Diagnosing Teacher Knowledge: A New Model

Attributes

Attribute Document in DTMR

The Diagnosing Teachers Multiplicative Reasoning Project (DTMR Project) is an exploratory project working to assess teachers' reasoning about multiplicative relations among quantities. The domain of multiplicative reasoning includes operations with fractions, decimals, ratios, percents, proportions, linear functions and more advanced topics. These topics are often taught discretely as separate units, however, Vernaud (1983, 1988) suggests they form an interconnected set of conceptual ideas. *Multiplicative conceptual field* is the term coined by Vergnaud and used by mathematics educators to refer to the domain of multiplicative reasoning and the connectedness of the topics within that domain.

Following Vergnaud, we identify components of reasoning that cut across topics which serve as unifying themes. These themes are not general categories that contain topics; instead they are components of reasoning that are used within a variety of topics. For example, fraction division and proportionality are often taught as separate topics, but incorporate many of the same unifying themes.

Rather than assessing teachers' knowledge of each topic or how to teach it, we identify components of reasoning that connect the topics within the multiplicative conceptual field and aim to diagnose teachers' understandings of these components. Psychometricians refer to these components as attributes. When teachers can recognize and flexibly employ a particular attribute, we say the teacher shows evidence of that particular attribute. To create such a diagnostic tool we first compile a document which synthesizes the definitions of the attributes we aim to assess. While the document outlines ten such attributes, we will highlight four of them for the purposes of later discussion.

Four Sample Attributes

Referent Unit

The ability to establish standard units of measurement and attend to the units to which numbers refer is vital to form a firm understanding of many topics within multiplicative reasoning (Iszák, Lobato, Orrill, Cohen, Templin, 2009). This concept is encompassed within the attribute known as referent unit. Consider, for example, a measurement division interpretation of the equation 6÷2/3=9, which can be accomplished with the question "How many two-thirds are in six?" What units are associated with each number within the equation? While the six refers to groups of one and 2/3 refers to a portion of a group of one, the referent unit changes when we interpret the quotient of the number sentence. In this case, nine does not describe groups of one. Rather, it refers to groups of 2/3. By acknowledging the standard units associated with each number within the equation, the referent unit attribute is being utilized.

An understanding of referent unit is absolutely vital so that teachers can engage in meaningful, clear communication with students when issues with fractions arise. Izsák (2008) describes a classroom situation in which a teacher's minimal understanding of referent unit created a communication barrier during a lesson on fraction multiplication. The teacher was using a number line to illustrate the solution to 1/5 of 2/3 (see Figure 1), with the first 1/5 of each third shaded as shown. Just as one shaded piece can be viewed as 1/15 when the whole is the entire number line, it can be interpreted as 1/5 when the whole is 1/3 of the number line. A student asked why the teacher was referring to a shaded piece as both 1/5 and 1/15, but the teacher could not provide an accurate, explicit answer to his question. The teacher's lack of understanding of referent unit inhibited her student from gaining a true understanding of a very important concept within multiplicative reasoning.

Figure 1. Illustration of 1/5 of 2/3 example.

Norming

Just as the referent unit attribute facilitates the interpretation of standard units, *norming* addresses situations in which it might be necessary to change a referent unit and interpret a quantity as a fraction of different wholes. As an example, we could consider the 6÷ 2/3=9 equation with a partitive division interpretation, as verbalized in a situation like this: “If I can run six feet in 2/3 of a second, how many feet can I run in a whole second?” One way to approach this problem would be to first find the number of feet I can run in 1/3 of a second and then multiply by 3 to find the number of feet in a whole second. The circled portion of the diagram in Figure 2 can represent 1/3 of a second, but in order to find the number of feet in 1/3 of a second, it is equally necessary to recognize this segment as representing 1/2 of 2/3 of a second. Interpreting this segment with fractions of different wholes shows clear evidence of the norming attribute.

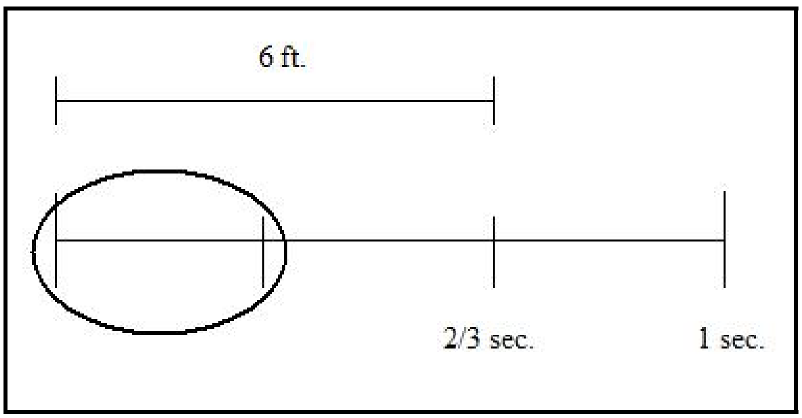


Figure 2. Partitive division interpretation of 6÷2/3=9.

Fractions can often be interpreted in a multitude of ways, and as such, it is absolutely vital for teachers to display flexibility in assigning fraction wholes. In the case of the example above, for instance, only a teacher with an understanding of norming could simultaneously recognize the shaded segment as 1/5 and 1/15. In addition to teaching multiple interpretations of a fraction situation, teachers also need to be able to appropriately assess students’ interpretations. A teacher with a firm understanding of norming can be sure that he or she is accurately assessing students’ understanding of how to assign and shift referent units within multiplicative reasoning situations.

Multiples of Unit Fractions

Teachers need ways to interpret the meaning of fractions and to communicate that meaning to students effectively. One such meaning involves interpreting fractions as multiples of unit fractions, or fractions with a one in the numerator. Learning about fractions in this way means interpreting the fraction 5/8 as five one-eighths or the fraction 5/4 as five one-fourths.

Teachers are accustomed to teaching fractions such as $\frac{2}{3}$ as, “two out of three.” We refer to this as an “N out of M” conception. This understanding of fractions quickly falls short when improper fractions are introduced since $\frac{4}{3}$ cannot be interpreted as, “4 out of 3.” Because four things cannot be taken out of three, a complete understanding requires the ability to interpret fractions as multiples, or repeated addition, of their unit fraction. If a teacher introduces $\frac{2}{3}$ as two one-thirds, then $\frac{4}{3}$ will make sense as four one-thirds. If teachers have this knowledge, fraction lessons can be taught using a method that encourages both proper and improper fractions to be interpreted in the same way. This deeper conceptual understanding of fractions is crucial for later concepts in algebra.

Composed Units

Teachers also need to have ways of thinking about proportional reasoning that are accessible to students who are just beginning to reason with ratios. This involves joining two different quantities to form a single unit. For example, in a story problem involving orange juice, the ability to form a unit composed of the amount of water and the amount of orange juice concentrate would look something like: 2 parts concentrate to 5 parts water, or 2:5.

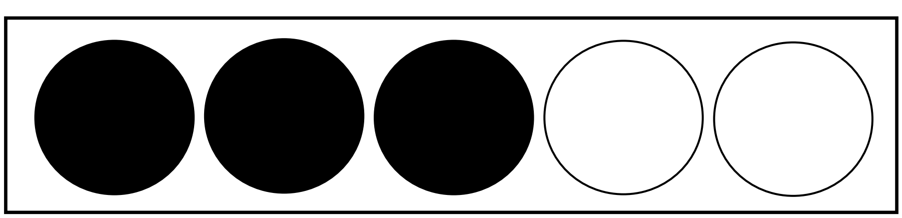
By forming a composed unit, teachers will better understand how to compare juices with different levels of “orangeyness,” know when to use additive and multiplicative comparisons, and complete various tasks using manipulations of the composed unit. Suppose the story problem reads, “One pitcher of orange juice contains ten cups of water and four cups of orange juice concentrate. How much water will be needed if you have three cups of orange juice concentrate?” This problem can be solved without using composed units as a simple proportion can be set up to solve $10/4 = x/3$. With whole number reasoning, one can solve for x without setting up a proportion by using the equation $10\cdot 3 = 4\cdot x$. Using this equation, one can solve for x and get 7.5. This procedural fluency, however, merely depends on the ability to cross-multiply and does not ever require the ability to set up a ratio, which is an important concept in proportional reasoning. The ability to form a composed unit requires this knowledge, and for this reason will provide a better foundation for teachers when reasoning proportionally. Using composed units, they will partition the 10/4 ratio to get the composed unit of 2½ / 1. Then iterating this unit will eventually reach 7½ / 3 to conclude that you need 7½ cups of water for three cups of orange juice concentrate. The ability to work with different quantities in the story problem in terms of this single composed unit enables greater flexibility and a deeper conceptual understanding of proportional reasoning. Composed unit reasoning is important because it preserves multiplicative comparison while remaining accessible to students who are just learning to reason proportionally.

Two Sample DTMR Items

With particular attributes in mind, we strategically design each test item in the DTMR. Through each test item we aim to derive information about the knowledge a teacher has regarding two or three attributes at the same time. In order to accomplish this, both the problem and the answer choices need to be crafted in such a way that requires the examinee to actually use the attributes in order to solve the problem correctly or demonstrate an error in the application of the attributes when an incorrect answer is obtained. The following two examples of test items demonstrate how this is accomplished.

Figure 3, The Circle Problem, provides a teacher with a figure which can be interpreted in multiple ways as the unit of measure is established.

Below is a group of 5 circles followed by three interpretations. Which of the interpretations are sensible?



I. The diagram can show 3/5.

II. The diagram can show 1 2/3.

III. The diagram can show 5/2.

Choices:

A I only.

B I and II only.

C I, II, and III.

D None of the above.

Figure 3. The Circle Item

The Circle Item assesses teachers' abilities to flexibly change their view of the whole. By establishing five circles as the whole, each circle is a fifth of the whole, thus the shaded circles are three fifths of the whole.

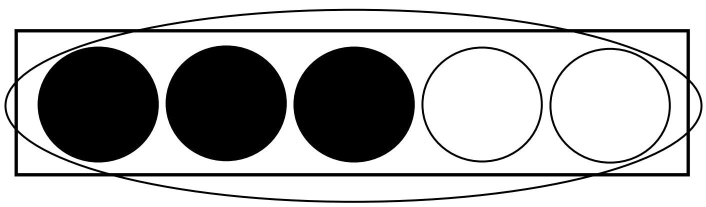


Figure 4. Three Fifths

Changing the unit of measure so that the three shaded circles are now viewed as the whole, all five circles combined can be interpreted as 1 2/3.

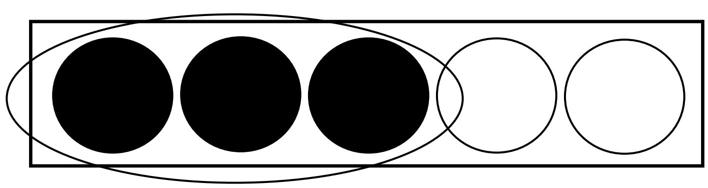


Figure 5. One and Two Thirds

Finally, by creating a whole out of the two white circles, all five circles combined can be interpreted as 5/2.

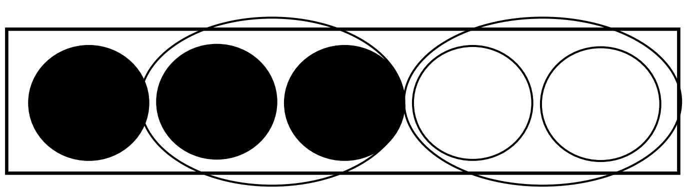
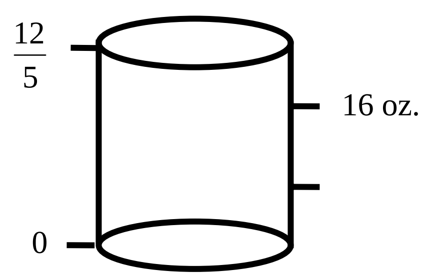


Figure 6. Five Halves

A teacher who chooses all three statements shows the ability to shift between different views of the whole and therefore exhibits evidence of the Norming Attribute. Simultaneously, while flexibly establishing different wholes with groups of circles, the teacher will also need to think of each individual circle in terms of a new unit fraction for each whole. For example, the ability to recognize 1 2/3 as five 1/3s and 5/2 as five 1/2s requires reasoning with multiples of unit fractions. Because choosing the correct answer for The Circle Item precludes a teacher to norm to a new whole and consider each circle as a new unit fraction with each interpretation, this item assesses two attributes at the same time.

Figure 7, The Brownie Item, presents teachers with a cooking problem that requires them to consider and manipulate ratios. A diagram accompanies the problem and anticipates that teachers will use it in their problem solving.

Milo is going to make a batch of his favorite chocolate brownies. He wants to make a batch that is 12/5 the amount of the original recipe. To make 2/3 of the batch he wants to make, he knows he needs 16 ounces of water. How much water is needed for the original brownie recipe?



Choices:

A. 8 ounces

B. 10 ounces

C. 15 ounces

D. 24 ounces

Figure 7. The Brownie Item

A teacher interpreting the diagram correctly can see that 12/5 of a batch uses 24 ounces of water. A composed unit is formed when she recognizes that each 1/5 of Milo's batch needs two ounces of water. The original recipe for the brownies calls for scaling up from 1/5 to 5/5. Two ounces iterated five times yields ten ounces of water which is needed for the original recipe. With this line of thinking, The Brownie Item targets composed unit reasoning. However, the teacher must also pay attention to referent units in order to form the composed unit correctly. Sixteen ounces is 2/3 of Milo's batch while this recipe is 12/5 of the original recipe. So the thirds in the problem refers to thirds of Milo's batch while the fifths refer to the original batch.

Other Attributes

The previously discussed items assess four attributes in particular, but these are not the only key ideas outlined within the attribute document. Several other attributes are needed for advanced reasoning within the multiplicative conceptual field. The DTMR document focuses on the attributes in Figure 8, which we have organized into three larger themes: reasoning with units; interpreting the meaning of rational numbers and reasoning proportionally; and making connections and judgments of appropriateness. The relationship between the DTMR test form and the attributes is a cyclical one. Just as the attributes are absolutely vital to the creation of the DTMR test items, the entire process of creating and revising the test forms also drives important clarifications and improvements of the attribute document.

Section		Attribute	Description
Reasoning with Units	1	Referent Units	Establishing standard units of measurement and attend to the units to which numbers refer
	2	Norming	Flexibly changing one's view of the "whole" in a given situation

	3	Nested Units	Creating multiple levels of partitioning in which units at each level have a fixed multiplicative relationship with units at other levels
Interpreting Rational Numbers & Proportional Reasoning	4	A One-Bths of M	Interpreting fractions as multiples, or repeated addition, of a unit fraction
	5	Reasoning Proportionally by Operating with Composed Units	Joining two different quantities to form a single unit
	6	Reasoning proportionally by Using Multiplicative Comparisons	Forming a multiplicative comparison between two quantities
	7	Reciprocal Relations of Relative Size	Conceiving A/Bths as A times as large as 1/Bth and conceiving 1/Bth as 1/Ath of A/Bths
Creating Relationships & Determining Appropriateness	8	Connections among Fractions, Ratios, Decimals, and Quotients	Making conceptual links across rational number types
	9	Equivalence	Making conceptual connections within a given number type
	10	Appropriateness	Making decisions regarding the relationship between a particular operation or type of reasoning and a given situation.

Figure 8. Attributes of the DTMR Project

We recognize that these ten attributes do not encompass all of the important components of reasoning within the multiplicative conceptual field. While the DTMR test form has the capability to assess multiple attributes, psychometric constraints limit this number to no more than ten. However, even with these limitations, we feel that a teacher proficient in these ten attributes is likely to have a strong understanding of multiplicative reasoning.

Test Development Process

Development of Items and the Q-Matrix

From the beginning, the attribute document drives the process of test development. Developing items that focus on specific attributes allows us to classify the demonstrated understandings of those taking the test. Assessments of this design are referred to as Diagnostic Classification Models (DCMs). In this model, solving an item correctly implies a presence of the targeted attributes while choosing an incorrect response indicates difficulty with the targeted attributes. In other words, alternate responses are not distracters disconnected from the attribute document, but rather they are answers which can demonstrate a lack of knowledge of an attribute. The teachers' responses to test items create a profile summarizing the mastery of each attribute.

With each answer choice of an item, both correct and incorrect, we attach information pertaining to the attributes that would be demonstrated in that answer choice. For example, in an item that tests for multiple attributes, we will distinguish that an incorrect response means a teacher demonstrates attributes x and y, but is not demonstrating attribute z. This method is called polytomous scoring. Scoring a test polytomously allows each answer to give information about the attributes providing a more complete analysis of teacher knowledge. However, the initial test will be scored dichotomously. Dichotomous scoring differs from polytomous scoring because it only looks at the attributes demonstrated in answering an item correctly. An incorrect answer does not give information about attributes other than a lack of demonstration.

After the creation of approximately 12 to 15 items that target a variety of attributes, we develop a Q-matrix (see Figure 9). The Q-matrix lists the targeted attributes for each test item and summarizes all attributes addressed by one test form. Items can load onto more than one attribute, and each attribute generally appears more than once throughout the test so that there are multiple opportunities to assess teacher knowledge in each area. This helps to verify the presence or absence of an attribute even in the event that an item appears context sensitive for the test taker.

	Sample Q-Matrix	Referent Unit	Norming	Multiples of Unit Fractions	Composed Unit Reasoning
	Name				
1	Circle Item		X	X	
2	Brownie Item	X			X

Figure 9. Sample Q-Matrix For Sample Items

For an example of how to read to the Q-Matrix, consider the first sample item presented, Circle Item. Looking at the row entitled "Circle Item," we can see that the columns marked include the targeted attributes: Norming and Multiples of Unit Fractions. Next, the Brownie Item attribute summary shows the use of referent unit and composed units. Once we have summarized the test form through its predicted attributes for each item, we are ready for the next step: teacher interviews.

Item Development Interviews

We use the process of teacher interviews to test the validity of our work on the Q matrix. Through the interviews we are able to compare the actual behaviors teachers use while problem solving to those we proposed on the Q matrix. By observing these behaviors, we determine which answers would provide us with an overestimation or underestimation of the knowledge teachers have in the targeted attributes. We take this information and use it to revise test items, as needed, so that the results we obtain will more accurately reflect the actual knowledge each teacher has.

To begin the teacher interview process, we ask a group of middle school math teachers to complete the test with no time limit. Afterwards, they participate in a videotaped interview to explain how they reasoned through each test item. Analysis of the interviews allows us to verify whether each teacher actually used the attributes we expected from the Q matrix. We discovered, however, that teachers often performed in ways we did not anticipate. Some teachers were able to solve many problems correctly without even using the attributes for which the items were designed (which would result in an overestimation of their abilities in the scoring process). Other teachers marked incorrect answers, but demonstrated an understanding of the attributes in the reasoning they articulated through their interviews (which would result in an underestimation of their abilities in the scoring process). With further analysis of the videos we concluded that there were reoccurring misinterpretations and methods of problem solving that created misleading results.

One item that was commonly misinterpreted by most teachers is the Circle Item. Most of the teachers we interviewed answered an item similar to the Circle Problem incorrectly. All of the teachers were able to interpret the diagram as showing 3/5 because they saw the three dark circles as three of the five circles provided. A few of the teachers were able to interpret the diagram as 1 2/3, when they interpreted the three shaded circles as one whole and the remaining two white circles as 2/3, regardless of the fact that the third circle of that set was not visible. However, the interpretation of 5/2 with this diagram was very difficult for most teachers. In the interviews, we discovered that teachers were describing the diagram as though the shading suggested that the dark circles could not be split up. We observed that some of the teachers who marked an answer which did not include 5/2 as one of the representations, had also expressed the idea that 5/2 was plausible, but made the decision to reject it. For this reason, we believe that our scoring might have underestimated the ability of other teachers who marked this item with an incorrect answer. We were not certain if teachers rejected statement III because they lacked understanding of multiples of unit fractions or because there was an issue with the shading of the diagram.

Through the interviews, we also observed that certain computational methods of problem solving were frequently used, which prevented us from correctly assessing the teachers' knowledge of the attributes. On an item like The Brownie Item, most teachers provided a correct answer, however, when we listened to teachers' explanations of how they solved the problem, we discovered that they were simply setting up an algorithm and solving by computation. While the ability to set up the correct algorithm and compute the correct answer indicates to us that the teachers have some mathematical understanding of the problem, it does not tell us anything about their knowledge of the targeted attributes. The display of attribute reasoning in a problem involving proportions is considered to be the proof that conceptual understanding of the problem exists. It is vital, therefore, that our teacher interview process accurately identifies when an item, such as The Brownie Item, can be easily solved by computational methods which replace attribute reasoning.

Analysis & Item Revisions

Although conducting teacher interviews may seem like the end of our test development process, it is only so *if* we have established the validity of the formerly created Q-matrix. Once we have collected a preliminary set of teacher interviews, there is a complex, but necessary, procedure that follows. Since our primary goal is to target the desired attributes, we do not begin by analyzing teacher by teacher. Instead, we watch the collected interviews one at a time and analyze item by item. We create transcripts of what is said by the teacher and the interviewer. This becomes helpful as we try to depict the exact problem-solving strategies each teacher was using for a particular item. We then infer attribute reasoning based on the problem-solving behaviors we observed. From our analysis we identify *particular* types of overestimating or underestimating issues. Finally, we brainstorm ideas on possible revisions that can address these reoccurring issues.

Let's revisit The Circle Item. When this item was included in the test form we had initially considered it to be in the medium difficulty level. Surprisingly, most of the teachers answered it incorrectly. As we analyzed several interviews we realized that this is an item in which we can mistakenly underestimate a teacher if we base our conclusions only on their answer choice. Listening to each teacher's reasoning helped us verify that most of them acquired a clear understanding of the intended attributes, yet, were misusing the provided diagram.

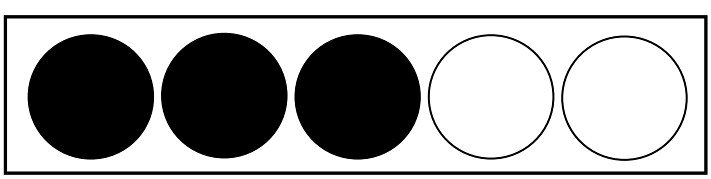


Figure 10. The Circle Item Diagram

Every teacher was able to interpret the diagram as showing 3/5. When working with diagrams, the most common practice among in-service teachers is to shade the regions they are referring to. Un-shaded regions are then considered to form the "whole". In the Circle Item we observed that several teachers were referring to the white circles as "empty circles." This makes 3/5 the most observable interpretation of the diagram. Consequently, statements II and III were difficult for teachers to consider. Few teachers were able to figuratively combine the three shaded circles to make it their "new whole" and interpreted the remaining two circles as 2/3 of another whole, however, they wanted to see the remaining two circles shaded "out of" a set of three circles in order to recognize 1 2/3 more clearly. The following diagram demonstrates one teacher's approach:

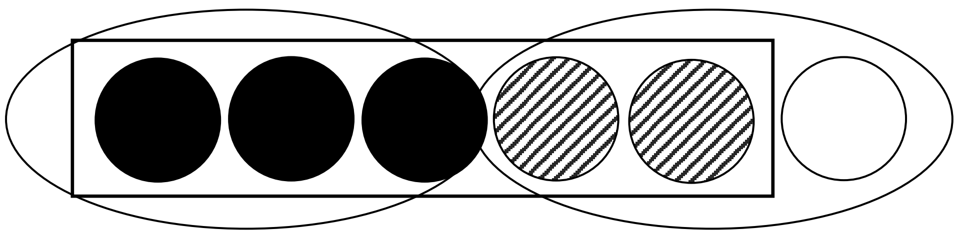


Figure 11. Teacher's Circle Diagram

Finally, the interpretation of 5/2 was the most difficult for teachers to accept. It is easier to recognize 5/2 if you first create a whole with the two white circles, however, we also found several teachers influenced by the order of the shaded regions. It is "unusual" to consider the two un-shaded, or empty, circles before the shaded regions. In order to make our diagram more flexible to all three interpretations, we altered the shading of our circles.

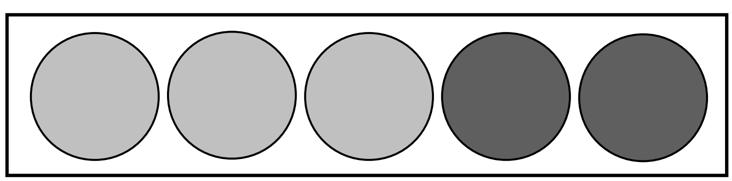


Figure 12. Revised Diagram for The Circle Item

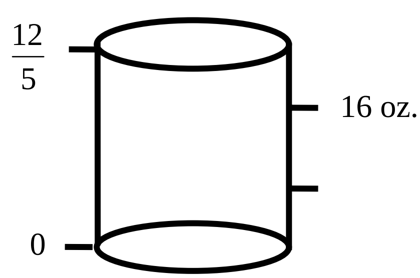
We realized that teachers' prior experience reading diagrams could hinder their flexibility to interpret and reinterpret a diagram; therefore, we decided to select two different shadings to separate the group of circles. We also shifted the traditional dark shading to the right in order to allow for flexibility in norming.

In other cases of item analysis, the revisions an item may call for are more complicated. After conducting and analyzing teacher interviews on The Brownie Item, we found most teachers restricted to solving the problem using computational strategies. Most teachers were able to arrive at the correct answer choice: 10 ounces. However, they relied heavily on computation by solving a proportion or setting up an equation similar to 12/5 = 24/x. Although they did not demonstrate an understanding of composed unit reasoning, can we conclude they lack this attribute? This is an example of an item in which we can mistakenly overestimate a teacher if we rely on a teacher's answer choice to conclude whether they have an attribute or not. After more analysis we observed that teachers took the same approach in similar items to The Brownie Item. If a teacher relies on computation as his/her first strategy and they arrive to a numerical answer that appears among the answer choices, then it is logical that the teacher will not attempt nor consider a second approach. After brainstorming possible revisions for this item, we concluded that we needed to change the answer choices provided so that *even* when teachers arrive to the answer (10 ounces) computationally it would not help them answer the item correctly.

We replaced the numeric values in the answer choices with statements of students' reasoning through the brownie problem. Previously, our answer choices were 8, 10, 15, and 24 ounces. The following figure shows the revision of The Brownie Item.

The following story problem was given to a group of students:

Milo is going to make a batch of his favorite chocolate brownies. He wants to make a batch that is 12/5 the amount of the original recipe. To make 2/3 of the batch he wants to make, he knows he needs 16 ounces of water. How much water is needed for the original brownie recipe?



Which of the following students demonstrates an appropriate approach?

A. Ted: 2/3 of the recipe takes 16 ounces, so to make the recipe he will need 3/3 or 24 ounces.

B. Alexa: 12/5 takes 24 ounces. Taking 1/12 of 12/5 and 24 means that 1/5 of the recipe takes 3 ounces. So Milo will need 3*5 or 15 ounces.

C. Monique: The common denominator of 12/15 and 2/3 is 15 so 10/15 will be 16 ounces and to get 15/15 he needs 5 more ounces so he needs 21 ounces.

Figure 13. Revision of The Brownie Item

As you can see above, each student took a different approach in solving the problem and each arrived to an incorrect answer, however, only one student had appropriate reasoning. Alexa in fact formed a composed unit with the number of ounces per 1/5 of the recipe but arrived to an incorrect answer because of a minor computational error. Our intention is to embed the attributes into these examples of students' reasoning. If a teacher can recognize which students use appropriate strategies we will be able to identify whether he/she has a thorough understanding of the Composed Unit Reasoning Attribute.

Now that we have illustrated the procedures that follow teacher interviews, it is easier to observe how this becomes an ongoing cycle. We conduct interviews, analyze teacher responses item by item, and make the necessary revisions. This process is repeated until we have established the validity of our Q-matrix.

What's Next?

Our goal by the end of the DTMR project is to have addressed content and construct validity in sufficient depth so that larger scale work and predictive validity studies may follow. We will proceed with the process of item development, item development interviews and revisions until we develop an adequate test form which demonstrates this validity with the Q-matrix. We aim to have a longer form of about 30 to 40 items to test out on a larger national sample of about 300-400 teachers.

We plan to incorporate into the project more psychometrics, which is a branch of psychology that looks at the theory, technique and interpretation of educational measurements. Simulation studies of data will be performed in order to learn more about the necessary sample sizes needed for accurate classification results. For instance, the simulation studies will be important for informing us how many people should be in the sample to obtain accurate estimates of item parameters for a given number of attributes, how many items should be included as common items on each form to ensure accurate equating of forms and how many times an attribute should be measured to obtain accurate estimates of attribute mastery. These simulation studies will examine both the dichotomously scored response models that we use in the present study and the polytomous models that we hope to use in future studies.

After we administer the test form to a large sample size, we will analyze the data by using diagnostic classification models. We will be particularly interested in identifying the various profiles, as established by patterns of responses, in our data. These profiles describe patterns of strengths and weaknesses based on the attributes that the items and the individual choices were designed to measure. We will also examine the relationships between the profiles observed in our sample of teachers and the information we will collect in the teacher background survey at the end of each test form.

Our test development process we use could serve as a model for developing further instruments for measuring other areas of multiplicative reasoning and reasoning in other domains. We will contribute our framework document and a summary of our model for test development so that others can build similar instruments. We will also prepare manuscripts that explain our approach to test development, results from the analyses of interviews and an accompanying framework document that both articulates the theoretical perspective on which we base the test form and includes a Q-Matrix specifying which aspects of multiplicative reasoning that the test emphasizes.

Another primary use of the final test could benefit professional development curriculum for middle grade schools. The test can be used as a tool to diagnose certain areas in which teachers are strong and weak so that educators can make effective use of professional development time. It also can be used as a pre and post assessment tool to gauge levels of understanding gained during professional development.