Perfect Powers Predominantly Produce Polygons

The Pythagoreans were avid admirers of 10, which they felt revealed metaphysical and musical dimensions when arranged as an equilateral triangle (like bowling pins). Generalizing this are the polygonal numbers $P_{s,n}$, with s sides and n dots per outermost side (recursively containing all smaller polygons with s sides, like Russian nested dolls). The Pythagoreans liked $P_{3,4} = 10$. Recently, it was proved in [**1**] that every cube is polygonal, which leads to the question of which perfect powers are polygonal. We exclude $s \leq 2$ and $n \leq 2$ to avoid triviality.

Theorem. *Let $a \in \mathbb{Z}^{>2}, b \in \mathbb{Z}^{\geq 2}$. Then $a^b = P_{s,a}$, for $s = 2\left(\frac{a^{b-1}-1}{a-1} + 1\right)$.*

Proof. Using the well-known formula

$$P_{s,n} = \frac{(s-2)n^2 - (s-4)n}{2},$$

we see $P_{s,a} = \frac{2a^2(\frac{a^{b-1}-1}{a-1}) - 2a(\frac{a^{b-1}-1}{a-1}) + 2a}{2} = a^2(\frac{a^{b-1}-1}{a-1}) - a(\frac{a^{b-1}-1}{a-1}) + a = a(\frac{a^{b-1}-1}{a-1})(a-1) + a = a(a^{b-1} - 1) + a = a^b - a + a = a^b$. ■

This result does not apply for $a = 2$, since it supplies a trivial solution with $n = a = 2$. For $a = 2$ and $b = mn$, we can write $2^b = (2^m)^n$ and apply our theorem. However, we conjecture that 2^b is not polygonal for prime b, i.e. there are no $s, n \in \mathbb{Z}^{>2}$ with $2^b = P_{s,n}$. This is supported experimentally (see [**2**]).

REFERENCES

1. Roger B. Nelsen (2025) Cubes are Polygonal, The College Mathematics Journal, 56:3, 231-232 `https://doi.org/10.1080/07468342.2024.2401288`.
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