The catenary degrees of elements in numerical monoids of embedding dimension three

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1 Introduction

Definition 1.1. *Let* $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$*. A* numerical monoid S *is a subset of* $\mathbb{N}_0$ *such that*

1. $0 \in S$,
2. $s_1 + s_2 \in S$ *for all* $s_1, s_2 \in S$ *and*
3. $|\mathbb{N}_0 \setminus S| < \infty$.

An element $a \in S$ *is an* atom *if there does not exist nonzero* $b, c \in S$ *such that* $b + c = a$*. If* $a_1, a_2, \ldots, a_k$ *are the atoms of a numerical semigroup* S*, then we write*

$$S = \langle a_1, a_2, \ldots, a_k \rangle = \{a_1 n_1 + a_2 n_2 + \cdots + a_k n_k : n_1, n_2, \ldots, n_k \in \mathbb{N}_0\}.$$

For elements $n_1, n_2 \in S$*, we say that* n_1 divides n_2 *if* $n_2 - n_1 \in S$*.*

Definition 1.2. *Let* $S = \langle n_1, n_2, \ldots, n_k \rangle$*. We say that* $f = (f_1, f_2, \ldots, f_k)$ *with* $f_1, f_2, \ldots, f_k \in \mathbb{N}_0$ *is a* factorization *of* $s \in S$ *if* $f_1 n_1 + f_2 n_2 + \cdots + f_k n_k = s$*. The* length *of* f *is denoted* $|f|$ *where* $|f| = f_1 + f_2 + \cdots + f_k$*. We denote the set of all factorizations of* s *as* $Z(s)$*.*

Definition 1.3. *Let* $S = \langle n_1, n_2, \ldots, n_k \rangle$*. Let* $f = (f_1, f_2, \ldots, f_k)$ *and* $\bar{f} = (\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_k)$ *be two factorizations of* $s \in S$*.*

- *Let* $\gcd(f, \bar{f}) = (\min\{f_1, \bar{f}_1\}, \min\{f_2, \bar{f}_2\}, \ldots, \min\{f_k, \bar{f}_k\})$*.*
- *We denote the* distance *between* f *and* $\bar{f}$ *as* $d(f, \bar{f})$ *where*

$$d(f, \bar{f}) = \max\{|f - \gcd(f, \bar{f})|, |\bar{f} - \gcd(f, \bar{f})|\}.$$

- *A sequence of factorizations* $z_1, \ldots, z_r$ *is an* N-chain *if* $d(z_i, z_{i+1}) \leq N$ *for* $1 \leq i \leq r - 1$*.*

- *For $s \in S$, the* catenary degree *of s, denoted $c(s)$, is the minimal N such that there is an N-chain between any two factorizations of s.*
- *We define the* catenary degree *of S, denoted $c(S)$, as*
$$c(S) = \sup\{c(s) : s \in S\}.$$

Definition 1.4. *Given a numerical monoid S and an element $n \in S$, let ∇_n denote the graph whose vertex set is $Z(n)$ and such that any two factorizations $z_1, z_2 \in Z(n)$ are connected by an edge only if $\gcd(z_1, z_2) \neq 0$. An element $m \in S$ is a* Betti element *of S if ∇_m is not connected.*

Definition 1.5. *Given a numerical monoid $S = \langle n_1, \ldots, n_k\rangle$, let $\varphi : \mathbb{N}_0^k \to S$ be given by*
$$\varphi(x_1, \ldots, x_k) = x_1 n_1 + \cdots + x_k n_k.$$
A minimal presentation *for S is a subset $\rho \subset \mathbb{N}_0^k \times \mathbb{N}_0^k$ such that whenever $\varphi(z) = \varphi(z')$, there exists $z = z_0, \ldots, z_r = z' \in \mathbb{N}_0^k$ such that $(z_i, z_{i+1}) = (a_i+u_i, b_i+u_i)$ for some $u_i \in \mathbb{N}_0^k$ and either $(a_i, b_i) \in \rho$ or $(b_i, a_i) \in \rho$.*

Definition 1.6. *Let S be a numerical monoid. Define the* catenary set $C(S)$ *of S to be*
$$C(S) = \{c(s) : s \in S\}.$$

The following lemma can be found in [3].

Lemma 1.7. *Let $S = \langle n_1, n_2, n_3\rangle$ be a numerical monoid with embedding dimension three. Any Betti element $m \in S$ can be written in the form*
$$c_i n_i = r_{ij} n_j + r_{ik} n_k,$$
where $\{i, j, k\} = \{1, 2, 3\}$ and $c_i = \min\{k > 0 : kn_i \in \langle n_j, n_k\rangle\}$.

Lemma 1.8. *Let $S = \langle n_1, n_2, n_3\rangle$ be a numerical monoid with embedding dimension three. If, for each $i \in \{1, 2, 3\}$, $n_j, n_k \nmid c_i n_i$, then $|Z(c_i n_i)| = 2$ for each i. Furthermore, $c(c_i n_i) = \max\{c_i, r_{ij} + r_{ik}\}$, where $c_i n_i = r_{ij} n_j + r_{ik} n_k$.*

Proof. Suppose, for each $i \in \{1, 2, 3\}$, $n_j, n_k \nmid c_i n_i$. Furthermore, suppose $|Z(c_i n_i)| > 2$ for some i. Then there exist $f, g, h \in Z(c_i n_i)$ such that $f_i = c_i$, $f_j = 0$ and $f_k = 0$. By the minimality of c_i, $g_i = 0$ and $h_i = 0$. This implies that $c_i n_i = g_j n_j + g_k n_k = h_j n_j + h_k n_k$. Since $n_j, n_k \nmid c_i n_i$, $\gcd(g, h) \neq 0$. Subtracting $\gcd(g, h)$ from each of g and h gives the equation $sn_j = tn_k$. Since this element is divisible by both n_j and n_k, it is divisible by $u = \operatorname{lcm}(n_j, n_k)$. This gives
$$\frac{u}{n_j} n_j = \frac{u}{n_k} n_k \leq sn_j = tn_k < c_i n_i.$$
Since the only factorization of $c_i n_i$ involving n_i is $c_i n_i$, u has no factorizations involving n_i. Moreover, by minimality of u, u is a Betti element whose only factorizations are $\frac{u}{n_j} n_j$ and $\frac{u}{n_k} n_k$. This means that $c_j = \frac{u}{n_j}$, contradicting the assumption that $n_k \nmid c_j n_j$. Therefore, $|Z(c_i n_i)| = 2$. The second claim follows directly from the first. □

2 Catenary Sets Can Be Arbitrarily Large

Lemma 2.1. *Let n be an odd integer. Then n and $3n-4$ are coprime. Thus, if $n \geq 5$, $S = \langle n, 3n-8, 3n-4 \rangle$ is a numerical monoid.*

Proof. Since $\gcd(a, ma+b) = \gcd(a, b)$ for any $a, b, m \in \mathbb{Z}$, we have that $\gcd(n, 3n-4) = \gcd(n, -4)$. Furthermore, n is odd and, therefore, $\gcd(n, -4) = 1$. Thus, $\gcd(n, 3n-4) = 1$. $\square$

Lemma 2.2. *Let n be an odd integer greater than or equal to 5 and let $S = \langle n, 3n-8, 3n-4 \rangle$. The Betti elements of S are*

$$u = \left(\frac{3n-5}{2}\right) n, v = \left(\frac{n+1}{2}\right)(3n-8) \text{ and } w = 2(3n-4).$$

Furthermore,

$$c(u) = \frac{3n-5}{2}, c(v) = \frac{3n-9}{2} \text{ and } c(w) = 4.$$

Proof. We have

$$\begin{aligned} u = \left(\frac{3n-5}{2}\right) n &= \left(\frac{3n-8+3}{2}\right)(n-1) + \frac{3n-5}{2} \\ &= \left(\frac{3n-8}{2}\right)(n-1) + \frac{3(n-1)}{2} + \frac{3n-5}{2} \\ &= \left(\frac{n-1}{2}\right)(3n-8) + (3n-4), \end{aligned}$$

$$\begin{aligned} v = \left(\frac{n+1}{2}\right)(3n-8) &= \left(\frac{3n-8-3}{2}\right) n + \frac{3n}{2} + \frac{3n-8}{2} \\ &= \left(\frac{3n-11}{2}\right) n + (3n-4), \end{aligned}$$

and

$$w = 2(3n-4) = 3n + (3n-8).$$

Suppose $kn = a(3n-8) + b(3n-4)$, where $0 < k < \frac{3n-5}{2}$, $0 \leq a$ and $0 \leq b$. Clearly, then, $a < \frac{3n-5}{2}$. Reducing modulo n, we have $-8a - 4b \equiv 0 \bmod n$ and, since n is odd, $2a + b \equiv 0 \bmod n$. Therefore, $2a + b = rn$ for some $r \in \mathbb{Z}$. But $0 \leq a$ and $0 \leq b$, so $0 \leq 2a + b = rn$. Thus, $0 \leq r$. Furthermore, a and b cannot be simultaneously zero, because $0 < k$, so $r \neq 0$, i.e., $1 \leq r$. Now,

$$\begin{aligned} kn = a(3n-8) + b(3n-4) &= a(6n-8) - 3an + b(3n-4) \\ &= 2a(3n-4) + b(3n-4) - 3an \\ &= (2a+b)(3n-4) - 3an \\ &= rn(3n-4) - 3an, \end{aligned}$$

so

$$k = r(3n-4) - 3a > r(3n-4) - 3\left(\frac{3n-5}{2}\right) = \left(r - \frac{3}{2}\right)(3n-4) + \frac{3}{2}.$$

If $r \geq 2$, then $k > \left(r - \frac{3}{2}\right)(3n-5) + \frac{3}{2} = (2r-3)\left(\frac{3n-5}{2}\right) + \frac{3}{2} > \frac{3n-5}{2}$. By contradiction, $r < 2$, so $r = 1$. Then $2a + b = n$, so $3a = \frac{3}{2}(n-b)$ and

$$k = 3n - 4 - 3a = 3n - 4 - \frac{3}{2}(n-b) = 3n - 4 - \frac{3}{2}n + \frac{3}{2}b < \frac{3n-5}{2}.$$

It follows that $\frac{3}{2}n + \frac{3}{2}b - 4 < \frac{3}{2}n - \frac{5}{2}$, which implies that $\frac{3}{2}b < \frac{3}{2}$, i.e., $b < 1$. Thus, $b = 0$, so $n = 2a + b = 2a$. However, n is odd. By contradiction, $\frac{3n-5}{2} = \min\{k > 0 : kn \in \langle 3n-8, 3n-4\rangle\}$. Therefore, by Lemma 1.7, $u = \left(\frac{3n-5}{2}\right)n$ is a Betti element of S.

Now suppose $k(3n-8) = an + b(3n-4)$, where $0 < k < \frac{n+1}{2}$, $0 \leq a$ and $0 \leq b$. Clearly, then, $b < \frac{n+1}{2}$. Reducing modulo n, we have $4b \equiv 8k \bmod n$ and, since n is odd, $b \equiv 2k \bmod n$. Therefore, $b = 2k + rn$ for some $r \in \mathbb{Z}$. Since $0 \leq b < \frac{n+1}{2}$ and $0 < k < \frac{n+1}{2}$, it follows that $r = 0$ or $r = -1$. Furthermore, since $k < \frac{n+1}{2}$ and n is odd, $k \leq \frac{n-1}{2}$. Suppose $r = -1$. Then $0 \leq b = 2k - n \leq n - 1 - n = -1$. By contradiction, $r = 0$, i.e., $b = 2k$. Then $an + b(3n-4) = an + k(6n-8) > an + k(3n-8) \geq k(3n-8)$. Again, by contradiction, $\frac{n+1}{2} = \min\{k > 0 : k(3n-8) \in \langle n, 3n-4\rangle\}$, so by Lemma 1.7, $v = \left(\frac{n+1}{2}\right)(3n-8)$ is a Betti element of S.

Finally, suppose $3n - 4 = an + b(3n-8)$, where $0 \leq a$ and $0 \leq b$. Reducing modulo n, we have $-4 \equiv -8b \bmod n$ or $8b - 4 \equiv 0 \bmod n$ and, since n is odd, $2b - 1 \equiv 0 \bmod n$. Therefore, $2b - 1 = rn$ for some $r \in \mathbb{Z}$. Suppose $b = 0$. Then $-1 = rn$. But $n \geq 5$ and $r \in \mathbb{Z}$, so by contradiction, $b > 0$. Thus $rn = 2b - 1 > 0$, implying that $r \geq 1$. It follows that $2b - 1 = rn \geq n$, so $b \geq \frac{n+1}{2}$. Consequently,

$$\begin{aligned} an + b(3n-8) \geq \frac{n+1}{2}(3n-8) > \frac{n(3n-8)}{2} & \\ > 2(3n-8) & \\ = 6n - 16 & \\ \geq 3n + 15 - 16 = 3n - 1 > 3n - 4. & \end{aligned}$$

By contradiction, $3n-4 \neq an+b(3n-8)$. Therefore, $2 = \min\{k > 0 : k(3n-4) \in \langle n, 3n-8\rangle\}$, so by Lemma 1.7, $w = 2(3n-4)$ is a Betti element of S.

Finally, by Lemma 1.8,

$$c(u) = \max\left\{\frac{3n-5}{2}, \frac{n-1}{2} + 1\right\} = \frac{3n-5}{2},$$

$$c(v) = \max\left\{\frac{n+1}{2}, \frac{3n-11}{2} + 1\right\} = \frac{3n-9}{2}$$

and

$$c(w) = \max\{2, 3+1\} = 4,$$

as desired. $\square$

Theorem 2.3. *Let n be an odd integer greater than or equal to 5 and let $S = \langle n, 3n-8, 3n-4\rangle$. Then*

$$\left\{0, 4, n-2, n-1, n, \ldots, \frac{3n-11}{2}, \frac{3n-9}{2}, \frac{3n-5}{2}\right\} \subset C(S).$$

Proof. Since $c(0) = 0$, $0 \in C(S)$. Furthermore, by Lemma 2.2,

$$\left\{4, \frac{3n-9}{2}, \frac{3n-5}{2}\right\} \subset C(S).$$

Therefore, it remains to show that, if $n \geq 7$, then

$$\left\{n-2, n-1, n, \ldots, \frac{3n-11}{2}\right\} \subset C(S).$$

If $n \geq 7$, let $s_k = \left(\frac{n+1}{2} + k\right)(3n-8)$, for $1 \leq k \leq \frac{n-5}{2}$. For a given k, we claim that s_k has exactly $k+2$ distinct factorizations:

$$\begin{aligned} z_0 &= \left(0, \frac{n+1}{2} + k, 0\right) \text{ and} \\ z_i &= \left(\frac{3n-5}{2} - 3i, k-i+1, 2i-1\right), \text{ for } 1 \leq i \leq k+1. \end{aligned}$$

We have

$$\begin{aligned} s_1 &= \left(\frac{n+3}{2}\right)(3n-8) \\ &= \left(\frac{3n-11}{2}\right)n + (3n-8) + (3n-4) \\ &= \left(\frac{3n-17}{2}\right)n + 3(3n-4). \end{aligned}$$

Thus, s_1 has at least three distinct factorizations:

$$z_0 = \left(0, \frac{n+3}{2}, 0\right), z_1 = \left(\frac{3n-11}{2}, 1, 1\right) \text{ and } z_2 = \left(\frac{3n-17}{2}, 0, 3\right).$$

Suppose there exists another factorization $z = (p, q, r)$ of s_1. Since $\frac{3n-17}{2} < 3n-4$, $3 < n$ and $\gcd(n, 3n-4) = 1$, it follows that z_2 is the only factorization of s_1 whose second entry is zero. Therefore, if $q = 0$, then $z = z_2$. Otherwise, $q \neq 0$ and, consequently, $z' = (p, q-1, r)$ is a factorization of $v = \left(\frac{n+1}{2}\right)(3n-8)$, which is a Betti element of S. Then, $z' = \left(0, \frac{n+1}{2}, 0\right)$ or $z' = \left(\frac{3n-11}{2}, 0, 1\right)$, which implies that $z = z_0$ or $z = z_1$. Therefore, z_0, z_1 and z_2 are the three unique distinct factorizations of s_1.

Now assume that s_k has exactly $k+2$ distinct factorizations of the form given above, for some k such that $1 \leq k \leq \frac{n-7}{2}$. The element s_{k+1} has at least

$k+3$ distinct factorizations:

$$z_0 = \left(0, \frac{n+1}{2} + (k+1), 0\right) \text{ and}$$
$$z_i = \left(\frac{3n-5}{2} - 3i, (k+1) - i + 1, 2i - 1\right), \text{ for } 1 \le i \le k+2.$$

Again, suppose there exists another factorization $z = (p, q, r)$ of s_{k+1}. Since $\frac{3n-5}{2} - 3(k+2) < \frac{3n-5}{2} - 3 = \frac{3n-11}{2} < 3n-4$, $2(k+2) - 1 = 2k+3 \le n-7+3 = n-4 < n$ and $\gcd(n, 3n-4) = 1$, it follows that z_{k+2} is the only factorization of s_{k+1} whose second entry is zero. Therefore, if $q = 0$, then $z = z_{k+2}$. Otherwise, $q \ne 0$ and, consequently, $z' = (p, q-1, r)$ is a factorization of s_k. By the induction hypothesis, it follows that $z' = \left(0, \frac{n+1}{2} + k, 0\right)$ or $z' = \left(\frac{3n-5}{2} - 3j, k-j+1, 2j-1\right)$ for some j such that $1 \le j \le k+1$. But that implies that $z = z_0$ or $z = z_i$ for some i such that $1 \le i \le k+1$. Therefore, $z_0, z_1, \ldots, z_{k+2}$ are the $k+3$ unique distinct factorizations of s_{k+1}.

For $1 \le i \le k$, $d(z_i, z_{i+1}) = \max\{3+1, 2\} = 4$. For $1 \le i \le k+1$, $d(z_0, z_i) = \max\left\{\frac{n+1}{2} + k - (k-i+1), \frac{3n-5}{2} - 3i + 2i - 1\right\} = \max\{\frac{n-1}{2} + i, \frac{3n-7}{2} - i\}$. Suppose $\frac{n-1}{2} + i > \frac{3n-7}{2} - i$. Then $2i > \frac{3n-7-n+1}{2} = \frac{2n-6}{2} = n-3$, implying that $i > \frac{n-3}{2}$. However, $i \le k+1 \le \frac{n-5}{2} + 1 = \frac{n-3}{2}$. By contradiction, $\frac{n-1}{2} + i \le \frac{3n-7}{2} - i$, so $d(z_0, z_i) = \frac{3n-7}{2} - i$. Therefore, $d(z_0, z_i) < d(z_0, z_j)$ whenever $1 \le j < i \le k+1$. It follows that if $N < d(z_0, z_{k+1})$, then there does not exist an N-chain from z_0 to any other factorization of s_k. Therefore, $c(s_k) \ge d(z_0, z_{k+1}) = \frac{3n-7}{2} - (k+1) = \frac{3n-9}{2} - k$. However, for $1 \le i \le k$, $d(z_i, z_{i+1}) = 4 < n-2 = \frac{2n-4}{2} = \frac{3n-9}{2} - \frac{n-5}{2} \le \frac{3n-9}{2} - k$. Thus, given any two factorizations z_i, z_j of s_k, with $i < j$, if $N = \frac{3n-9}{2} - k$, then there exists an N-chain from z_i to z_j, namely, $z_i, z_{i+1}, \ldots, z_j$ if $i \ne 0$, otherwise, $z_j, z_{j+1}, \ldots, z_{k+1}, z_0 = z_i$. Therefore, $c(s_k) \le \frac{3n-9}{2} - k$ and, thus, $c(s_k) = \frac{3n-9}{2} - k$. Consequently, for $1 \le k \le \frac{n-5}{2}$, $c(s_k) = \frac{3n-9}{2} - k \in C(S)$, i.e.,

$$\left\{n-2, n-1, n, \ldots, \frac{3n-11}{2}\right\} \subset C(S),$$

which concludes the proof. □

Corollary 2.4. *There exist numerical monoids with arbitrarily large catenary sets.*

3 The Minimum Catenary Degree

Lemma 3.1. *Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid. Suppose $s \in S$ and $|Z(s)| \ge 2$. Let B be the set of Betti elements of S that divide s. Then, given $(f_1, \ldots, f_k) = f \in Z(s)$, there exists $g \in Z(s)$ such that*

$$d(f, g) \ge b = \min\{c(m) : m \in B\}.$$

Proof. First, we will show that there exists $(h_1, \dots, h_k) = h \in Z(m)$ for some $m \in B$ such that $h_i \leq f_i$ for $1 \leq i \leq k$.

Let

$$X = \{(x_1, \dots, x_k) : x_i \leq f_i \text{ for } 1 \leq i \leq k\}$$

and let

$$F = \{x \in X : |Z(\varphi(x))| \geq 2\}.$$

Note that $f \in F$, so that $F \neq \emptyset$. Therefore, choose $h \in F$ such that $|h| \leq |z|$ for all $z \in F$. Let $m = \varphi(h)$. Since $h \in F$, $|Z(m)| = |Z(\varphi(h))| \geq 2$.

Suppose ∇_m is connected. Then there exists a factorization $j \in Z(m)$ such that $\gcd(h, j) \neq 0$. Let $h' = h - \gcd(h, j)$, $j' = j - \gcd(h, j)$ and $m' = \varphi(h')$. Then $h' \in Z(m')$ and $j' \in Z(m')$, so $|Z(m')| \geq 2$. It follows that $h' \in F$. Furthermore,

$$|h'| = |h - \gcd(h, j)| < |h|.$$

However, $h' \in F$, so $|h| \leq |h'|$, implying that ∇_m is not connected. Therefore, m is a Betti element of S. Since $h \in F$, $h_i \leq f_i$ for $1 \leq i \leq k$, i.e., $f - h \in \mathbb{N}_0^k$. Thus, $s - m = \varphi(f) - \varphi(h) = \varphi(f - h) \in S$, i.e., m divides s, so $m \in B$.

Since removing edges of weight $c(m)$ and greater disconnects the factorization graph of m, every factorization of m is connected to another factorization by an edge of weight greater than or equal to $c(m)$. Therefore, there exists $j \in Z(m)$ such that $d(h, j) \geq c(m) \geq b$. Let $g = j + f - h$. Then

$$\varphi(g) = \varphi(j + f - h) = \varphi(j) + \varphi(f) - \varphi(h) = m + s - m = s,$$

so $g \in Z(s)$. Furthermore, $d(f, g) = d(h + f - h, j + f - h) = d(h, j) \geq b$. $\square$

Proposition 3.2. *Let S be a numerical monoid and let $n \in S$. Furthermore, let B be the set of Betti elements of S that divide n and let $b = \min\{c(m) : m \in B\}$. If $f_1, f_2 \in Z(n)$ and $d(f_1, f_2) < b$, then there exists $f_3 \in Z(n)$ such that* $\max\{|f_1|, |f_2|\} < |f_3|$.

Proof. Suppose $f_1, f_2 \in Z(n)$ for some $n \in S$ and $d(f_1, f_2) < b$. Let

$$f_i' = f_i - \gcd(f_1, f_2)$$

for $i = 1, 2$ and let $n' = \varphi(f_1')$. Furthermore, let B' be the set of Betti elements of S that divide n' and let $b' = \min\{c(m) : m \in B'\}$. By Lemma 3.1, there exists $f_3' \in Z(n')$ such that $d(f_1', f_3') \geq b'$. Suppose $m' \in B'$. Then $n' - m' \in S$ and, since

$$n - n' = \varphi(f_1) - \varphi(f_1') = \varphi(f_1 - f_1') = \varphi(\gcd(f_1, f_2)) \in S,$$

we have $n - m' = n - n' + n' - m' \in S$. Therefore, $m' \in B$, i.e., $B' \subset B$. It follows that $\{c(m) : m \in B'\} \subset \{c(m) : m \in B\}$ and, thus, that

$$b' = \min\{c(m) : m \in B'\} \geq \min\{c(m) : m \in B\} = b.$$

Consequently, $d(f_1', f_3') \geq b' \geq b$.

Since $\gcd(f_1', f_2') = 0$, then

$$b > d(f_1, f_2) = d(f_1', f_2') = \max\{|f_1'|, |f_2'|\}.$$

Furthermore,

$$b \le d(f_1', f_3') = \max\left\{\left|\frac{f_1'}{\gcd(f_1', f_3')}\right|, \left|\frac{f_3'}{\gcd(f_1', f_3')}\right|\right\}.$$

But $|f_1'| \le \max\{|f_1'|, |f_2'|\} < b$, so

$$\left|\frac{f_1'}{\gcd(f_1', f_3')}\right| \le |f_1'| < b.$$

Therefore,

$$b \le d(f_1', f_3') = \left|\frac{f_3'}{\gcd(f_1', f_3')}\right| \le |f_3'|.$$

Consequently,

$$\max\{|f_1'|, |f_2'|\} < b \le |f_3'|.$$

Let $f_3 = f_3' + \gcd(f_1, f_2)$. Since

$$\varphi(\gcd(f_1, f_2)) = \varphi(f_1 - f_1') = \varphi(f_1) - \varphi(f_1') = n - n',$$

then

$$\varphi(f_3) = \varphi(f_3' + \gcd(f_1, f_2)) = \varphi(f_3') + \varphi(\gcd(f_1, f_2)) = n' + n - n' = n.$$

Therefore, $f_3 \in Z(n)$. Furthermore,

$$\begin{aligned} |f_i| &= |f_i' + \gcd(f_1, f_2)| \\ &= |f_i'| + |\gcd(f_1, f_2)| \\ &\le \max\{|f_1'|, |f_2'|\} + |\gcd(f_1, f_2)| \\ &< |f_3'| + |\gcd(f_1, f_2)| \\ &= |f_3' + \gcd(f_1, f_2)| = |f_3| \end{aligned}$$

for $i = 1, 2$. Therefore, $\max\{|f_1|, |f_2|\} < |f_3|$. □

Theorem 3.3. *Let S be a numerical monoid and let $s \in S$ such that $|Z(s)| \ge 2$. Furthermore, let B be the set of Betti elements of S that divide s and let $b = \min\{c(m) : m \in B\}$. Then $c(s) \ge b$.*

Proof. Let $V \subset Z(s)$ denote the set of factorizations v of s for which there exists $v' \in Z(s)$ with $d(v, v') < b$. Suppose $V = \emptyset$. Then $d(z, z') \ge b$ for all $z, z' \in Z(s)$, and it follows that $c(s) \ge b$. Otherwise, $V \ne \emptyset$, so choose $w \in V$ such that $|w| \ge |v|$ for all $v \in V$. Since $w \in V$, there exists $w' \in Z(s)$ such that $d(w, w') < b$. By Proposition 3.2, there exists $w'' \in Z(s)$ such that $\max\{|w|, |w'|\} < |w''|$. Therefore, $|w| \le \max\{|w|, |w'|\} < |w''|$. Since $|w| \ge |v|$ for all $v \in V$, it follows that $w'' \notin V$. Consequently, $d(w'', z) \ge b$ for all $z \in Z(s)$. Thus, $c(s) \ge b$. □

Corollary 3.4. *Let S be a numerical monoid and let $s \in S$ such that $c(s) > 0$. Furthermore, let B be the set of Betti elements of S and let $b = \min\{c(m) : m \in B\}$. Then $c(s) \geq b$.*

Corollary 3.5. *Let S be a numerical monoid and let B be the set of Betti elements of S. Then $C(S) = \{0, c\}$ if and only if $c(m) = c$ for all $m \in B$.*

4 The Single Betti Element Case

We will now investigate the case when our numerical semigroups have a single Betti element, in order words, when $c_1n_1 = c_2n_2 = c_3n_3$. When this is the case, our numerical semigroup is more "well behaved" thus giving it properties not always found in other cases. We now take a look at some of these properties.

First, by applying an ordering to our generators, we quickly see that we have an ordering on our c_i's as well.

Lemma 4.1. *Let $S = \langle n_1, n_2, n_3 \rangle$ be a numerical semigroup of embedding dimension three where $n_1 < n_2 < n_3$ such that S has a single Betti element. Then $c_1 > c_2 > c_3 > 1$.*

Proof. Since S has a single Betti element, then $c_1n_1 = c_2n_2 = c_3n_3$. First consider $c_1n_1 = c_2n_2$. Since $n_1 < n_2$ then $c_2n_2 = c_1n_1 < c_1n_2 \implies c_2n_2 < c_1n_2 \implies c_1 > c_2$. Similarly, $c_2 > c_3$. Also, $c_3 > 1$ because by definition $c_3 > 0$ and if $c_3 = 1$ that would imply n_3 can be decomposed into n_1 and n_2 atoms which would make n_3 no longer an atom, a contradiction. Thus $c_1 > c_2 > c_3 > 1$. □

Now, we take a closer look at the structure of the single Betti element. We will show that if our semigroup does have exactly one Betti element, then that Betti element has exactly 3 factorizations which we can explicitly state.

Lemma 4.2. *Let S be a numerical semigroup of embedding dimension three with a single Betti element. Then the Betti element has only three factorizations (i.e. $(c_1, 0, 0), (0, c_2, 0), (0, 0, c_3)$).*

Proof. First we show that c_1n_1, c_2n_2, c_3n_3 are the only factorizations of the single Betti element. So suppose,

$$c_1n_1 = a_2n_2 + a_3n_3.$$

where $a_2, a_3 > 0$. From this, we see that $a_2 < c_2$ and $a_3 < c_3$ because if we suppose $a_2 \geq c_2$, then $a_2n_2 \geq c_2n_2 = c_1n_1$ which would force a_3n_3 to be non-positive which forces a_3 to be non-positive, a contradiction (if $a_3 = 0$, then that would force $a_2 = c_2$ which gives us $c_1n_1 = c_2n_2$ which isn't a new factorization). Now, since $c_1n_1 = c_2n_2$ we can write

$$c_2n_2 = a_2n_2 + a_3n_3$$

which simplifies to

$$(c_2 - a_2)n_2 = a_3 n_3.$$

Since $c_2 - a_2$ and a_3 are positive, we have a contradiction of the minimality condition of c_2. Thus, we can't factor the Betti element in terms of two of the generators. Also, the Betti element can't be factored into three generators or else, that factorization would have an edge to every other factorization in the Betti element's ∇ graph, thus making it connected, which is a contradiction by the definition of a Betti element. Thus, $(c_1, 0, 0), (0, c_2, 0), (0, 0, c_3)$ are the only factorizations of the Betti element. □

Given that we know what the factorizations of our Betti element is now, we can now find out what our minimal presentation is now.

Lemma 4.3. *Let S be a numerical semigroup of embedding dimension three with a single Betti element. Then $\{((c_1, 0, 0), (0, c_2, 0)), ((c_1, 0, 0), (0, 0, c_3))\}$ is a minimal presentation for S.*

Proof. From lemma 4.2 we know that the Betti element has three factorizations $(c_1, 0, 0), (0, c_2, 0), (0, 0, c_3)$. Taking the pairwise dot product of each of these gives us zero so the ∇ graph of our Betti element consists of three vertices and no edges. Clearly, one edge is not enough to connect the graph. However, out of the three possible edges that can be drawn, choosing any two of them will connect the ∇ graph. Choose the edge between $(c_1, 0, 0)$ and $(0, c_2, 0)$ and also choose the edge between $(c_1, 0, 0)$ and $(0, 0, c_3)$. These connect the ∇ graph and this is the only Betti element so $\{((c_1, 0, 0), (0, c_2, 0)), ((c_1, 0, 0), (0, 0, c_3))\}$ is a minimal presentation. □

Also with our factorizations we can easily figure out the catenary degree of our single Betti element.

Lemma 4.4. *Let S be a numerical semigroup of embedding dimension three with a single Betti element b. Then $c(b) = c_1$.*

Proof. From lemma 4.2 we have that our Betti element has three factorizations $(c_1, 0, 0)$, $(0, c_2, 0)$, and $(0, 0, c_3)$. We now consider the distances between these factorizations. We will use the inequalities found in Lemma 4.1.

$$d((c_1, 0, 0), (0, c_2, 0)) = \max\{|(c_1, 0, 0)|, |(0, c_2, 0)|\} = \max\{c_1, c_2\} = c_1$$

$$d((c_1, 0, 0), (0, 0, c_3)) = \max\{|(c_1, 0, 0)|, |(0, 0, c_3)|\} = \max\{c_1, c_3\} = c_1$$

$$d((0, c_2, 0), (0, 0, c_3)) = \max\{|(0, c_2, 0)|, |(0, 0, c_3)|\} = \max\{c_2, c_3\} = c_2$$

We now apply the algorithm for computing the catenary degree. Since $c_1 > c_2$, we remove the two edges with weight c_1 which disconnects the graph since a single edge can not connect a graph of three vertices. Thus, $c(b) = c_1$. □

One might recall Corollary 3.5 which states that our catenary set has a single non-zero element when the catenary degree of all the Betti elements of a numerical semigroup are equal. Well, this is trivially true when we have a single Betti element, so we have the following as a result of that corollary and the previous lemma.

Corollary 4.5. *Let S be a numerical semigroup of embedding dimension three with a single Betti element. Then $C(S) = \{0, c_1\}$.*

Proof. Since S has a single Betti element, it is trivial that all the Betti elements have the same catenary degree, namely c_1 according to Lemma 4.4. Thus, by Corollary 3.5, $C(S) = \{0, c_1\}$. □

We introduce a lemma that gives us a quick bound on the weight of any swap.

Lemma 4.6. *Let (r, s, t) be a swap. Then $w(r, s, t) \geq \max\{|r|, |s|, |t|\}$.*

Proof. Without loss of generality, let $r, s \leq 0$ and $t \geq 0$. Then $w(r, s, t) = \max\{|r + s|, |t|\} = \max\{|r| + |s|, |t|\} \geq \max\{|r|, |s|, |t|\}$. □

An interesting question to ask is whether or not there exists edge weights greater than the minimum nonzero catenary degree and less than the maximum catenary degree that are not in the catenary set of a numerical semigroup and if such edges do exist, what form do they have? In the case of the single Betti element case, if such edges exist, they have a very predictable form as seen in the next theorem.

Theorem 4.7. *Let S be a numerical semigroup of embedding dimension three with a single Betti element. Then all edge weights between factorizations of elements in S less than $c(S)$ have the form kc_2 for $k \in \mathbb{Z}^+$.*

Proof. Recall that all edge weights are of the form $\{w(x\alpha + y\beta) | x, y \in \mathbb{Z}\}$ where $\alpha \neq \beta$ are two vector differences in the minimal presentation (i.e. a fundamental swap). From Lemma 4.3, the minimal presentation of S is

$$\{((c_1, 0, 0), (0, c_2, 0)), ((c_1, 0, 0), (0, 0, c_3))\}.$$

Thus, all edge weights are of the form $\{w(x(c_1, -c_2, 0) + y(c_1, 0, -c_3)) | x, y \in \mathbb{Z}\}$. Making use of Lemma 4.6, we can simplify: $w(x(c_1, -c_2, 0) + y(c_1, 0, -c_3)) = w(c_1(x + y), -xc_2, -yc_3) \geq |c_1(x + y)|$. First consider the case when $x \neq -y$. Then $x + y \neq 0$ so $|x + y| \geq 1$. Thus, $|c_1(x + y)| = |c_1||x + y| \geq |c_1| = c_1 = c(S)$ (the last equality is a result of Lemma 4.5). Thus, if $x \neq -y$, then we get edge weights greater than or equal to $C(S)$. We want the edge weights less than $C(S)$ so this can only occur when $x = -y$. So considering the case when $x = -y$, we have $w(x(c_1, -c_2, 0) + y(c_1, 0, -c_3)) = w(x(c_1, -c_2, 0) - x(c_1, 0, -c_3)) = w(0, -xc_2, xc_3) = |x|w(0, -c_2, c_3) = |x| \max\{c_2, c_3\}$. By Lemma 4.1, $c_2 > c_3$ so $|x| \max\{c_2, c_3\} = |x|c_2 = kc_2$ for some $k \in \mathbb{Z}^+$. □

It is interesting to note that one can use the above theorem to easily prove that $C(S) = \{0, c_1\}$ if S is a numerical semigroup of embedding dimension three with a single Betti element. However, since the result immediately follows from a previous corollary, we omit this proof. We leave it as an exercise to the reader to try to prove $C(S) = \{0, c_1\}$ as a result of the above theorem (Hint: One can make use of the fact that the c_i's are pairwise coprime, a result proved later in this section.)

Now we will look at a method of constructing these semigroups that have a single Betti element. It is interesting to note that this construction only generates semigroups of embedding dimension three with a single Betti element and moreover, this construction actually generates all of them.

Theorem 4.8. *The following two statements are equivalent*

1. $c_1 > c_2 > c_3 > 1$
   c_1, c_2, c_3 *are pairwise coprime*
   $S = \langle c_2c_3, c_1c_3, c_1c_2 \rangle$

2. $c_1n_1 = c_2n_2 = c_3n_3$
   $n_1 < n_2 < n_3$
   S *is numerical semigroup of embedding dimension three*
   $c_i = \min\{r > 0 | rn_i \in \langle n_j, n_k \rangle\}$ *for*
   $\{i, j, k\} = \{1, 2, 3\}$

Proof. First, we show $1 \implies 2$.

Clearly, $c_1n_1 = c_1(c_2c_3) = c_2(c_1c_3) = c_2n_2 = c_3(c_1c_2) = c_3n_3$ so $c_1n_1 = c_2c_2 = c_3n_3$.

Since $c_1 > c_2$, then $c_2n_2 = c_1n_1 > c_2n_1$ so simplifying gives us $n_2 > n_1$. Similarly, $n_3 > n_2$ so $n_1 < n_2 < n_3$.

Next, we show the c_i's truly are minimal. Suppose $r_1n_1 = r_2n_2 + r_3n_3$ where $0 < r < c_1$. Substituing, we get $r_1c_2c_3 = r_2c_1c_3 + r_3c_1c_2$. Taking this modulo c_1, we get $r_1c_2c_3 = 0 \mod c_1$. Thus, $c_1 | r_1c_2c_3$. However, $\gcd(c_1, c_2) = \gcd(c_1, c_3) = 1$ so it must be the case that $c_1 | r$ a contradiction since r is less than c_1 and non-zero. Thus, c_1 is minimal and similar argument can be used to show c_2, c_3 are minimal.

Next, we show that S is primitive by showing $\gcd(n_1, n_2, n_3) = 1$. Notice, $\gcd(n_1, n_2, n_3) = \gcd(\gcd(n_1, n_2), n_3) = \gcd(\gcd(c_2c_3, c_1c_3), c_1c_2)$. Since $\gcd(c_1, c_2) = 1$, then $\gcd(c_1c_3, c_2c_3) = c_3$ so $\gcd(c_3, c_1) = 1$ and $\gcd(c_3, c_2) = 1$ imply

$$\gcd(\gcd(c_2c_3, c_1c_3), c_1c_2) = \gcd(c_3, c_1c_2) = 1.$$

Lastly, we show that this numerical semigroup is really of embedding dimension three. Suppose one of the atoms c_i for $i \in \{1, 2, 3\}$ can be decomposed into the other two atoms. But that would imply $c_i = 1$ a contradiction since all the c_i's are greater than 1.

Now we show $2 \implies 1$

First we show c_1, c_2, c_3 are pairwise coprime. Suppose by the contrary that they weren't. Then, WLOG, let $\gcd(c_1, c_2) = d$ where $d > 1$. Then $d | c_1$

and $d|c_2$ so we can write $a_1d = c_1, a_2d = c_2$ for $a_1, a_2, \in \mathbb{Z}$. Also, we know $a_1 < c_1$ and $a_2 < c_2$ (we can't have equality since $d > 1$). Rewriting, we have $a_1 = \frac{c_1}{d}, a_2 = \frac{c_2}{d}$. Now, we divide out an equation by d and get $c_1n_1 = c_2n_2 \implies \frac{c_1}{d}n_1 = \frac{c_2}{d}n_2 \implies a_1n_1 = a_2n_2$. However, $a_1 < c_1$ so this contradicts the minimality of c_1. Thus, c_1, c_2, c_3 are pairwise coprime.

The fact that $c_1 > c_2 > c_3 > 1$ follows from Lemma 4.1.

Lastly we show $n_1 = c_2c_3, n_2 = c_1c_3, n_3 = c_1c_2$. Take the equation $c_1n_1 = c_2n_2$. From that we can get $c_1n_1 = 0 \mod c_2$ which implies $c_2|c_1n_1$. Since $\gcd(c_1, c_2) = 1$, we have $c_2|n_1$. Simarlarly, $c_3|n_1$. And since $\gcd(c_2, c_3) = 1$, we have $n_1 = kc_2c_3, k \in \mathbb{Z}$. Similarly, $n_2 = lc_1c_3$ and $n_3 = mc_1c_2$ for $l, m \in \mathbb{Z}$. Notice that we have to satisfy the condition $c_1n_1 = c_2n_2 = c_3n_3$. Take $c_1n_1 = c_2n_2$. Plugging in n_1, n_2, we have $c_1(kc_2c_3) = c_2(lc_1c_3)$ and after simplifying we get $k = l$. Using this argument again gives us $k = l = m$. We know $k = 1$ or else $\gcd(n_1, n_2, n_3) = \gcd(kc_2c_3, lc_1c_3, mc_1c_2) = \gcd(kc_2c_3, kc_1c_3, kc_1c_2) \geq k$ which is a contradiction since $\gcd(n_1, n_2, n_3)$ needs to be 1 for S to be a numerical semigroup. Thus, $n_1 = c_2c_3, n_2 = c_1c_3, n_3 = c_1c_3$. □

We know that all numerical semigroups of embedding dimension three with a single Betti element have a catenary set of the form $\{0, c\}$. So a natural question to ask is whether or not, for any given c, if there exists a semigroup of embedding dimension three that has catenary set $\{0, c\}$. For c large enough we will show that this is actually the case. Moreover, using the construction technique above, for c large enough, we can actually construct an explicit numerical semigroup with the desired catenary set.

Notice that our construction uses the fact that the c_i's are all pairwise coprime and construction proof proves existence so this motivates the following lemma whose purpose will become more apparent in the next theorem.

Lemma 4.9. *Let $n \geq 7$. Then there exists two primes $x, y < n$ that do not divide n.*

Proof. If n is prime, then choose $x = 2$ and $y = 3$. Since n is prime, clearly $x$and y do not divide it so we are done. So suppose n is not prime. Then the number of primes less than n is the same as the number of primes less than or equal to n which we'll denote as $\pi(n)$. Also, the number of primes less than n that divide n is the same as the number of primes less than or equal to n that divide n which we'll denote as $\omega(n)$. Thus, it suffices to show that $\pi(n) - \omega(n) \geq 2$ for $n \geq 7$.

It can easily be shown that $\omega(n) \leq \log_2 n$. Suppose by the contrary that $\omega(n) > \log_2 n$. Then when we write n as a prime factorization $p_1^{r_1}p_2^{r_2} \cdots p_k^{r_k}$ where $k = \omega(n) > \log_2 n$. Clearly, $n = p_1^{r_1}p_2^{r_2} \cdots p_k^{r_k} \geq p_1p_2 \cdots p_k$. Also, since 2 is the smallest prime, $p_1p_2 \cdots p_k \geq 2^k$. Since $k > \log_2 n$, then $2^k > 2^{log_2 n} = n$, thus $n > n$, a contradiction. (It is interesting to note that if $\Omega(n)$ is the number of primes including multiplicity, then $\log_2(n)$ is the tightest possible bound since it reaches equality for infinitely many n).

According to [6], for $n \geq 55$, $\pi(n) > \frac{n}{\ln(n)+2}$.

Since $n \geq 7$, we can use those two bounds ($\pi(n) > \frac{n}{\ln(n)+2}$ and $\omega(n) \leq \log_2 n$) and write $\pi(n) - \omega(n) > \frac{n}{\ln(n)+2} - \log_2(n)$ for $n \geq 55$. Thus, it is also true for $n \geq 65$. We want to show that $\frac{n}{\ln(n)+2} - \log_2(n) \geq 2$ for $n \geq 65$. First, we claim that $\sqrt{n} \geq \ln n + 2$ for $n \geq 65$. We show this by showing that the derivative of the $\sqrt{n}$ function is greater than that of the $\ln n + 2$ function for $n \geq 65$ and also $\sqrt{65} \geq \ln(65) + 2$ which guarantees that $\sqrt{n} \geq \ln n$ for $n \geq 65$. We set the equality $\frac{d}{dx}(\sqrt{n}) \geq \frac{d}{dx}(\ln(n) + 2) \to \frac{1}{2}n^{\frac{-1}{2}} \geq n^{-1}$ which simplifies to $\sqrt{n} \geq 2$ which is true for $n \geq 4$. Also, one can verify with a calculator that $\sqrt{65} \geq \ln(65) + 2$, thus, we have shown $\sqrt{n} \geq \ln(n) + 2$ for $n \geq 65$. Since n is positive for $n \geq 65$,we have $\sqrt{n} \geq \ln(n) + 2 \to \sqrt{n}/n \geq (\ln(n) + 2)/n \to n/\sqrt{n} \leq n/(\ln(n) + 2) \to \sqrt{n} \leq n/(\ln(n) + 2)$. From this, we can write $\frac{n}{\ln(n)+2} - \log_2(n) \geq \sqrt{n} - \log_2(n)$. We show this is an increasing function for $n \geq 65$ by showing it's derivative is positive for $n \geq 65$. Setting the inequality: $\frac{d}{dx}(\sqrt{n} - \log_2(n)) \geq 0 \to \frac{1}{2}n^{-1/2} - \frac{1}{n\ln(2)} \geq 0$ which simplifies to $n \geq \frac{4}{(\ln(2))^2} \approx 8.3$. Thus, since this is an increasing function for $n \geq \frac{4}{(\ln(2))^2}$ and since $\sqrt{65} - \log_2(65) > 2$, we know $\sqrt{n} - \log_2(n) \geq 2$ for $n \geq 65$. Thus, we know $\pi(n) - \omega(n) \geq 2$ for $n \geq 65$.

We still need to show this is true for $7 \leq n < 65$. Since this is a finite number of cases to consider, one can write a computer program to verify. We used the program Sage which is python-based. First we defined the $\pi(n)$ and $\omega(n)$ functions,

Code 4.10.

```
def Pi(n):
    return len([i for i in (2..n) if (i+0).is_prime()])
def Omega(n):
    return len([i for i in (2..n) if (i+0).is_prime() and n%i==0])
```

Next, we check for which values of $7 \leq n < 65$ have the property that $\pi(n) - \omega(n) < 2$.

Code 4.11.

```
[i for i in range(7,65) if Pi(i)-Omega(i)<2]
```

We expect this code to return an empty list which it does. Thus, this verifies for $7 \leq n < 65$ that $\pi(n) - \omega(n) \geq 2$ and we had shown that was true earlier as well for $n \geq 65$. Thus, $\pi(n) - \omega(n) \geq 2$ for $n \geq 7$, concluding our proof. $\square$

It is interesting to note that if one wants to, they can extend this lemma to show that for any P, for all n sufficiently large, that there exists P smaller primes that don't divide n.

Now that we have the lemma that we need, we can easily prove what we claimed earlier: that for large enough c, there always exists a numerical semi-group group of embedding dimension three such that $C(S) = \{0, c\}$.

Theorem 4.12. *Let $c > 2$. Then there exists a numerical semigroup S in embedding dimension 3 such that $C(S) = \{0, c\}$.*

Proof. For $c < 7$, we can explicity find numerical semigroups with this property (these need to be found and stated later) so suppose $c \geq 7$. Let $c = c_1$ and pick c_2, c_3 such that they are pairwise coprime and $c_1 > c_2 > c_3 > 1$. It suffices to pick c_2, c_3 such that they are primes less than c_1 that do not divide c_1, thus satisfying the inequality and coprime conditions. By Lemma 4.9, such c_2, c_3 always exists. Now let $S = \langle n_1, n_2, n_3 \rangle = \langle c_2c_3, c_1c_3, c_1c_2 \rangle$. Then by Theorem 4.8, S is a numerical semigroup of embedding dimension 3 such that each c_i agrees with the usual definition of c_i, $c_1n_1 = c_2n_2 = c_3n_3$, and $n_1 < n_2 < n_3$. Thus, from Corollary 4.5, we have that $C(S) = \{0, c_1\} = \{0, c\}$. □

5 Minimal Presentations of Subsets

We introduce the definition for a minimal presentation with respect to a subset of the numerical semigroup. We make a slight modification found in [1].

Definition 5.1. *If ρ is a minimal presentation for S', a subset of a numerical semigroup S (that is, ρ is a minimal system of generators of σ as a congruence), then whenever $z \in S'$ and $z\sigma z'$, there exists $z_0, \ldots, z_k \in \mathbb{N}^p$ in such a way that $z = z_0\sigma z_1 \ldots \sigma z_{k-1}\sigma z_k = z'$ and $(z_i, z_{i+1}) = (a_i + u_i, b_i + u_i)$ for some $u_i \in \mathbb{N}^p$ and $(a_i, b_i) \in \rho \cup \rho^{-1}$. Moreover, no proper subset of ρ generates σ as a congruence.*

When it comes to minimal presentations of a subset of a numerical semigroup S, it is clear it is going to be a subset of the minimal presentation of S. However, is there ever a case where the minimal presentation of a subset of S is a *proper* subset of S? The answer is yes for particular subsets.

Proposition 5.2. *Let S be a numerical semigroup and let $S_{\mathfrak{B}} \subseteq S$ such that $\mathfrak{B}$ is a subset of the Betti elements of S and $S_{\mathfrak{B}} = \{s \in S | \forall b \in \mathfrak{B} : s - b \notin S\}$. Let $p = \cup_{n \in S} p_n$ be a minimal presentation of S where the p_n's are constructed as they usually are in [1]. Then, $p_{\mathfrak{B}} = p \setminus \cup_{n \in \mathfrak{B}} p_n$ is a minimal presentation for $S_{\mathfrak{B}}$.*

Proof. Suppose by the contrary that for $(a_j, b_j) \in p_n$ for $n \in \mathfrak{B}$, that $(a_j, b_j) \in p_{\mathfrak{B}}$. Then there exists an element $s \in S_{\mathfrak{B}}$ such that there exists two factorizations f_a and f_b such that there exists a path $f_1 \to f_2 \cdots \to f_{k-1} \to f_k$ where $a = 1, b = k$, and each $(f_i, f_{i+1}) = (a_i + u_i, b_i + u_i)$ for some $u_i \in \mathbb{N}^p$ and $(a_i, b_i) \in p_{\mathfrak{B}} \cup p_{\mathfrak{B}}^{-1}$ for $i = 1, 2, \ldots, k-1$ and for some $j \in \{1, 2, \ldots, k-1\}$, $(f_j, f_{j+1}) = (a_j + u_j, b_j + u_j)$ for some $u_j \in \mathbb{N}^p$. If such an s didn't exist, then $p_{\mathfrak{B}}$ could be generated without p_n which would make $p_{\mathfrak{B}}$ no longer minimal, a contradiction. We can write out $f_j = a_j + u_j$ and $f_{j+1} = b_j + u_j$ where a_j, b_j are factorizations of n. We see that $u_j = f_j - a_j$. Since u_j is composed of non-negative components, then $\phi(f_j) - \phi(a_j) \in S \implies s - n \in S$. However, $n \in \mathfrak{B}$ and s has the property that $\forall n \in \mathfrak{B}, s - n \notin S$, a contradiction. Thus, $p_{\mathfrak{B}} = p \setminus \cup_{n \in \mathfrak{B}} p_n$ is a presentation for $S_{\mathfrak{B}}$. □

From this point on, whenever $S_{\mathfrak{B}}$ is used, it will be as defined in the previous proposition (Proposition 5.2) where $\mathfrak{B}$ is some subset of Betti elements of S. Also, whenever $\rho_{\mathfrak{B}}$ is used, it is assumed to be the minimal presentation for $S_{\mathfrak{B}}$.

We now see that minimal presentation of a subset of S can depend on the Betti elements that divide the elements in that subset. Evidence from data shows that the catenary degree of an element is bounded above by the catenary degrees of the Betti elements that divide it. Since the catenary degree is heavily influenced by the minimal presentation and since the minimal presentation is dependent on the Betti elements that divide elements of the subset in question, this provides us with enough motivation to continue with this "sub-minimal presentation" idea which will eventually evolve throughout this section into a result that agrees with the generated data.

Now we show that the highest catenary degree attained in $S_{\mathfrak{B}}$ is bounded by $S_{\mathfrak{B}}$'s minimal presentation. The proof for this lemma is nearly identical to the result in [1].

Lemma 5.3. *Let* $|\rho_{\mathfrak{B}}| = \max\{|a||(a,b) \in \rho_{\mathfrak{B}} \cup \rho_{\mathfrak{B}}^{-1}$ *for some* $b \in \mathbb{N}^p\}$*. Then* $c(S_{\mathfrak{B}}) \leq |\rho_{\mathfrak{B}}|$*.*

Proof. Since $\rho_{\mathfrak{B}}$ is a minimal presentation for $S_{\mathfrak{B}}$ then whenever $z \in S_{\mathfrak{B}}$ and $z\sigma z'$ then there exists $z_0, \ldots, z_k \in \mathbb{N}^p$ in such a way that $z = z_0\sigma z_1 \sigma \ldots \sigma z_{k-1}\sigma z_k$ and $(z_i, z_{i+1}) = (a_i + u_i, b_i + u_i)$ for some $u_i \in \mathbb{N}^p$ and $(a_i, b_i) \in \rho_{\mathfrak{B}} \cup \rho_{\mathfrak{B}}^{-1}$. Notice that if $(a,b) \in \rho_{\mathfrak{B}} \cup \rho_{\mathfrak{B}}^{-1}$ then $\gcd(a,b) = 0$. Thus, $(a,b) = \max\{|a|,|b|\}$. Observe that $d(a+u, b+u) = d(a,b)$ so in the above chain, the distance between adjacent elements is bounded by $\max\{|a||(a,b) \in \rho_{\mathfrak{B}} \cup \rho_{\mathfrak{B}}^{-1}$ for some $b \in \mathbb{N}^p\}$. □

We now introduce $\mu_{\mathfrak{B}}(n)$ which is a function that takes Betti elements and looks at the edges that connect disconnected components. Note that this function is dependent on $\mathfrak{B}$.

Definition 5.4. *First, we define* $\mu_{\mathfrak{B}}(n)$*. Let* $n \in S_{\mathfrak{B}}$ *be such that* G_n *is not connected and let* $\mathfrak{R}_1^n, \ldots, \mathfrak{R}_{k_n}^n$ *be its different* $\mathfrak{R}$*-classes. We set* $\mu_{\mathfrak{B}}(n) = \max\{r_1^n, \ldots, r_{k_n}^n\}$ *where* $r_i^n = \min\{|z| : z \in \mathfrak{R}_i^n\}$*. We define* $\mu_{\mathfrak{B}}(S_{\mathfrak{B}}) = \max\{\mu_{\mathfrak{B}}(n) | n \in S_{\mathfrak{B}}$ *and* G_n *not connected* $\}$*.*

So in colloquial terms, we look at the shortest factorization of each disconnected component and then take the max over that. It is interesting to note that if you take the factorization graph of an element and "squish" the factorizations based on their $\mathfrak{R}$-class, and keep all the edges, then the catenary degree of this new multigraph is precisely $\mu_{\mathfrak{B}}(n)$.

We now prove a result that is similar to the result in [1] and follows nearly a similar argument being careful of the fact that we are using minimal presentations of $S_{\mathfrak{B}}$ for some $\mathfrak{B}$.

Lemma 5.5. *Let* $S_{\mathfrak{B}}$ *be a subset of a numerical semigroup* S *constructed in the usual manner. Then* $c(S_{\mathfrak{B}}) = \mu_{\mathfrak{B}}(S_{\mathfrak{B}})$*.*

Proof. Construct ρ in the following way. For every $n \in S_{\mathfrak{B}}$ such that G_n is not connected, choose $(z_1, \ldots, z_{k_n}^n) \in \mathfrak{R}_1^n \times \cdots \times \mathfrak{R}_{k_n}^n$ such that $|z_i^n| = r_i^n$ for

$i \in \{1, \dots, k_n\}$. Take $\rho_n = \{(z_1^n, z_2^n), (z_1^n, z_3^n), \dots, (z_1^n, z_{k_n}^n)\}$. If G_n is connected, then set $\rho_n = \{\}$. Then, from an earlier lemma, we see that $\rho = \cup_{n \in S_{\mathfrak{B}}} \rho_n$ is a minimal presentation for S. In view of Lemma 5.3, we deduce $c(S_{\mathfrak{B}}) \leq \mu_{\mathfrak{B}}(S_{\mathfrak{B}})$.

Let $n \in S_{\mathfrak{B}}$ be such that $\mu_{\mathfrak{B}}(S_{\mathfrak{B}}) = \mu_{\mathfrak{B}}(n)$ and assume without loss of generality that $\mu_{\mathfrak{B}}(n) = |z_1^n|$. If $c(S_{\mathfrak{B}}) < \mu_{\mathfrak{B}}(S_{\mathfrak{B}})$, then $c(n) < |z_1^n|$, or in other words, factorizations of n can be joined by c-chains for some $c < |z_1^n|$. Let $z = z_1^n$ and $z' = z_2^n$. Since z and z' are different factorizations of n, there must be a chain $z_1, \dots, z_k$ of factorizations of n with $z_1 = z, z_k = z'$ and $d(z_i, z_{i+1}) \leq c$. As z and z' are in different $\mathfrak{R}$ classes, there exists $i \in \{1, \dots, k\}$ such that $z = z_1, \dots, z_i \in \mathfrak{R}_1^n$ and $z_{i+1} \notin \mathfrak{R}_1^n$. From the definition of $\mathfrak{R}$-class, this in particular implies that $\operatorname{supp}(z_i) \cap \operatorname{supp}(z_{i+1})$ is empty. Hence, $d(z_i, z_{i+1}) = \max\{|z_i|, |z_{i+1}|\}$. As $z_i \in \mathfrak{R}_i^n$ and $|z_i^n| = r_1^n = \min\{|z| : z \in \mathfrak{R}_1^n\}$, we get that $|z_1^n| \leq |z_i|$. But then we obtain $|z_1^n| \leq \max\{|z_i|, |z_{i+1}|\} = d(z_i, z_{i+1}) \leq c$, contradicting that $c < |z_1^n|$. □

Finally, we show the main result of this section: that the catenary degree of an element is bounded above by the catenary degrees of the Betti elements that divide it.

Theorem 5.6. *Let $S_{\mathfrak{B}}$ be a subset of a numerical semigroup S constructed in the usual manner and let $s \in S_{\mathfrak{B}}$. Then, $c(s) \leq \max\{c(b) | b$ is a Betti element in $S_{\mathfrak{B}}\}$.*

Proof. Let $b \in S_{\mathfrak{B}}$ such that $\mu_{\mathfrak{B}}(S_{\mathfrak{B}}) = \mu(b)$. Recall that by definition of μ, b must be a Betti element. Suppose that $c(b) < \mu(b)$. It was shown in Lemma 5.5 that this leads to a contradiction. Thus, $c(b) \geq \mu(b)$. So, by Lemma 5.5, $c(S_{\mathfrak{B}}) = \mu_{\mathfrak{B}}(S_{\mathfrak{B}}) = \mu(b) \leq c(b)$. However, $b \in S_{\mathfrak{B}}$ so it is not possible for $c(S_{\mathfrak{B}}) < c(b)$, thus it must be the case that $c(S_{\mathfrak{B}}) = c(b)$. For any other Betti element $b' \in S_{\mathfrak{B}}$, $c(b') \leq c(b)$ otherwise if $c(b') > c(b)$ then $c(S_{\mathfrak{B}}) = c(b) < c(b')$ a contradiction. Thus, $c(S_{\mathfrak{B}}) = \max\{b | b \text{ is a Betti element in } S_{\mathfrak{B}}\}$. For any element $s \in S_{\mathfrak{B}}$, $c(s) \leq c(S_{\mathfrak{B}})$ thus

$$c(s) \leq c(S_{\mathfrak{B}}) = \max\{b | b \text{ is a Betti element in } S_{\mathfrak{B}}\},$$

as desired. □

Moreover, by picking $\mathfrak{B} = \emptyset$, we prove the known result that the highest catenary degree of a numerical semigroup is always attained at a Betti element.

Corollary 5.7. *Let S be a numerical semigroup and let $s \in S$. Then $c(s) \leq \max\{c(b) | b$ is a Betti element in $S\}$*

Proof. Let $\mathfrak{B} = \emptyset$. Then $S_{\mathfrak{B}} = \{s \in S | \forall b \in \mathfrak{B} : s - b \notin S\} = \{s \in S | \forall b \in \emptyset : s - b \notin S\} = \{s \in S\} = S$. Since S can be written as $S_{\mathfrak{B}}$ where $\mathfrak{B}$ is a subset of the Betti elements in S, then $c(s) \leq \max\{b | b \text{ is a Betti element in } S\}$ follows from Theorem 5.6. □

6 Thriftiness and Catenary Inequalities

This section needs some work. It was originally assumed that

$$c(c_in_i) = \max\{c_i, r_{ij} + r_{ik}\}$$

but that isn't always the case. Thus, some of the proofs involving thriftiness are no longer correct as they use an old definition of thriftiness that was defined using the catenary degree (which was assumed to be that maximum). Any comments on this issue will appear in red text like this in addition to proofs that need to be repaired.

We will now look into the idea of thrifiness. One of biggest problems in computing catenary degrees in a abstract sense is that sometimes we can do a swap with respect to one generator whose weight is less than the weight of the "cheapest" fundamental swap with respect to that same generator. In other words, even if we choose our minimal presentation carefully, the fundamental swaps that arise from that aren't necessarily the cheapest swaps. The reason that these cheaper or "thrifty" swaps become as issue is because they allow one to construct "shortcuts" in the paths between factorizations connected by fundamental swaps.

Since issues arise from thrify swaps, can we gain any immediate results if we assume the semigroups we are looking at are not thrifty? The answer is yes. Although many of the results in this section use the thriftiness hypothesis in their respective proofs, we believe many of these results still hold regardless of thriftiness so showing that is a possibility for future work.

We begin with a concise definition of thriftiness.

Definition 6.1. *Let $S = \langle n_1, n_2, n_3\rangle$ be a numerical semigroup of embedding dimension three and let $\{i, j, k\} = \{1, 2, 3\}$. Consider the element $c_in_i = r_{ij}n_j + r_{ik}n_k$. There may be many choices for the r's so choose r_{ij} and r_{ik} such that $r_{ij} + r_{ik}$ is minimized. Denote $w_i = \max\{c_i, r_{ij} + r_{ik}\}$, i.e., let w_i be the cheapest weight of a fundamental swap involving the factorization c_in_i. We call a semigroup n_i-*thrifty if there exists a $d \in \mathbb{Z}^+$ with $d \neq c_i$ there exists $x, y \in \mathbb{N}$ where $dn_i = xn_j + yn_k$ and $\max\{d, x + y\} \leq w_i$. If $\max\{d, x + y\} < w_i$, we call the semigroup *strictly n_i-thrifty.*

We can use a little bit of graph theory sometimes to get a bound on the catenary. The following lemma gives us a technique to obtain a bound on the catenary degree by observing the edges emanating from a single factorization.

Lemma 6.2. *Let s be in a numerical semigroup S. Let f be a factorization of s. Then $c(s) \geq \min\{E\}$ where E is the set of edge weights emanating from f.*

Proof. If the algorithm for computing catenary degree were allowed to run, then when $e := \min\{E\}$ all the edges from f is removed thus making it disconnected from the rest of the graph. However, it's possible the graph might have already been disconnected earlier in the algorithm. Thus, $c(s) \geq \min\{E\}$. $\square$

Given that the c_i's are minimal it would be alarming if the catenary degree could somehow be less than the c_i's themselves. Luckily, we are always guaranteed by the following lemma that this never occurs.

Theorem 6.3. *Let $S = \langle n_1, n_2, n_3 \rangle$ be a numerical semigroup of embedding dimension three. Let $s \in S$. Then $c(s) \geq \min\{c_1, c_2, c_3\}$.*

Proof. First, let $\{i, j, k\} = \{1, 2, 3\}$. Consider the types of swaps that can occur between two factorizations. In embedding dimension three, the only swaps that can occur is swapping n_i-type atoms for n_j-type and n_k-type atoms. Suppose between factorizations f_1 and f_2 of s we swap x n_i-atoms for y n_j-atoms and z n_k-atoms. Then $d(f_1, f_2) = \max\{x, y+z\} \geq x$. Note that $x \geq c_i$ by minimality of c_i. And $c_i \geq \min\{c_1, c_2, c_3\}$. Thus, $d(f_1, f_2) \geq \min\{c_1, c_2, c_3\}$. But, f_1 and f_2 were chosen arbitrarily so this is true between any two factorizations, thus all edges are of weight greater than or equal to $\min\{c_1, c_2, c_3\}$ thus making it impossible for the catenary degree to be less than $\min\{c_1, c_2, c_3\}$. Therefore, $c(s) \geq \min\{c_1, c_2, c_3\}$. □

If we look at elements that are only supported in certain components can we give bounds on its catenary degree using only the c_i's that pertain to those components? The answer according to the following lemma is yes in the case where we look at the catenary degree of multiples of generators.

Lemma 6.4. *Let $k \in \mathbb{N}$. $c(kn_i) \geq c_i$ or $c(kn_i) = 0$ for $i = 1, 2, 3$.*

Proof. WLOG, let $i = 1$. If $c(kn_1)$ has only one factorization, then $c(kn_1) = 0$ so suppose $c(kn_1)$ has more than one factorization. We know that $(k, 0, 0)$ is at least one factorization. The only type of swap we can make is swapping x n_1-atoms for y n_2-atoms and z n_3-atoms where $x \leq k$. We also note that $x \geq c_1$ or else x would violate the minimality of c_1. The weight edge of such a swap would be $\max\{x, y+z\} \geq x$. Thus, all edge weights from $(k, 0, 0)$ are $\geq x$. However, x is always greater than c_1, so all the edge weights from $(k, 0, 0)$ are $\geq c_1$, i.e., $min\{E\} \geq c_1$ where E is as in Lemma 6.2. Thus, by Lemma 6.2, $c(kn_1) \geq \min\{E\} \geq c_1$. □

Using thriftiness, we can get a similar result to Lemma 6.4 that is in terms of the catenary degree of the corresponding Betti element.

Lemma 6.5. *Let $i = 1, 2$ or 3 and let $S = \langle n_1, n_2, n_3 \rangle$ be a numerical semigroup of embedding dimension three such that it is not strictly n_i thrifty. Let $k \in \mathbb{N}$. Then $c(kn_i) \geq c(c_i n_i)$ or $c(kn_i) = 0$.*

Proof. WLOG, let $i = 1$. If $c(kn_1)$ has only one factorization, then $c(kn_1) = 0$ so suppose $c(kn_1)$ has more than one factorization. We know that $(k, 0, 0)$ is at least one factorization. The only type of swap we can make is swapping x n_1-atoms for y n_2-atoms and z n_3-atoms where $x \leq k$. We also note that $x \geq c_1$ or else x would violate the minimality of c_1. The weight edge of such a swap would be $\max\{x, y+z\}$. By the non strictly n_1 thriftiness of S, $\max\{x, y+z\} \geq c(c_i n_i)$. Thus, every edge weight emanating from $(k, 0, 0)$ is $\geq c(c_i n_i)$, i.e., $\min\{E\} \geq c(c_i n_i)$ where E is as in Lemma 6.2. Thus, by Lemma 6.2, $c(kn_1) \geq \min\{E\} \geq c(c_i n_i)$. □

We can combine the previous result (Lemma 6.5) and the fact that the highest catenary degree is attained at a Betti element (Lemma 5.7) to get the following result.

Theorem 6.6. *Let $S = \langle n_1, n_2, n_3 \rangle$ with $n_1 < n_2 < n_3$ be a numerical semigroup with of embedding dimension three. Let i be such that*

$$\max\{c(c_1 n_1), c(c_2 n_2), c(c_3 n_3)\} = c(c_i n_i).$$

Let $k \in \mathbb{N}$. If S is not strictly n_i-thrifty, then $c(kn_i) = c(c_i n_i)$ or $c(kn_i) = 0$.

Proof. Suppose that S is not strictly n_i-thrifty. By (Lemma 6.5), $c(kn_i) \geq c(c_i n_i)$ or $c(kn_i) = 0$. Suppose that $c(kn_i) > c(c_i n_i)$. However, we know that $c(S) = c(\max\{c(c_1 n_1), c(c_2 n_2), c(c_3 n_3)\})$ by Lemma 5.7. Thus, $c(kn_i) > c(c_i n_i) = c(\max\{c(c_1 n_1), c(c_2 n_2), c(c_3 n_3)\}) = c(S)$ but $c(kn_i) > c(S)$ is a contradiction. Thus, it must be that $c(kn_i) = c(c_i n_i)$ or 0. □

Consider the case where we have a numerical semigroup of embedding dimension three with three distinct Betti elements. Now suppose that we knew the catenary degree of two of those Betti elements. Can the catenary degree of the third be anything? The answer is no, as the following lemma suggests that the catenary degrees of the three Betti elements bound each other in a triangle inequality sort of manner.

Lemma 6.7. *Let $S = \langle n_1, n_2, n_3 \rangle$ be a numerical semigroup of embedding dimension three where $|C(B)| = 3$ where B is the set of Betti elements. Then for $\{i, j, k\} = \{1, 2, 3\}$, $c(c_i n_i) < c(c_j n_j) + c(c_k n_k)$.*

Proof. Without loss of generality, let $i = 1, j = 2, k = 3$. So we want to show $c(c_1 n_1) < c(c_2 n_2) + c(c_3 n_3)$. Since the three Betti elements are distinct, we know the following is true:

$$c_1 = r_{21} + r_{31},$$

$$c_2 = r_{12} + r_{32},$$

$$c_3 = r_{13} + r_{23}.$$

We will rearrange these formulas for convenience later:

$$c_1 = r_{21} + r_{31},$$

$$r_{12} = c_2 - r_{32},$$

$$r_{13} = c_3 - r_{23}.$$

We know that $c(c_1 n_1) = \max\{c_1, r_{12} + r_{13}\}$ by Lemma 1.8. We use the set of above equations to substitute and we get

$$\max\{c_1, r_{12} + r_{13}\} = \max\{r_{21} + r_{31}, c_2 - r_{32} + c_3 - r_{23}\}.$$

First, suppose $\max\{r_{21}+r_{31}, c_2-r_{32}+c_3-r_{23}\} = r_{21}+r_{31}$. We have

$$\begin{aligned}
c(c_1n_1) &= \max\{c_1, r_{12}+r_{13}\} = \max\{r_{21}+r_{31}, c_2-r_{32}+c_3-r_{23}\}\\
&= r_{21}+r_{31} < (r_{21}+r_{23}) + (r_{31}+r_{32})\\
&\leq \max\{c_2, r_{21}+r_{23}\} + \max\{c_3, r_{31}+r_{32}\}\\
&= c(c_2n_2) + c(c_3n_3).
\end{aligned}$$

Now, suppose $\max\{r_{21}+r_{31}, c_2-r_{32}+c_3-r_{23}\} = c_2-r_{32}+c_3-r_{23}$. Then

$$\begin{aligned}
c(c_1n_1) &= \max\{c_1, r_{12}+r_{13}\} = \max\{r_{21}+r_{31}, c_2-r_{32}+c_3-r_{23}\}\\
&= c_2-r_{32}+c_3-r_{23} < c_2+c_3\\
&\leq \max\{c_2, r_{21}+r_{23}\} + \max\{c_3, r_{31}+r_{32}\}\\
&= c(c_2n_2) + c(c_3n_3).
\end{aligned}$$

Thus, in either case, $c(c_1n_1) < c(c_2n_2) + c(c_3n_3)$. $\square$

7 Conjectures, Problems and Examples

Below we have our list of current outstanding conjectures. In what follows, let S be a numerical monoid of embedding dimension three.

Conjecture 7.1. *Let*

$$G = \{cn_i : c_i \leq c \leq c(S), 1 \leq i \leq 3\}.$$

It follows that $C(G) = C(S)$.

Conjecture 7.2. *Let $s \in S$. Then there exists a generator n_i that can be subtracted from s such that $c(s-n_i) \geq c(s)$. Similarly, there exists a generator n_j that can be subtracted from s such that $0 < c(s-n_i) \leq c(s)$.*

Conjecture 7.3. *Let S have three distinct Betti elements with distinct catenary degrees where c_in_i and c_jn_j are the Betti elements with the higher catenary degree and the c_kn_k is the other Betti element. Then the periodicity starts at*

$$\max\{Ap(S, c_in_i) \cap Ap(S, c_jn_j) \cap \mathbb{N}\backslash Ap(S, c_kn_k)\}.$$

There are similar results if the Betti elements or catenary degrees of Betti elements coincide.

Conjecture 7.4. *Let S be a numerical semigroup such that it is not n_i-thrifty and c_in_i is the only Betti element that has catenary degree that achieves the lowest nonzero catenary degree of S. Then there are a finite number of elements that hit the smallest nonzero catenary degree.*

Problem 7.5. *Characterize those monoids S in which the set of elements whose catenary degree is the minimum nonzero value in $C(S)$ is symmetric.*

A proof of the following conjecture should not be difficult.

Conjecture 7.6. *Let S be a numerical semigroup of embedding dimension three with a single Betti element. Then every element has a triangle number of factorizations.*

Conjecture 7.7. *Let $S = \langle 4, 6, 4n+1 \rangle$. Then $C(S) = \{0, 3, c_n\}$ for some c_n, and $c_n < c_{n+1}$.*

Conjecture 7.8. *There is at most one nonzero catenary degree which is achieved by finitely many elements.*

The following examples each demonstrate some interesting behavior. The descriptions given are not rigorous.

Example 7.9. $S = \langle 12, 27, 29 \rangle$ has 2 Betti elements, and the one with the higher catenary degree has dropdowns after it.

Example 7.10. $S = \langle 11, 29, 32 \rangle$ has a dropdown of a different height.

Example 7.11. For $S = \langle 11, 13, 19 \rangle$, the set of elements whose catenary degree is the minimum nonzero value in $C(S)$ is not symmetric. Also, there exists an element with three factorizations that hits the least non-zero catenary degree.

Example 7.12. $S = \langle 7, 11, 17 \rangle$ has three distinct Betti elements, but only two nonzero possible catenary degrees.

Example 7.13. $S = \langle 11, 31, 37 \rangle$ has minimum catenary degree occurring at the greatest, rather than least, Betti element.

Example 7.14. $S = \langle 9, 40, 47 \rangle$ has eventual catenary degree period of $9 \cdot 40$.

Example 7.15. For $S = \langle 17, 41, 43, 59, 61 \rangle$, the catenary degree of the monoid is 8 and is achieved at exactly one element, namely 208.

8 Code Appendix

```
load('/media/sf_Desktop/NumericalSemigroup.sage')

def getCatDegrees(s, n):
    l = []
    m = []
    for t in range(1,n):
        if s.Contains(t):
            l.append(s.CatenaryDegree(t))
            #print (t, s.CatenaryDegree(t)), s.
                Factorizations(t)
            m.append((t, s.CatenaryDegree(t)))
    #show(list_plot(m))
```

```
    return (list_plot(m),Set(l))

def getCatDegrees2(s):
    catSet = {0}
    r = getRelations(s)
    a = r[0]
    b = r[1]
    c = r[2]
    print a, b, c
    weightDict = {weight(a): a, weight(b): b, weight(c):
        c}
    x = weightDict[max(weight(a), weight(b), weight(c))]
        #let x be the tuple with highest weight
    y = weightDict[min(weight(a), weight(b), weight(c))]
        #let y be the tuple with lowest weight
    print x, y
    catSet.add(weight(x))
    catSet.add(weight(y))
    catSet.add(weight(tupleAdd(x,y)))
    k = 1
    currentWeight = weight(tupleAdd(x,y))
    while currentWeight > weight(tupleAdd(x,tupleMult(k
        +1, y))):
        print k
        catSet.add(weight(tupleAdd(x,tupleMult(k+1, y))))
        currentWeight = weight(tupleAdd(x,tupleMult(k+1,
            y)))
        k += 1
    return catSet

def sortFactorizations(s, n):
    l = []
    m = []
    p = []
    for t in range(1,n):
        if s.Contains(t):
            if len(s.Factorizations(t)) == 1:
                l.append(t)
            elif len(s.LengthSet(t)) == 1:
                m.append(t)
            else:
                p.append(t)
    return l,m,p

def CatDegreesEmbDimThree():
    length = len(NumSemigroups)
```

```
    count = 0
    for s in NumSemigroups:
        count += 1
        if tuple(s.gens) in TestedSemigroups.keys():
            print s.gens, TestedSemigroups[tuple(s.gens)]
            continue
        l = []
        for t in range(1, 5*s.frob):
            if s.Contains(t):
                l.append(s.CatenaryDegree(t))
        l = Set(l)
        print s.gens, l
        if count % 300 == 0:
            print str(100.*(count/length))+"%"
        TestedSemigroups[tuple(s.gens)] = l

def getRelations(s):
    presentation = s.MinimalPresentation()
    tuples = []
    for i in presentation:
        tuple = tupleAdd(i[0], tupleMult(-1, i[1]))
        count = 0
        for j in tuple:
            if j < 0:
                count += 1
        if count == 1:
            tuple = tupleMult(-1, tuple)
        tuples.append(tuple)
    return tuples

def tupleAdd(t1, t2):
    return tuple([t1[i]+t2[i] for i in range(0,3)])

def tupleMult(n, t):
    return tuple([n*t[i] for i in  range(0,3)])

def weight(t):
    pos = 0
    neg = 0
    for i in range(0,3):
        if t[i] > 0:
            pos += t[i]
        else:
            neg += -t[i]
    return max(pos, neg)
```

```
def getEquations(S):
    e = [0,0,0]
    presentations = S.MinimalPresentation()
    for pres in presentations:
        for index, tuple in enumerate(pres):
            if tuple[1] == 0 and tuple[2] == 0:
                t1 = tuple
                t2 = pres[(index+1)%2]
                e[0] =  str(t1[0])+"*"+str(S.gens[0])+" =
                     "+str(t2[1])+"*"+str(S.gens[1])+" +
                    "+str(t2[2])+"*"+str(S.gens[2])

            if tuple[0] == 0 and tuple[2] == 0:
                t1 = tuple
                t2 = pres[(index+1)%2]
                e[1] =  str(t1[1])+"*"+str(S.gens[1])+" =
                     "+str(t2[0])+"*"+str(S.gens[0])+" +
                    "+str(t2[2])+"*"+str(S.gens[2])

            if tuple[0] == 0 and tuple[1] == 0:
                t1 = tuple
                t2 = pres[(index+1)%2]
                e[2] =  str(t1[2])+"*"+str(S.gens[2])+" =
                     "+str(t2[0])+"*"+str(S.gens[0])+" +
                    "+str(t2[1])+"*"+str(S.gens[1])

    return str(e[0])+"\n"+str(e[1])+"\n"+str(e[2])

def gcd_test(n):
    coprime = n.coprime_integers(n)
    i = 1
    j = 2
    while i < len(coprime):
        j = i+1
        while j < len(coprime):
            if gcd(i,j) == 1:
                return (coprime[i],coprime[j])
            j += 1
        i += 1
    return false
```

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