Pollock's Generalized Tetrahedral Numbers Conjecture

The nth tetrahedral number $Te_n = \binom{n+2}{3}$ represents the sum of the first n triangular numbers. In the song "The Twelve Days of Christmas," Te_n counts the total number of gifts received after day n.

A longstanding conjecture of Pollock (from [**4**]) is that every positive integer may be expressed as the sum of at most five tetrahedral numbers. To date, only 241 positive integers have been found requiring five tetrahedral numbers (see [**3**]). Recently, progress has been made (in [**1**]) on a related conjecture of Pollock from the same 19th century paper.

Here we instead consider generalized tetrahedral numbers $Te_n = \frac{(n+2)(n+1)n}{6}$, defined for all integers n. These are the generalized binomial coefficients $\binom{n+2}{3}$, as popularized in [**2**]. With these we can prove the following.

Theorem. *Every integer may be expressed as the sum of at most four generalized tetrahedral numbers.*

Proof. For arbitrary $n \in \mathbb{Z}$, we have $Te_n + Te_{n-2} + Te_{-n-1} + Te_{-n-1} = \frac{1}{6}((n+2)(n+1)n + n(n-1)(n-2) + 2(-n+1)(-n)(-n-1)) = n$. ■

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