and adjoining another at a smaller simplex and continuing this process. Then starting at a point, inflating to the first simplex, then inflating a shared simplex, we can build up to the THICC. □

• Words are hard, need to know more vocab

Definition 110. (The Inflation Algorithm) We define the *Inflation Algorithm* for THICCs Δ as follows: let v be the number of vertices in Δ. Then, let $S_1 = \langle 1 \rangle$,

$$\begin{aligned} S_i &= a_{i-1} S_{i-1} + b_{i-1} \langle 1 \rangle \\ &= \left\langle \prod_{j=1}^{i-1} a_i, \left(\prod_{j=2}^{i-1} a_i \right) b_1, \left(\prod_{j=3}^{i-1} a_i \right) b_2 b_1, \cdots, b_{i-1} \right\rangle \end{aligned}$$

where $b_i = m_{i-1} - \sum_{x \in X_{i-1}} x$, and X_i is the set of generators corresponding to the face we wish to inflate in the ith iteration, $m_i = a_{i-1} m_{i-1}$, and $a_i > v$ is a distinct prime so that $m_i - \sum_{x \in X_i} x \in S_i$.

Lemma 111. *For every iteration of the inflation algorithm, $b_i \in S_i$.*

Proof. We proceed by induction on the number of iterations already completed. For $S_1 = \langle 1 \rangle$, m_1 is prime and greater than the number of vertices in Δ. If there is only one vertex in Δ, then there are no more iterations to be done. If there are more vertices, then $m_1 > 1$, and since any $X_1 = \{1\}$, we have $m_1 - \sum_{x \in X_1} x \in S$. Now suppose for $m_i = \prod_{j=1}^{i-1} m_j$, we have $m_i - \sum_{x \in X_i} x \in S$. Then in $S_{i+1} = \langle a_i n_1, \ldots, a_i n_i, n_{i+1} \rangle$, where $n_{i+1} = m_i - \sum_{x \in X_i} x$ for some $X_i \subseteq \{n_1, \ldots, n_k\}$.
If $n_{i+1} \notin X_{i+1}$, then denote $X'_{i+1} = \{\frac{x}{a_i} : x \in X_{i+1}\}$. Then

$$m_{i+1} - \sum_{x \in X_{i+1}} x = a_i \left(m_i - \sum_{x \in X'_{i+1}} x \right)$$

By inductive hypothesis, $m_i - \sum_{x \in X'_{i+1}} x \in S_i$, so $a_i \left(m_i - \sum_{x \in X'_{i+1}} x \right) \in S_{i+1}$.
If $n_{i+1} \in X_{i+1}$, then the face that is being inflated was created in the previous iteration; furthermore, denoting $X''_{i+1} = \{\frac{x}{a_i} : x \in X_{i+1}, x \neq n_{i+1}\}$, we have $X''_{i+1} \subseteq X_i$. Then we have

$$\begin{aligned} m_{i+1} - \sum_{x \in X_{i+1}} x = a_i m_i - a_i \sum_{x \in X''_{i+1}} x - n_{i+1} &= a_i m_i - a_i \sum_{x \in X''_{i+1}} x + (a_i - 1) n_{i+1} - a_i n_{i+1} \\ &= a_i m_i - a_i \sum_{x \in X''_{i+1}} x + (a_i - 1) n_{i+1} - a_i (m_i - \sum_{x \in X_i} x) \\ &= a_i \left(\sum_{x \in X_i} x - \sum_{x \in X''_{i+1}} x \right) + (a_i - 1) n_{i+1} \end{aligned}$$

Since $X''_{i+1} \subseteq X_i$, then $\sum_{x \in X_i} x - \sum_{x \in X''_{i+1}} x \geq 0$ and is the sum of some $\{n_1, \ldots, n_i\}$, so $a_i \left(\sum_{x \in X_i} x - \sum_{x \in X''_{i+1}} x \right) \in S_{i+1}$, hence $m_{i+1} - \sum_{x \in X_{i+1}} x \in S_{i+1}$. □

Lemma 112. *Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical semigroup made by the inflation algorithm. Then for any $X \subseteq \{n_1, \ldots, n_k\}$, $\sum_{x \in X} x$ is uniquely factored.*

Proof. We proceed by induction on the number of generators. For $i = 1$ and $S_1 = \langle 1 \rangle$, $0, 1$ are both uniquely factored in S_1. Now suppose for $S_i = \langle n'_1, \ldots, n'_i \rangle$, for any $X \subseteq \{n'_1, \ldots, n'_i\}, \sum_{x \in X} x$ is uniquely factored in S_i. Then let $S_{i+1} = \langle n_1, \ldots, n_i, n_{i+1} \rangle$, and $X \subseteq \{n_1, \ldots, n_i, n_{i+1}\}$, with $a_i n'_j = n_j$ for $1 \leq j \leq i$. Since $m_i - \sum_{x \in X_i} \in S_i$, and $n_{i+1} = m_i - \sum_{x \in X_i} x$ for some $X_i \subseteq \{n'_1, \ldots, n'_i\}$, we have $n_{i+1} = \sum_{i=1}^{k} d_j n'_j$. If $n_{i+1} \notin X_{i+1}$, then

$$\sum_{x \in X_{i+1}} x = \sum_{j=1}^{i+1} c_j n_j$$

implies $c_{i+1} \equiv 0 \pmod{a_i}$. Denoting $X' = \{\frac{x}{a_i} \mid x \in X_{i+1}\}$ and $c' = \frac{c_{i+1}}{a_i}$, we have

$$\sum_{x \in X'} x = \sum_{j=1}^{i} c_j n'_j + c' n_{i+1} = \sum_{j=1}^{i} c_j n'_j + c' \sum_{j=1}^{j} d_j n'_j = \sum_{j=1}^{i} (c_j + c' d_j) n'_j$$

By the inductive hypothesis, this is uniquely factored, specifically if $n_j \notin X'$ then $c_j + c'd_j = 0$. Since $c_j, d_j \geq 0$, then we have if $n_j \notin X'$ then $c_j = 0$ Hence $\sum_{x \in X_{i+1}} x$ is also uniquely factored.
If $n_{i+1} \in X_{i+1}$, denote $X'_{i+1} = X_{i+1} \backslash \{n_{i+1}\}$. Then if

$$\sum_{x \in X_{i+1}} x = \sum_{x \in X'_{i+1}} x + n_{i+1} = \sum_{j=1}^{i+1} c_j n_j$$

then $c_j \equiv 1 \pmod{a_i}$. so we have

$$\sum_{x \in X'_{i+1}} x = \sum_{j=1}^{i} c_j n_j + (c_{i+1} - 1) n_{i+1} = \sum_{j=1}^{i} c_j n_j + (c_{i+1} - 1) \sum_{j=1}^{j} d_j n'_j$$

Denoting $X' = \{\frac{x}{a_i} \mid x \in X'_{i+1}\}$ and $c' = \frac{c_{i+1}-1}{a}$, we have

$$\sum_{x \in X'} x = \sum_{j=1}^{i} (c_j + c'd_j) n'_j$$

By the inductive hypothesis, this is uniquely factored, specifically if $n_j \notin X'$ then $c_j + c'd_j = 0$. Since $c_j, d_j \geq 0$, then we have if $n_j \notin X'$ then $c_j = 0$ Hence $\sum_{x \in X_{i+1}} x$ is also uniquely factored. □

Corollary 113. *Every simplex of a numerical semigroup made by the inflation algorithm is accepted.*

Proof. Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical semigroup made by the inflation algorithm, and $X \subseteq \{n_1, \ldots, n_k\}$. Denoting $m = \sum_{x \in X} x$, by Lemma 112, m is uniquely factored, hence Δ_m is the simplex 2^X. □

Corollary 114. *Numerical semigroups made by the inflation algorithm and any $X \subseteq \{n_1, \ldots, n_k\}$ are valid for use in Proposition 105*

Proof. This follows directly from Corollary 113. □

Theorem 115. *For any arbitrary THICC, there exists a numerical semigroup S with $m \in S$, such that Δ_m is the THICC.*

Proof. By using the inflating algorithm, by Corollary 114, we can inflate any face, hence by Lemma 109, we can build a numerical semigroup that has any THICC as a simplicial complex. □

Proposition 116. *(Gluing Holes) Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical semigroup, $m \in S$, and the simplicial complex $\{\emptyset, \{n_i\}, \{n_j\}\}$ be accepted in S as $\Delta_{lcm(n_i, n_j)}$. Further, suppose that*

$$a = m - lcm(n_i, n_j) = \sum_{\ell=1}^{k} b_\ell n_\ell \in S$$

is nonzero. Then, picking p relatively prime to $m - lcm(n_1, n_2)$, Δ_{pm} in the semigroup

$$T = pS + a\langle 1 \rangle$$

is the simplicial complex Δ_m in S with an additional vertex with edges going to n_i and n_j.

• I would skip down to "Let's Just Glue Everything" (it's the most general)

Proof. Without loss of generality let the vertices be n_1 and n_2. It suffices to show that the only factorizations of pm using $m - \text{lcm}(n_1, n_2)$ involve exactly either one of pn_1 or pn_2. Write:

$$pm = c(m - \text{lcm}(n_1, n_2)) + \sum_{i=1}^{k} c_i p n_i$$

We have that $p \mid c$ and so we write $c = dp$:

$$dp(m - \text{lcm}(n_1, n_2)) + \sum_{i=1}^{k} c_i p n_i = p(m - \text{lcm}(n_1, n_2)) + p\text{lcm}(n_1, n_2)$$

Dividing through by p and rearranging terms,

$$(d-1)(m - \text{lcm}(n_1, n_2)) + \sum_{i=1}^{k} c_i n_i = \sum_{i=1}^{k} ((d-1)b_i + c_i) n_i = \text{lcm}(n_1, n_2)$$

Since $\text{lcm}(n_1, n_2)$ has factorizations only in terms of n_1 or n_2, we have that $c_i = 0$ for all $i \neq 1, 2$, and without loss of generality, $(d-1)b_1 + c_1 = 0$. Since $c_1 \geq 0$, we have $c_1 = 0$ and our factorization involves at most only $m - \text{lcm}(n_1, n_2)$ and pn_2; setting $d = 1$ gives the factorization that uses both. The same argument switching n_1 and n_2 yields the other factorization. □

Proposition 117. *(Gluing Holes II) Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical semigroup, $m \in S$, and the unfilled triangle with vertices $n_{i_1}, n_{i_2}, n_{i_3}$ be accepted in S as Δ_ℓ. Further, suppose*

$$a = m - \ell = \sum_{i=1}^{k} b_i n_i \in S$$

is nonzero. Then, picking p relatively prime to $m - \ell$, Δ_{pm} in the semigroup

$$T = pS + a\langle 1 \rangle$$

is the simplicial complex Δ_m in S with an additional vertex connected to every subset of $\{n_{i_1}, n_{i_2}, n_{i_3}\}$ of size 2.

Proof. Without loss of generality let the vertices be n_1, n_2, n_3. It suffices to show that any factorization of pm using $m - \ell$ must use exactly two of $\{n_1, n_2, n_3\}$. Write:

$$pm = c(m - \ell) + \sum_{i=1}^{k} c_i p n_i$$

We have that $p \mid c$ and so let $c = dp$:

$$dp(m - \ell) + \sum_{i=1}^{k} c_i p n_i = p(m - \ell) + p\ell$$

Dividing through by p and rearranging terms,

$$(d-1)(m - \ell) + \sum_{i=1}^{k} c_i n_i = \sum_{i=1}^{k} ((d-1)b_i + c_i) n_i = \ell$$

So $c_i = 0$ for all $i \neq 1, 2, 3$, and without loss of generality $(d-1)b_1 + c_1 = 0$, which implies $c_1 = 0$ and $d = 1$. Then we have that $c_2 n_2 + c_3 n_3 = \ell$, which gives us the factorization involving $m - \ell$, pn_2, and pn_3. Similar arguments give the other factorizations. □

Theorem 118. *(Holebound) Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical semigroup, $m \in S$, $N < k$. Suppose the N-skeleton with vertices $n_{i_1}, \ldots, n_{i_{N+1}}$ is accepted in S as Δ_ℓ. Further, let*

$$a = m - \ell = \sum_{i=1}^{k} b_i n_i \in S$$

have a factorization with $b_{i_1}, \ldots, b_{i_{N+1}} \neq 0$. Then, picking p relatively prime to $m - \ell$, Δ_{pm} in the semigroup

$$T = pS + a\langle 1 \rangle$$

is the simplicial complex Δ_m in S with an additional vertex connected to every subset of $\{n_{i_1}, \ldots, n_{i_{N+1}}\}$ of size N.

Proof. The same argument used in the above two propositions generalizes to N vertices. □

Definition 119. Let Δ be a simplicial complex and n be a vertex. We define the deletion of vertex n in Δ to be $\text{del}_n(\Delta) = \{f : f \in \Delta, n \notin f\}$, and the link of vertex n in Δ to be $\text{link}_n(\Delta) = \{f \backslash \{n\} : f \in \Delta, n \in f\}$.

Theorem 120. *(Let's Just Glue Everything) Let Δ be a simplicial complex, and $v_i \in \Delta$. Suppose there exists a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$ in which $del_{v_i}(\Delta)$ is accepted as m, $link_{v_i}(\Delta)$ is accepted as ℓ, and*

• This subsumes the previous gluing results so just look at this one!

$$a = m - \ell = \sum_{j=1}^{k} b_j n_j \in S$$

is nonzero in S. Then, picking p relatively prime to $m - \ell$, Δ_{pm} in the semigroup

$$T = pS + a\langle 1 \rangle$$

is Δ.

Proof. Consider any factorization of pm using $m - \ell$:

$$pm = c(m - \ell) + \sum_{j=1}^{k} c_j p n_j$$

We have that $p \mid c$ and so write $c = dp$:

$$p(m - \ell) + p\ell = pm = dp(m - \ell) + \sum_{j=1}^{k} c_j p n_j$$

Rearranging and dividing by p:

$$(d-1)(m - \ell) + \sum_{j=1}^{k} c_j n_j = \sum_{j=1}^{k} ((d-1)b_j + c_j) n_j = \ell$$

So $c_j = 0$ for all $\{n_j\} \notin \Delta$; in particular, $m - \ell$ does not connect to any vertices not in Δ. Further, setting $d = 1$ forces $\sum_{j=1}^{k} c_j n_j = \ell$, connecting $m - \ell$ to all the maximal faces in Δ_m; by how the link and deletion are defined, this new complex is exactly Δ. □

Theorem 121. *(All Euler Characteristics Are Possible, and Our First Serious Title) For every $z \in \mathbb{Z}$, there exists a numerical semigroup S and an $m \in S$ so that $\chi(\Delta_m) = z$.*

Proof. For $z = 0$, take any semigroup $S = \langle n_1, \ldots, n_k \rangle$ and any $m > F(S) + \sum_{i=1}^{k} n_i$. For any $z < 0$, take S to be a skeletal monoid with embedding dimension $|z-1|$, and m be the least common multiple of the generators; this gives the 1-skeleton on $|z-1|$ vertices, with Euler characteristic z.

For $z > 0$, we may build the semigroup S inductively. Let $S_1 = \langle n_1, n_2 \rangle$; we have that $\Delta_{n_1 n_2}$ is the 1-skeleton and $\Delta_{(n_1 n_2)^2}$ has the line connecting them. Using Proposition 116, pick p_1 relatively prime to $n_3 = n_1 n_2 (n_1 n_2 - 1)$, and $S_2 = \langle p_1 n_1, p_1 n_2, n_3 \rangle$. Then $\Delta_{p_1 (n_1 n_2)^2}$ is the unfilled triangle with Euler characteristic 1, and the two vertices $\{pn_1, pn_2\}$ are still accepted as $\Delta_{p_1 n_1 n_2}$, as $pn_1 pn_2 = \operatorname{lcm}(pn_1, pn_2)$, and any factorization involving n_3:

$$p_1 n_1 n_2 = pc_1 n_1 + pc_2 n_2 + c_3 n_1 n_2 (n_1 n_2 - 1)$$

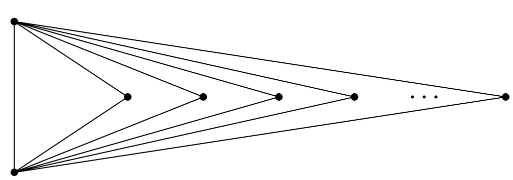
forces $p \mid c_3$ and the right to be larger than the left. So we may repeat the process as many times as the two vertices are still accepted, picking a prime p_{i-1} larger than all previous generators. Now, letting

$$P_i = \prod_{j=1}^{i-1} p_j$$

and having the base case above, suppose that

$$S_n = \left\langle P_n n_1, P_n n_2, \frac{P_n}{p_1} n_3, \ldots, \frac{P_n}{p_{n-1}} n_3 \right\rangle$$

still accepts the two vertices $\{P_n n_1, P_n n_2\}$ as $P_n n_1 n_2$, and $\Delta_{P_n (n_1 n_2)^2}$ is $n-1$ unfilled triangles that share a base. In particular, $\chi(\Delta_{P_n (n_1 n_2)^2}) = n - 1$. It suffices to show that the two vertices are still accepted in S_{n+1}, as then we may glue on another one-dimensional hole and increase the Euler characteristic by 1.



Consider any factorization of

$$P_{n+1} n_1 n_2 = c_1 P_{n+1} n_1 + c_2 P_{n+1} n_2 + \sum_{i=3}^{n} c_i \frac{P_{n+1}}{p_i} n_3$$

Since the p_i are all relatively prime, any nonzero c_i for $i \geq 3$ forces $p_i \mid c_i > 0$ and hence the right is bigger than the left. And since n_1 and n_2 are relatively prime (as S_1 is primitive), we cannot use both in making their least common multiple, completing the induction. Hence we may take $\chi(\Delta_{P_{z+1}(n_1 n_2)^2}) = z$ in S_{z+1} to get all positive Euler characteristics. $\square$

We now venture deeper into the realm of simplicial complexes, and discover the following definitions that will aid us greatly.

Lemma 122. *Let $S = \langle n_1, \dots n_k \rangle$ be a numerical semigroup, and $m \in S$ such that $n_i \in \Delta_m$. Then $link_{n_i}(\Delta_m) = del_{n_i}(\Delta_{m-n_i})$.*

Proof. Let f be a maximal face in $\text{link}_{n_i}(\Delta_m)$. Then $n_i \notin f$ and $f \cup \{n_i\} \in \Delta_m$. So we have

$$m = \sum_{x \in f} c_{x,f} x + c n_i \iff m - n_i = \sum_{x \in f} c_{x,f} x + (c-1) n_i$$

$c \geq 1$, and so $f \in \Delta_{m-n_i}$. Hence $f \in \text{del}_{n_i}(\Delta_{m-n_i})$ and $\text{link}_{n_i}(\Delta_m) \subseteq \text{del}_{n_i}(\Delta_{m-n_i})$. Let f be a maximal face in $\text{del}_{n_i}(\Delta_{m-n_i})$. Then $n_i \notin f$ and $m - n_i = \sum_{x \in f} c_{x,f} x$. Therefore $f \cup \{n_i\} \in \Delta_m$ and $f \in \text{link}_{n_i}(\Delta_m)$. Hence $\text{del}_{n_i}(\Delta_{m-n_i}) \subseteq \text{link}_{n_i}(\Delta_m)$. Hence $\text{link}_{n_i}(\Delta_m) = \text{del}_{n_i}(\Delta_{m-n_i})$. □

Theorem 123. *Let $h_0, \dots, h_d$ be a sequence of nonnegative integers. Then, there exists a semigroup S and an element $m \in S$ so that $H_i(\Delta_m) = h_i$ for $0 \leq i \leq d$.*

Proof. We will first show it is possible to construct a simplicial complex having n k-dimensional holes, with zero ℓ-homology for $k \neq \ell$. proceeding inductively. Let $S_1 = \langle n_1, \dots, n_k \rangle$ be skeletal, with $P = \text{lcm}(n_1, \dots, n_k)$ as usual. Then, we have that $\Delta_{(k-1)P}$ is the $k-1$ skeleton, and ΔP^2 is the complete graph on k vertices. Then, picking q_1 relatively prime to $P^2 - (k-1)P = (P-k+1)P$, we have by Theorem 120 that $\Delta_{q_1 P^2}$ in $S_2 = \langle q_1 n_1, \dots, q_1 n_k, (P-k+1)P \rangle$ has one k-dimensional hole. Since q_1 is relatively prime to $(P-k+1)P$, $(P-k+1)P$ cannot appear in any factorization of $p_1 P$, and we have again $p_1 P$ is the $k-1$ skeleton, and we may repeat the construction picking primes q_{i-1} larger than all previous generators; this gives the base case.

Now, let $m = (P-k+1)P$ and let

$$Q_i = \prod_{j=1}^{i-1} p_j$$

and suppose that

$$S_n = \left\langle Q_n n_1, Q_n n_2, \dots, Q_n n_k, \frac{Q_n}{q_1} m, \dots, \frac{Q_n}{q_{n-1}} m \right\rangle$$

still accepts the $k-1$-skeleton as $Q_n P$, and $\Delta_{Q_n P^2}$ has $n-1$ k-dimensional holes. Then, consider any factorization of

$$Q_{n+1}(k-1)P = \sum_{i=1}^{k} c_i Q_{n+1} n_i + \sum_{i=k+1}^{n} d_i \frac{Q_{n+1}}{q_i} m$$

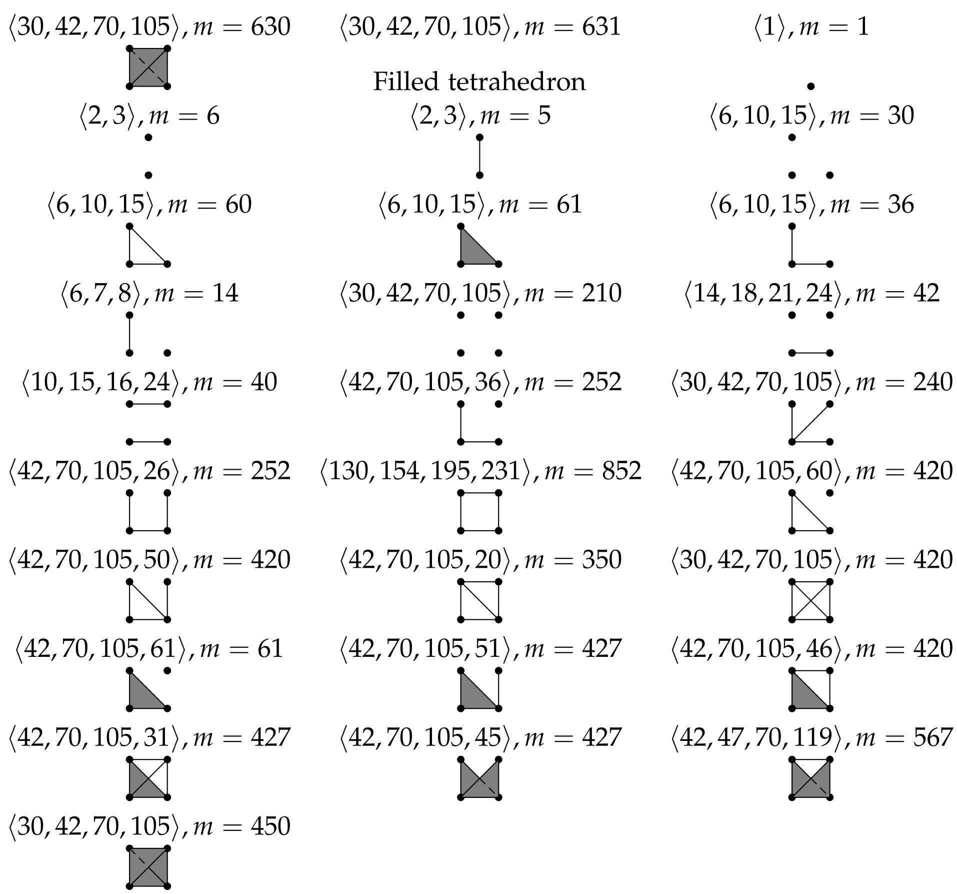
Since the q_i are all relatively prime, $q_i \mid d_i$ for all valid i, since $m > P$ any nonzero d_i makes the right larger than the left. The same contradiction arises if all $c_i > 0$; however setting $c_i = \frac{P}{n_i}$ for any $k-1$ coefficients gives a valid factorization. Hence the $k-1$-skeleton is again accepted as $P_{n+1}(k-1)P$, completing the induction.

So, starting with any skeletal monoid with embedding dimension k, we may take $\Delta_{Q_{n+1,k}P_k}$ in that semigroup to obtain a simplicial complex with n k-dimensional holes. P_k is uniquely determined by k relatively prime numbers, and P_{n+1} is uniquely determined by $n+1$ prime numbers. Hence, we may pick prime numbers so that $Q_{h_d,d}P_d$ are pairwise relatively prime for $0 \le i \le d$. Then, we may use Theorem 28 to create a new semigroup taking the disconnected union of all these complexes.

□

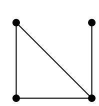
Theorem 124. *All simplicial complexes on up to four vertices occur in some numerical semigroup*

Proof. We give an example of each simplicial complex on up to four vertices:



□

Example 125. Consider the simplicial complex Δ:



The deletion of the bottom right vertex is given by a line segment and an isolated vertex, and the link is given by three isolated vertices. These complexes cannot arise in the same semigroup of embedding dimension 3, since the link implies the semigroup is skeletal, while the deletion implies it is not. However, even though the link cannot be accepted whenever the deletion arises, we may still construct Δ; in particular, $\Delta_{420} = \Delta$ in $S = \langle 42, 50, 70, 105\rangle$.

12. On the Subject of Graphs

Theorem 126. *(Realization of Bipartite Graphs) Let $K_{k,\ell}$ denote the complete bipartite graph from k to ℓ vertices. There exists a numerical semigroup U such that $K_{a,b}$ is accepted.*

Proof. Let $S = \langle n_1, \ldots, n_k\rangle$ be a skeletal monoid, with $P = \operatorname{lcm}(n_i)$ and $P/n_1 = 2$. Let $T = \langle m_1, \ldots, n_\ell\rangle$ also be a skeletal monoid, with $Q = \operatorname{lcm}(m_i)$, $Q/m_1 = 2$, and $\gcd(Q/m_i, P/n_j) = 1$ for all $(i,j) \neq (1,1)$. That is, let $\gcd(P,Q) = 2$. Then, picking relatively prime scalars $c_1, c_2 > 2$ which are also relatively prime to P and Q, consider the glued semigroup

$$U = \frac{Q}{2}c_1 S + Pc_2 T$$

and consider a factorization of $\frac{PQ}{2}c_1 + PQc_2$:

$$\frac{PQ}{2}c_1 + PQc_2 = \sum_{i=1}^{k} \frac{Q}{2}c_1 d_i n_i + \sum_{i=1}^{\ell} Pc_2 e_i m_i$$

Since $2 \mid Q$, we must have that $P \mid \sum_{i=1}^{k} d_i n_i$, which forces $\frac{P}{n_i} \mid d_i$ for each i, giving us

$$\frac{PQ}{2}c_1 + PQc_2 = \sum_{i=1}^{k} \frac{PQ}{2}c_1 d_i' + \sum_{i=1}^{\ell} Pc_2 e_i m_i$$

Since $2 \mid P$ we now must have that $Q \mid \sum_{i=1}^{\ell} e_i m_i$, giving us, after dividing out by $\frac{PQ}{2}$,

$$c_1 + 2c_2 = \sum_{i=1}^{k} c_1 d_i' + \sum_{i=1}^{\ell} 2c_2 e_i' = Dc_1 + 2Ec_2$$

where $D = \sum_{i=1}^{k} d_i'$ and $E = \sum_{i=1}^{\ell} e_i'$. So we have

$$(D-1)c_1 + (2E-2)c_2 = 0$$

If $D = E = 1$, this holds and furthermore, this is only a factorization using one generator from each semigroup, implying an edge in the graph. Now suppose without loss of generality $E > 1$. Then $2E - 2 > 0$ and $D - 1 < 0$. Since all summands of D are nonnegative, we have $D = 0$. Then we have

$$(2E-2)c_2 = c_1$$

Now $c_2 \mid c_1$ a contradiction. So $\Delta_{\frac{PQ}{2}c_1 + PQc_2}$ is exactly $K_{a,b}$. □

Definition 127. (Simplicial Hypergraphs) Let $k_1, \ldots, k_n \in \mathbb{N}$, and let X_i be a set with k_i vertices. We say the hypergraph $H_{k_1,\ldots,k_n}$ is the simplicial complex whose maximal faces are exactly the n-dimensional faces containing one element of each X_i.

• Hi Coneill! Fix this definition please!

Theorem 128. *(Realization of 3-Hypergraphs) Let $k_1, k_2, k_3 \geq 3$. Then there exists a numerical semigroup T so that H_{k_1,k_2,k_3} is accepted.*

Proof. Let $S_1 = \langle n_1, \dots, n_{k_1} \rangle$, $S_2 = \langle m_1, \dots, m_{k_2} \rangle$, $S_3 = \langle \ell_1, \dots, \ell_{k_3} \rangle$ be skeletal with $P = \mathrm{lcm}(n_i)$, $Q = \mathrm{lcm}(m_i)$, and $R = \mathrm{lcm}(\ell_i)$. Further, let $\gcd(P, Q) = 2$, $\gcd(P, R) = 3$, and $\gcd(Q, R) = 1$. Then, pick $c_1, c_2, c_3 > 3$ pairwise relatively prime and relatively prime to P, Q, R, and consider the glued semigroup

$$T = \frac{QR}{6} c_2 c_3 S_1 + \frac{PR}{3} c_1 c_3 S_2 + \frac{PQ}{2} c_1 c_2 S_3$$

Consider a factorization of $\frac{PQR}{6} c_2 c_3 + \frac{PQR}{3} c_1 c_3 + \frac{PQR}{2} c_1 c_2$:

$$\frac{PQR}{6} c_2 c_3 + \frac{PQR}{3} c_1 c_3 + \frac{PQR}{2} c_1 c_2 = \sum_{i=1}^{k_1} \frac{QR}{6} c_2 c_3 d_i n_i + \sum_{i=1}^{k_2} \frac{PR}{3} c_1 c_3 e_i m_i + \sum_{i=1}^{k_3} \frac{PQ}{2} c_1 c_2 f_i \ell_i$$

Since $3 \mid R$ and $2 \mid Q$, we have $P \mid \sum_{i=1}^{k_1} d_i n_i$ and so $\frac{P}{n_i} \mid d_i$. Hence,

$$\frac{PQR}{6} c_2 c_3 + \frac{PQR}{3} c_1 c_3 + \frac{PQR}{2} c_1 c_2 = \sum_{i=1}^{k_1} \frac{PQR}{6} c_2 c_3 d_i' + \sum_{i=1}^{k_2} \frac{PR}{3} c_1 c_3 e_i m_i + \sum_{i=1}^{k_3} \frac{PQ}{2} c_1 c_2 f_i \ell_i$$

since $6 \mid P$ and $3 \mid R$, we have that $Q \mid \sum_{i=1}^{k_2} e_i m_i$, and, repeating the argument one more time, $R \mid \sum_{i=1}^{k_3} f_i \ell_i$. Now, we have

$$\frac{PQR}{6} c_2 c_3 + \frac{PQR}{3} c_1 c_3 + \frac{PQR}{2} c_1 c_2 = D \frac{PQR}{6} c_2 c_3 + E \frac{PQR}{3} c_1 c_3 + F \frac{PQR}{3} c_1 c_2$$

where $D = \sum_{i=1}^{k_1} d_i'$ and E, F are defined similarly. Dividing through by $\frac{PQR}{6}$, we get

$$c_2 c_3 + 2 c_1 c_3 + 3 c_1 c_2 = D c_2 c_3 + 2E c_1 c_3 + 3F c_1 c_2$$
$$(D-1) c_2 c_3 + (2E-2) c_1 c_3 + (3F-3) c_1 c_2 = 0$$

It now suffices to show $D = E = F = 1$. Suppose towards a contradiction that $E > 1$ and $F \geq 1$; then we must have that $D < 1$, which implies $D = 0$. Hence,

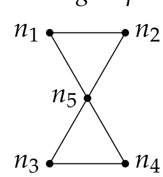
$$c_2 c_3 = (2E-2) c_1 c_3 + (3F-3) c_1 c_2$$

which implies $c_1 \mid c_2 c_3$, a contradiction. Now, if instead $E > 1$ and $F = 0$, then we have

$$3 c_1 c_2 = (D-1) c_2 c_3 + (2E-2) c_2 c_3$$

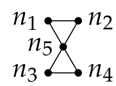
which implies $c_3 \mid 3 c_1 c_2$, another contradiction. In particular, if any of $D, E, F = 0$, we arrive at a similar contradiction. From the above $E \leq 1$, and we may repeat this argument for D, F to show $D, F \leq 1$. Since each must be nonzero, $D = E = F = 1$ as desired. ∎

Theorem 129. *There is no numerical semigroup such that*



is accepted.

Proof. Without loss of generality we may assume that S has exactly 5 generators, since we may remove any generators that do are not in the complex above. If S is no longer primitive, we may scale down both the element and the generators by the greatest common divisor. Suppose towards a contradiction that



is accepted in $S = \langle n_1, n_2, n_3, n_4, n_5 \rangle$ as m. Then

$$\begin{aligned} m &= a_1 n_1 + a_2 n_2 = b_1 n_1 + b_5 n_5 = c_2 n_2 + c_5 n_5 \\ &= d_3 n_3 + d_4 n_4 = e_3 n_3 + e_5 n_5 = f_4 n_4 + f_5 n_5 \end{aligned}$$

We will prove $a_1 = b_1$ and by symmetry $a_2 = c_2, d_3 = e_3$ and $d_4 = f_4$. Then we will have $a_2 n_2 = b_5 n_5, a_1 n_1 = c_5 n_5, d_4 n_4 = e_5 n_5$ and $d_3 n_3 = f_5 n_5$. Then suppose without loss of generality $b_5 < e_5$. Then $m = e_3 n_3 + e_5 n_5 = e_3 n_3 + (e_5 - b_5) n_5 + a_2 n_2$, a contradiction, since that triangle is not in the simplicial complex. Thus $e_5 = b_5 = f_5 = c_5$. Then $m = b_1 n_1 + b_5 n_5 = b_1 n_1 + e_5 n_5 = b_1 n_1 + d_4 n_4$, a contradiction as that edge is not in the simplicial complex.

We now stick to our word and prove $a_1 = b_1$. Suppose towards a contradiction that $a_1 > b_1$. Then

$$m = a_1 n_1 + a_2 n_2 = b_1 n_1 + b_5 n_5 \iff (a_1 - b_1) n_1 + a_2 n_2 = b_5 n_5$$

If $b_5 \leq e_5$, we would have

$$m = (e_5 - b_5) n_5 + e_3 n_3 + (a_1 - b_1) n_1 + a_2 n_2$$

a contradiction since we do not have the triangle n_1, n_2, n_3 in the simplicial complex. Hence $b_5 > e_5$ and by symmetry $b_5 > f_5$. Then

$$m = b_1 n_1 + b_5 n_5 = e_3 n_3 + e_5 n_5 \iff b_1 n_1 + (b_5 - e_5) n_5 = e_3 n_3$$

If $e_3 \leq d_3$, we would have

$$m = (d_3 - e_3) n_3 + d_4 n_4 + b_1 n_1 + (b_5 - e_5) n_5$$

a contradiction, since we don't have the triangle n_1, n_4, n_5 in the simplicial complex. Hence $e_3 > d_3$. Then

$$(e_3 - d_3) n_3 + e_5 n_5 = d_4 n_4$$

and we have $d_4 \geq f_4$. Note that $d_4 = f_4$ is valid. Now, since $b_5 > f_5$, then

$$b_1 n_1 + (b_5 - f_5) n_5 = f_4 n_4$$

and we have $f_4 > d_4$, a contradiction.

Now suppose that $a_1 < b_1$. Then

$$(b_1 - a_1) n_1 + b_5 n_5 = a_2 n_2$$

and $a_2 > c_2$. Then

$$a_1 n_1 + (a_2 - c_2) n_2 = c_5 n_5$$

and $c_5 > e_5, f_5$. Then

$$c_2 n_2 + (c_5 - e_5) n_5 = e_3 n_3$$

and $e_3 > d_3$. Then

$$(e_3 - d_3)n_3 + e_5 n_5 = d_4 n_4$$

and $d_4 \geq f_4$. Note that $d_4 = f_4$ is valid. Now, since $c_5 > f_5$, then

$$c_2 n_2 + (c_5 - f_5)n_5 = f_4 n_4$$

and $f_4 > d_4$, a contradiction. Hence $a_1 = b_1$ as promised. ☐

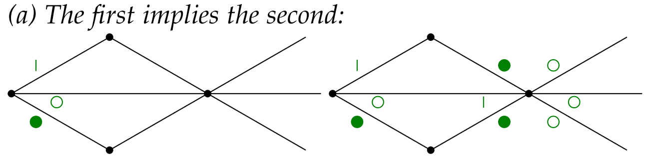
13. On the Subject of Dodgeball

Definition 130. (Dodgeball Labelling) Let G be a graph. We must assign each vertex-edge pairing either a green line, a green hollow circle, or a green filled-in circle, such that each vertex has exactly one green line. Further, we may assign any number of blue lines to any number of vertex-edge pairings; such an assignment is called a *dodgeball labelling*. Further, the dodgeball labelling given by no green hollow circles is called the *trivial labelling*.

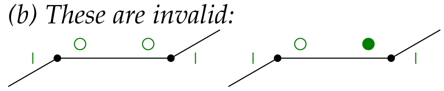
Let Δ_m be a simplicial complex who is a graph. For a vertex n_i, we consider a green line on a vertex-edge $(n_i, \{n_i, n_j\})$ pairing corresponding to a factorization $m = a_i n_i + a_j n_j$ to denote that a_i is the largest coefficient of n_i in any factorization of m using n_i. A green hollow circle coming out of a vertex edge $(n_i, \{n_i, n_k\})$, with respect to a green line, denotes that the factorization $m = b_i n_i + b_k n_k$ has strictly smaller coefficient for n_i, i.e. $b_i < a_i$.. A green filled-in circle is marked for coefficients of factorizations that may be strictly smaller or may be equal to the coefficient corresponding to the green line.

For a fixed vertex n_i, the same number of blue lines on two vertex-edge pairings with vertex n_i indicates that the coefficient of n_i in the two corresponding factorizations are the same.

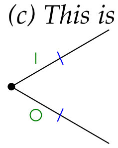
Theorem 131. *With these conventions, we have the following implications for labelling:*
(a) The first implies the second:



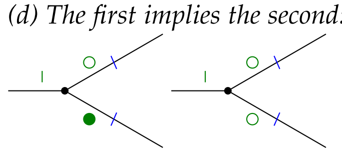
(b) These are invalid:



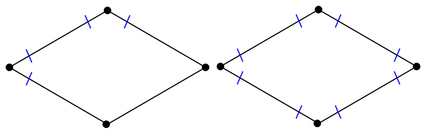
(c) This is invalid:



(d) The first implies the second:

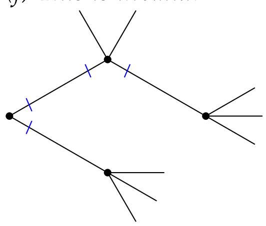


(e) The first implies the second

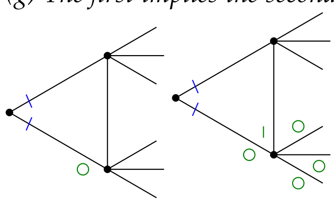


(f) This is invalid:

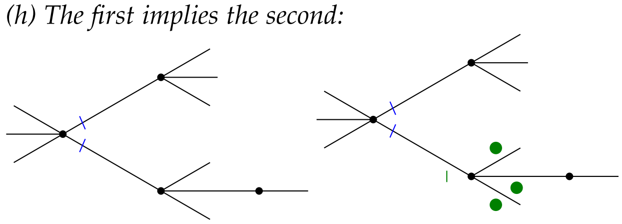
• A labelling with all blue lines that is not entirely made of 4-cycles is impossible



(g) The first implies the second:



(h) The first implies the second:



Proof. (a) The first figure implies there are factorizations:

$$m = a_1 n_1 + a_2 n_2 = b_1 n_1 + b_3 n_3 = c_1 n_1 + c_4 n_4$$

such that $b_1 < c_1 \leq a_1$. Further, we have factorizations involving n_3:

$$m = d_3 n_3 + d_5 n_5 = e_3 n_3 + e_6 n_6 = f_3 n_3 + f_7 n_7$$
$$= g_2 n_2 + g_3 n_3 = h_3 n_3 + h_4 n_4$$

Now, if $b_3 \leq d_3$, since

$$b_1 n_1 + b_3 n_3 = d_3 n_3 + d_5 n_5$$

$$b_1 n_1 = (d_3 - b_3) n_3 + d_5 n_5$$

And we may substitute:

$$m = (a_1 - b_1) n_1 + a_2 n_2 + b_1 n_1 = (a_1 - b_1) n_1 + a_2 n_2 + (d_3 - b_3) n_3 + d_5 n_5$$

giving us a factorization of m using strictly more than two vertices, a contradiction. A similar argument works to show $b_3 > e_3, f_3$, and $b_3 \geq g_3, h_3$, giving us all the markings we desired.

(b) We have the factorizations:

$$m = a_1 n_1 + a_i n_i = b_1 n_1 + b_2 n_2 = c_2 n_2 + c_j n_j$$

for some unlabelled arbitrary n_i, n_j, and $b_1 < a_1$ and $b_2 \leq c_2$. But together these imply:

$$(c_2 - b_2)n_2 + c_j n_j = b_1 n_1$$

$$(a_1 - b_1)n_1 + b_1 n_1 + a_i n_i = (a_1 - b_1)n_1 + (c_2 - b_2)n_2 + c_j n_j + a_i n_i = m$$

giving a factorization with four vertices if both circles are hollow, and with three vertices if one is not; either is impossible.

(c) This is obviously impossible from how we have defined dodgeball.
(d) This implication is obvious from how we have defined dodgeball.
(e) Starting with the first figure, any placement of the green line on a vertex without blue lines results in a contradiction, appealing to (a), (b), or (c).
(f) We begin with the factorizations

$$m = a_1 n_1 + a_2 n_2 = b_1 n_1 + b_3 n_3 = c_1 n_1 + c_4 n_4$$

We know $a_1 = c_1$ and $a_2 = b_2$. It follows $a_1 n_1 + a_2 n_2 = a_1 n_1 + c_4 n_4$, and thus $a_2 n_2 = c_4 n_4$. Similarly, $a_2 n_2 + b_3 n_3 = a_1 n_1 + a_2 n_2$, so $b_3 n_3 = a_1 n_1 = c_1 n_1$. Therefore,

$$m = c_1 n_1 + c_4 n_4 = b_3 n_3 + c_4 n_4$$

a contradiction since there is no factorization containing n_3 and n_4.
(g) For the vertex where the green circle is marked, placing the green line anywhere else results in a contradiction appealing to (a) and (c).
(h) We have the factorizations:

$$m = a_1 n_1 + a_2 n_2 = b_2 n_2 + b_3 n_3 = c_3 n_3 + c_4 n_4$$

where without loss of generality we consider only one other vertex attached to n_3. If $d_3 > b_3$, then,

$$(d_3 - b_3)n_3 + d_5 n_5 = b_2 n_2 = a_2 n_2$$

So we have

$$a_1 n_1 + (d_3 - b_3)n_3 + d_5 n_5 = m$$

a factorization of m using three vertices, a contradiction. Hence $b_3 \geq d_3$, and this holds for all other edges coming out of n_3.

□

Proposition 132. *If a graph has no valid dodgeball labelling, the graph is not accepted by any semigroup.*

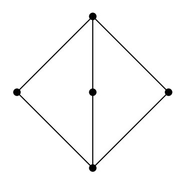
Proof. We prove the contrapositive: if a graph is accepted by some semigroup, then we may simply label the graph according to its factorizations in that semigroup. □

Proposition 133. *Let G be a graph accepted in $S = \langle n_1, \ldots, n_k \rangle$ as m and $n_i \in \Delta_m$. Then $del_{n_i}(\Delta_m)$ is accepted in some semigroup*

Proof. If $\text{del}_{n_i}(\Delta_m)$ does not disconnect any vertices, then for any factorization of m that uses n_i, the other generator is in another factorization. Hence (Δ_m) in $S'\langle n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_k \rangle$ has all the faces in (Δ_m) in S did except those containting n_i, i.e. (Δ_m) in S' is $\text{del}_{n_i}(\Delta_m)$.

If $\text{del}_{n_i}(\Delta_m)$ disconnects vertices, let X be the set of those vertices. Then let $G' = \text{del}_{x \in X}(\Delta_m)$. Since each vertex in G' is disconnected in $\text{del}_{n_i}(\Delta_m)$, then deleting any of those does not disconnect vertices unless $G = \Delta_m$ is a star with n_i as the central vertex. In that case we know both the star and deletion of n_i appear in skeletal monoids. If G is not a star, then by the previous argument, G' is accepted. Then $\text{del}_{n_i}(G')$ is accepted in some semigroup. Then using Corollary 27, glue on v vertices where $v = |X|$. This new simplicial complex is exactly $\text{del}_{n_i}(\Delta_m)$. □

Example 134. The diamond



is accepted in $S = \langle 50, 70, 75, 105, 42 \rangle$ as 360.

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[1] Claire Kiers, Christopher O'Neill, and Vadim Ponomarenko, *Numerical semigroups on compound sequences*, Comm. Algebra **44** (2016), no. 9, 3842–3852. MR 3503387

[2] J. C. Rosales and P. A. Garcí a Sánchez, *Numerical semigroups*, Developments in Mathematics, vol. 20, Springer, New York, 2009. MR 2549780