Maximal-ly Surprising Triangle Functions

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Abstract

We look for functions that, evaluated symmetically on the angles of a triangle and added, achieve their maximum at surprising values.
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It is well-known (e.g. in [1]) that for any plane triangle with angles $\theta_1, \theta_2, \theta_3$, the function $\cos\theta_1 + \cos\theta_2 + \cos\theta_3 \leq \frac{3}{2}$, where $\frac{3}{2}$ is achieved with an equilateral triangle. This symmetric outcome is unsurprising, since the function is symmetric in the three angles.

We consider therefore the question of finding simple, familiar, "nice" functions $f(x)$ on $[0, \pi]$, such that the triangle function $f(\theta_1) + f(\theta_2) + f(\theta_3)$ achieves its maximum on a surprising triangle, i.e. neither equilateral nor degenerate. Such a function $f(x)$ (also the triangle function $f(\theta_1) + f(\theta_2) + f(\theta_3)$) would be surprising at its maximum, i.e. maximal-ly surprising. The above example shows that $f(x) = \cos(x)$ is not maximal-ly surprising, and perhaps it may seem that such functions do not exist.

The natural approach to finding maximal values would be with Lagrange multipliers. We seek maximal values of $f(x)+f(y)+f(z)$, on the triangle formed by intersecting the plane $x + y + z = \pi$ with the first orthant. Unsurprising maxima would be found on the boundary (i.e. if $xyz = 0$), or at the center (i.e. $x = y = z = \frac{\pi}{3}$).

Here we will consider functions $f(x) = \cos^m(x)$, for all natural m. We will show that all odd $m > 1$ are maximal-ly surprising, while even m are not. Further, the triangle function at these surprising triangles approaches a limit as odd $m \to \infty$, and we will determine this limiting value, which is $2^{2/3} - 2^{-4/3}$. The first few maximal-ly surprising triangles are illustrated in Figure 1, below.

First, we need a technical lemma. It will allow us to rule out maxima that are nondegenerate and scalene.

Lemma 1. *Let $m \in \mathbb{N}$, and set $g(x) = \cos^m(x)\sin(x)$. Suppose $\theta_1, \theta_2, \theta_3 \in [0, \pi]$ are distinct with $\theta_1 + \theta_2 + \theta_3 = \pi$ and $g(\theta_1) = g(\theta_2) = g(\theta_3)$. Then $\theta_1\theta_2\theta_3 = 0$.*

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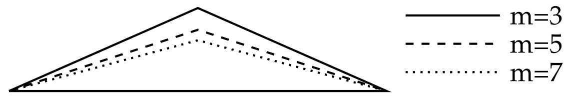

Figure 1: The maximal-ly surprising triangles for $f(x) = \cos^m(x)$ at $m = 3, 5, 7$.

Proof. We first prove that $g(x)$ is unimodal on $[0, \pi/2]$. We calculate $g'(x) = -m\cos^{m-1}(x)\sin^2(x) + cos^{m+1}(x) = \cos^{m-1}(x)(-m\sin^2(x) + \cos^2(x)) = \cos^{m-1}(x)(-m + (m+1)\cos^2(x))$. Note that $\cos^{m-1}(x) > 0$ on $[0, \pi/2)$, while $-m + (m+1)\cos^2(x)$ is monotone decreasing from 1 down to $-m$. Hence $g(x)$ monotonically increases from $g(0) = 0$ to some maximum achieved at some $x^\star$, then monotonically decreases down to $g(\pi/2) = 0$. Also $g(x)$ is positive on $[0, \pi/2)$.

Next, we observe that if m is even then $g(\pi - x) = g(x)$, so $g(x)$ is unimodal and positive on $(\pi/2, \pi]$ as well. On the other hand, if m is odd then $g(\pi - x) = -g(x)$, so $-g(x)$ is unimodal and positive on $(\pi/2, \pi]$.

Now, suppose m is even. Each horizontal line $y = M$ crosses $g(x)$ in 0, 2, 3, or 4 places. 3 crossings occurs only for $M = 0$, and 2 crossings occurs only if M is that unique maximum value, achieved once in $[0, \pi/2)$ and again in $(\pi/2, \pi]$. Suppose now that $g(\theta_1) = g(\theta_2) = g(\theta_3)$, for distinct $\theta_1, \theta_2, \theta_3$. If $M = 0$ then $\{\theta_1, \theta_2, \theta_3\} = \{0, \pi/2, \pi\}$ so $\theta_1\theta_2\theta_3 = 0$. Otherwise the three θ's are chosen from $\{x, \pi - x\} \cup \{y, \pi - y\}$ for some x, y. By the pigeonhole principle, two must be chosen from the same set, so without loss we have $\theta_1 = \pi - \theta_2$. But now $\pi = \theta_1 + \theta_2 + \theta_3 = (\pi - \theta_2) + \theta_2 + \theta_3$, so $\theta_3 = 0$ and hence $\theta_1\theta_2\theta_3 = 0$.

The case of m odd is simpler. Since $g(x)$ is unimodal and positive on $[0, \pi/2)$, and negative on $(\pi/2, \pi]$, if $g(x) = M > 0$ has at most two distinct solutions, both in $[0, \pi/2)$. Similarly, $g(x) = M < 0$ has at most two distinct solutions, both in $(\pi/2, \pi]$. Hence $g(x) = M$ can have three distinct solutions only for $M = 0$, and again we have $\{\theta_1, \theta_2, \theta_3\} = \{0, \pi/2, \pi\}$ so $\theta_1\theta_2\theta_3 = 0$. □

We now present a technical lemma that locates the roots of a particular polynomial. Its proof is tedious and computational, and is deferred until later.

Lemma 2. *Let $m \in \mathbb{N}$ be odd with $m \geq 3$. Set $f(x) = x^{m-2} - 2(2x^2 - 1)^{m-1}$, $x_1 = 1 - \frac{\ln 2}{3(m-1)}$, and $x_2 = 1 - \frac{\ln 2}{3m}$. Then $f(x)$ has two roots in $(0, 1)$: one root is $\frac{1}{2}$, and the second root is contained in the interval (x_1, x_2) because $f(x_1) > 0 > f(x_2)$.*

Now we are ready for the main result.

Theorem 3. *Let $m \in \mathbb{N}$ with $m \geq 2$, and set $f(x) = \cos^m(x)$. If m is even, then $f(x)$ is not maximal-ly surprising, i.e. there is no maximal-ly surprising triangle. If instead m is odd, then $f(x)$ is maximal-ly surprising. Further, there is exactly one maximal-ly surprising triangle, whose triangle sum $f(\theta_1) + f(\theta_2) + f(\theta_3)$ is within $\frac{0.88}{m-1}$ of the limiting value $2^{2/3} - 2^{-4/3} \approx 1.190550789$.*

Proof. We first consider the domain boundary, i.e. $\theta_1 + \theta_2 + \theta_3 = \pi$ in the first orthant. On that boundary, without loss of generality, $\theta_3 = 0$, so $\theta_2 = \pi - \theta_1$, and our triangle function is $f(0)+f(\theta_1)+f(\pi-\theta_1) = 1+\cos^m(\theta_1)+(-1)^m\cos^m(\theta_1)$. If m is even, then this is $1+2\cos^m(\theta_1)$, which has maximum 3, which is the global maximum since each of $f(\theta_i)$ achieves its maximum there. No interior maxima can beat this, so even m causes $f(x)$ to be not maximal-ly surprising. We assume henceforth that m is odd. Now, this function is constant 1 on the entire boundary, which we will show is not maximal.

We consider the Lagrangian $L = f(\theta_1)+f(\theta_2)+f(\theta_3)+\lambda(\theta_1+\theta_2+\theta_3-\pi)$, with gradient $\nabla L =$

$$(-m\cos^{m-1}(\theta_1)\sin(\theta_1)+\lambda, -m\cos^{m-1}(\theta_2)\sin(\theta_2)+\lambda, -m\cos^{m-1}(\theta_3)\sin(\theta_3)+\lambda).$$

Setting $\nabla L = 0$ and rearranging, we find that we need both $\theta_1+\theta_2+\theta_3 = \pi$ and $\cos^{m-1}(\theta_1)\sin(\theta_1) = \cos^{m-1}(\theta_2)\sin(\theta_2) = \cos^{m-1}(\theta_3)\sin(\theta_3)$. Applying Lemma 1, we find that if $\theta_1, \theta_2, \theta_3$ are distinct, then $\theta_1\theta_2\theta_3 = 0$, which is on the boundary.

Since we are looking for interior Lagrangian zeroes, we may assume without loss of generality that our angles are $\theta = \theta_1 = \theta_2$ and $\pi - 2\theta = \theta_3$. Set $x = \cos\theta$, and note that $\cos\theta_3 = -\cos(2\theta) = -(2\cos^2(\theta)-1) = -2x^2+1$. So now we have reduced our problem to finding zeroes of the polynomial $g(x) = 2x^m + (-2x^2+1)^m$. Since $0 \le \theta \le \frac{\pi}{2}$, we have $x = \cos\theta \in (0,1)$, where the endpoints are excluded since this would not be interior.

Now, $g'(x) = 2mx^{m-1} + (-4x)m(-2x^2+1)^{m-1} = 2mx(x^{m-2} - 2(-2x^2+1)^{m-1})$, whose zeroes coincide with the zeroes of $h(x) = x^{m-2} - 2(2x^2-1)^{m-1}$. We now apply Lemma 2 to $h(x)$, concluding that there are two zeroes: $x = \frac{1}{2}$, and another $x' \in (1 - \frac{\ln 2}{3(m-1)}, 1 - \frac{\ln 2}{3m})$.

Now, $x = \frac{1}{2}$ corresponds to an equilateral triangle, so this would not be maximal-ly surprising if it were maximal. However, it is not maximal, since the triangle function evaluates to $3(\frac{1}{2})^m < 1$, even less than on the boundary.

Turning now to the other zero of $h(x)$, x', we set $c = \frac{\ln 2}{3}$ and compute

$$\begin{aligned}
f(x') &= 2(x')^m + \left(1 - 2(x')^2\right)^m \\
&\le 2\left(1 - \frac{c}{m}\right)^m + \left(1 - 2\left(1 - \frac{c}{m-1}\right)^2\right)^m \\
&= 2\left(1 - \frac{c}{m}\right)^m - \left(1 - \frac{4c}{m-1} + \frac{2c^2}{(m-1)^2}\right)^m \\
&\le 2\left(1 - \frac{c}{m}\right)^m - \left(1 - \frac{4c}{m-1}\right)^m \\
&= 2\left(1 - \frac{c}{m}\right)^m - \left(1 - \frac{4c}{m-1}\right)^{m-1}\left(1 - \frac{4c}{m-1}\right) \\
&\le 2e^{-c} - e^{-4c - \frac{c^2}{2(m-1-c)}}\left(1 - \frac{4c}{m-1}\right),
\end{aligned} \tag{1}$$

where in the last step we used the standard bounds $e^{-c-\frac{c^2}{2(x-c)}} \le (1 - \frac{c}{x})^x \le e^{-c}$

(valid for $x > c$). In the other direction, we have

$$\begin{aligned} f(x') &= 2(x')^m + \left(1 - 2(x')^2\right)^m \\ &\geq 2\left(1 - \frac{c}{m-1}\right)^m + \left(1 - 2\left(1 - \frac{c}{m}\right)^2\right)^m \\ &= 2\left(1 - \frac{c}{m-1}\right)^{m-1}\left(1 - \frac{c}{m-1}\right) - \left(1 - \frac{4c}{m} + \frac{2c^2}{m^2}\right)^m \\ &\geq 2e^{-c-\frac{c^2}{2(m-1-c)}}\left(1 - \frac{c}{m-1}\right) - e^{-4c+\frac{2c^2}{m}}. \end{aligned} \tag{2}$$

Note that as $m \to \infty$, both upper and lower bounds approach $2e^{-c} - e^{-4c} = 2^{2/3} - 2^{-4/3}$. It only remains to estimate convergence rate.

We will bound the gap between the upper and lower bounds, which we call $E(m)$. We have

$$\begin{aligned} E(m) = 2e^{-c}&\left(1 - \left(1 - \frac{c}{m-1}\right)e^{-\frac{c^2}{2(m-1-c)}}\right) \\ &+ e^{-4c}\left(e^{\frac{2c^2}{m}} - \left(1 - \frac{4c}{m-1}\right)e^{-\frac{c^2}{2(m-1-c)}}\right) \end{aligned} \tag{3}$$

Set $x_1 = \frac{c^2}{2(m-1-c)}$ and $x_2 = \frac{2c^2}{m}$. Now, $x_1 = \frac{c^2}{2(m-1)}\left(1 - \frac{c}{m-1}\right)^{-1}$. Since $\left(1 - \frac{c}{m-1}\right)^{-1} \leq \left(1 - \frac{\log 2}{6}\right)^{-1} \leq 1.14$, so $x_1 \leq 1.14\frac{c^2}{2(m-1)}$. Also, $x_2 \leq \frac{2c^2}{m-1}$.

Since $m \geq 3$ and $c = \frac{\ln 2}{3}$ we have $x_1, x_2 \in (0, 0.05)$. We multiply $1 - e^{-x_1} \leq x_1$ on both sides by $1 - \frac{c}{m-1}$ and rearrange to find

$$1 - \left(1 - \frac{c}{m-1}\right)e^{-x_1} \leq \frac{c}{m-1}(1-x) + x_1 \leq \frac{c}{m-1} + 1.14\frac{c^2}{2(m-1)}.$$

We now multiply $1 - e^{-x_1} \leq x_1$ on both sides by $1 - \frac{4c}{m-1}$ and rearrange to find

$$1 - \left(1 - \frac{4c}{m-1}\right)e^{-x_1} \leq \frac{4c}{m-1}(1-x_1) + x_1,$$

and therefore

$$\begin{aligned} e^{x_2} - \left(1 - \frac{4c}{m-1}\right)e^{-x_1} &\leq \frac{4c}{m-1}(1-x_1) + x_1 + (e^{x_2} - 1) \\ &\leq \frac{4c}{m-1} + 1.14\frac{c^2}{2(m-1)} + 2x_2 \\ &\leq \frac{4c}{m-1} + 1.14\frac{c^2}{2(m-1)} + \frac{4c^2}{m-1}. \end{aligned} \tag{4}$$

Putting it all together, we find

$$E(m) \leq \frac{2e^{-c}(c + 0.57c^2) + e^{-4c}(4c + 0.57c^2 + 4c^2)}{m-1} \leq \frac{0.88}{m-1}.$$

□

We close by inviting the reader to look for other maximal-ly surprising triangle functions, which may provide other magical values, like $2^{2/3} - 2^{-4/3}$.

References

[1] Range of sum of cosines of angles of a triangle using vectors?, published 9/28/2016, `https://math.stackexchange.com/q/1944727`.

1 Appendix: Proof of Lemma 2

Lastly, we turn to the proof of Lemma 2. Surely there are better proofs than the one which follows, but after many attempts this is the nicest we could find.

Proof of Lemma 2. We have $f(x) \geq 0$ exactly when $x^{m-2} \geq 2(2x^2-1)^{m-1}$. Provided $x \neq \frac{1}{\sqrt{2}}$ (and since $m-1$ is even), this is equivalent to $\frac{x^{m-2}}{2(2x^2-1)^{m-1}} \geq 1$. Hence, provided $x \neq \frac{1}{\sqrt{2}}$, $f(x)$ has the same sign as $g(x) = \ln \frac{x^{m-2}}{2(2x^2-1)^{m-1}} = (m-2)\ln x - \ln 2 - (m-1)\ln(2x^2-1)$.

Next, we prove that $g(x)$ is monotone in x. Taking its derivative, we get $\frac{dg(x)}{dx} = \frac{m-2}{x} - \frac{4x(m-1)}{2x^2-1} = \frac{(m-2)(2x^2-1)-4x^2(m-1)}{x(2x^2-1)} = \frac{-2mx^2-(m-2)}{x(2x^2-1)}$. If $x \in (0, \frac{1}{\sqrt{2}})$, then both numerator and denominator are negative so $g(x) > 0$. If instead $x \in (\frac{1}{\sqrt{2}}, 1)$, then the numerator is negative and the denominator is positive so $g(x) < 0$. Hence $f(x)$ is unimodular, increasing from $f(0) = -2$ up to a (positive) maximum, then decreasing down to $f(1) = -1$, and therefore has two roots. One root is easily found as $\frac{1}{2}$, since $f(\frac{1}{2}) = 2^{-m+2} - 2^{-1}(-2^{-1})^{m-1} = 0$ (since m is odd). The theorem will be complete once we find the other root, by proving that $f(x_1) > 0 > f(x_2)$. To do this, we will need the well-known bounds $-u - u^2 \leq \ln(1-u) < -u$, which hold for all $u \in (0, \frac{1}{2})$. Note that since $\frac{\ln 2}{3m} < \frac{\ln 2}{3(m-1)} \leq \frac{\ln 2}{6} < \frac{1}{8}$, each of $1 - x_1, 1 - x_2, 1 - (2x_1^2 - 1), 1 - (2x_2^2 - 1)$ lie in $(0, \frac{1}{2})$. Set $\delta_i = 1 - x_i$ for convenience.

Now we will prove that $g(x_1) > 0$, by calculating

$$\begin{aligned} g(x_1) &= (m-2)\ln(1-\delta_1) - \ln 2 - (m-1)\ln(1-(4\delta_1 - 2\delta_1^2)) \\ &\geq -\ln 2 + (m-2)(-\delta_1 - \delta_1^2) + (m-1)(4\delta_1 - 2\delta_1^2) \\ &= -\ln 2 + \delta_1(3m-2) + \delta_1^2(-3m+4) \\ &= \frac{-(\ln 2)(3m-3)}{3m-3} + \frac{(\ln 2)(3m-2)}{3m-3} + \delta_1^2(-3m+4) \\ &= \frac{\ln 2}{3m-3} + \delta_1^2(-3m+4) \\ &= \delta_1 + \delta_1^2(-3m+4) \\ &= \delta_1\left(\frac{3m-3}{3m-3} + \frac{(\ln 2)(-3m+4)}{3m-3}\right) \\ &= \frac{\delta_1}{3m-3}(3m(1-\ln 2) + (4\ln 2 - 3)) \\ &> 0, \end{aligned} \tag{5}$$

where the final inequality follows since $3m(1-\ln 2) \geq 9(1-\ln 2) \approx 2.76$, while $4\ln 2 - 3 \approx -0.23$.

To prove that $g(x_2) < 0$, we will recall that $x_2 = 1 - \frac{\ln 2}{3m}$ and instead treat $g(x_2) := h(m)$ as a function in m. We will prove $h(3) < 0$, $\lim_{m\to\infty} h(m) = 0$, and $h(m)$ is increasing on $(3,\infty)$. This will prove that $h(m)$ is negative on $(3,\infty)$, so in particular it is negative at the desired m.

Taking $m = 3$, we compute $g(1 - \frac{\ln 2}{9}) = h(3) \approx -0.071 < 0$. For large m, we note that $\delta_2 = \frac{\ln 2}{3m} \to 0$, and compute

$$\begin{aligned} h(m) &= (m-2)\ln(1-\delta_2) - \ln 2 - (m-1)\ln(1 - 4\delta_2 + 2\delta_2^2) \\ &= -(m-2)\delta_2 + O(\delta_2^2) - \ln 2 + 4(m-1)\delta_2 + O(\delta_2^2) \\ &= (3m-2)\delta_2 - \ln 2 + O(\delta_2^2) \\ &= \frac{(3m-2)(\ln 2)}{3m} - \ln 2 + O(\delta_2^2) \\ &= \frac{-2\ln 2}{3m} + O(\delta_2^2). \end{aligned} \tag{6}$$

This proves that $\lim_{m\to\infty} h(m) = 0$, as desired. Next, we need the bound $\ln t \geq \frac{2(t-1)}{t+1}$, which holds for all $t \geq 1$ with equality only for $t = 1$. This is proved because by rearranging $(t-1)^2 \geq 0$ to $t^2 + 2t + 1 \geq 4t$ and hence $\frac{1}{t} \geq \frac{4}{(t+1)^2}$, so the function $\ln t - \frac{2(t-1)}{t+1}$ is 0 at $t = 1$ and has positive derivative in $(1,\infty)$.

We set $y_2 = 2x_2^2 - 1$. An easy calculus exercise shows that $2x^2 - x - 1 < 0$ on $(0,1)$, so $x_2 > y_2$. We calculate $x_2' = \frac{\ln 2}{3m^2} = \frac{\delta_2}{m}$, $y_2' = 4x_2x_2' = \frac{4x_2\delta_2}{m}$. Recalling that $h(m) = (m-2)\ln(x_2) - \ln 2 - (m-1)\ln(y_2)$, we are now ready to calculate

$$
\begin{aligned}
h'(m) &= \ln x_2 + \frac{m-2}{x_2} x_2' - \ln y_2 - \frac{m-1}{y_2} y_2' \\
&= \ln(x_2/y_2) + \frac{(m-2)x_2'}{x_2} - \frac{4(m-1)x_2 x_2'}{y_2} \\
&\geq \frac{2(x_2-y_2)}{x_2+y_2} + \frac{(m-2)x_2'}{x_2} - \frac{4(m-1)x_2 x_2'}{y_2},
\end{aligned} \tag{7}
$$

where in the last step we used the bound $\ln t \geq \frac{2(t-1)}{t+1}$ with $t = \frac{x_2}{y_2} > 1$. We continue as

$$
\begin{aligned}
&\frac{2(x_2-y_2)}{x_2+y_2} + \frac{(m-2)x_2'}{x_2} - \frac{4(m-1)x_2 x_2'}{y_2} \\
&= \frac{1}{(x_2+y_2)x_2 y_2} \left(2(x_2-y_2)x_2 y_2 + (m-2)x_2'(x_2+y_2)y_2 - 4(m-1)x_2 x_2'(x_2+y_2)x_2\right) \\
&= \frac{1}{(x_2+y_2)x_2 y_2 m} \big(2(x_2-y_2)x_2 y_2 m + (m-2)\delta_2(x_2+y_2)y_2 \\
&\qquad\qquad\qquad - 4(m-1)x_2\delta_2(x_2+y_2)x_2\big) \\
&= \frac{1}{(x_2+y_2)x_2 y_2 m} p(\delta_2),
\end{aligned} \tag{8}
$$

where

$$
\begin{aligned}
p(\delta_2) &= 2(x_2-y_2)x_2 y_2 m + (m-2)\delta_2(x_2+y_2)y_2 - 4(m-1)x_2\delta_2(x_2+y_2)x_2 \\
&= \delta_2 \left(\frac{\ln 2}{3}(4\delta_2^3 - 18\delta_2^2 + 26\delta_2 - 11) + 4\delta_2^2 - 10\delta_2 + 4\right),
\end{aligned} \tag{9}
$$

where we use $x_2 = 1 - \delta_2, y_2 = 1 - 4\delta_2 + 2\delta_2^2$, and $m = \frac{\ln 2}{3}\frac{1}{\delta_2}$ and simplify. Now δ_2 lies in $(0, \frac{\ln(2)}{9})$, and it is a routine calculus exercise to prove that cubic polynomial $q(\delta_2) := \frac{p(\delta_2)}{\delta_2}$ is decreasing on the interval $(0, \frac{\ln(2)}{9})$, and hence is bounded below by $q(\frac{\ln(2)}{9}) \approx 1.15$. In particular, $p(\delta_2) > 0$, so $h'(m) > 0$. Since $h(3) < 0$ and $\lim_{m\to\infty} h(m) = 0$, in fact $h(m) < 0$ on $(3, \infty)$, and hence $g(x_2) < 0$ and $f(x_2) < 0$, as desired.

□