Wolstenholme's Congruence for Lucas sequences

The recent paper [**1**] in this Monthly showed that $\sum_{k=1}^{p-1} \frac{1}{F_k^2} \equiv 0 \pmod{F_p}$ for any odd prime p, a variation of a well-known result of Wolstenholme. Below we show a similar result involving both Fibonacci numbers and Lucas numbers. Recall that the Lucas sequence L_n has the same recursive relationship as the Fibonacci sequence, but with different starting values, namely $L_1 = 1$, $L_2 = 3$. Here the statement $a/b \equiv 0 \pmod{n}$ means that $\gcd(b, n) = 1$ and $a \equiv 0 \pmod{n}$.

Theorem. *If p is an odd prime, then* $\sum_{k=1}^{p-1} \frac{1}{L_k^2} \equiv 0 \pmod{F_p}$.

Proof. We rewrite the sum as follows:

$$\sum_{k=1}^{p-1} \frac{1}{L_k^2} = \sum_{k=1}^{\frac{p-1}{2}} \left(\frac{1}{L_k^2} + \frac{1}{L_{p-k}^2} \right) = \sum_{k=1}^{\frac{p-1}{2}} \frac{L_k^2 + L_{p-k}^2}{L_k^2 L_{p-k}^2}.$$

It suffices to show that each term of this sum is congruent to $0 \pmod{F_p}$. Let $1 \le k \le \frac{p-1}{2}$. We first show that each denominator is coprime to F_p. Since p is an odd prime, we have that $\gcd(2k, p) = 1$. By strong divisibility of the Fibonacci sequence, we get $\gcd(F_{2k}, F_p) = F_1 = 1$. Applying the identity $F_{2k} = F_k L_k$ then yields $\gcd(L_k, F_p) = 1$; hence $\gcd(L_k^2 L_{p-k}^2, F_p) = 1$ as required.

We now show that $L_k^2 + L_{p-k}^2 \equiv 0 \pmod{F_p}$. Applying the identity $L_n = F_{n+1} + F_{n-1}$ gives

$$\begin{aligned} L_k^2 + L_{p-k}^2 &= (F_{k+1} + F_{k-1})^2 + (F_{p-k-1} + F_{p-k+1})^2 \\ &= (F_{k+1}^2 + F_{p-(k+1)}^2) + (F_{k-1}^2 + F_{p-(k-1)}^2) \\ &\quad + 2(F_{k+1}F_{k-1} + F_{p-k-1}F_{p-k+1}). \end{aligned} \tag{1}$$

Applying Cassini's identity, $F_{n-1}F_{n+1} = (-1)^n + F_n^2$, to the final term gives

$$F_{k+1}F_{k-1} + F_{p-k-1}F_{p-k+1} = (F_k^2 + F_{p-k}^2) + ((-1)^k + (-1)^{p-k}). \tag{2}$$

Since p is odd, it follows from Catalan's identity that $F_n^2 + F_{p-n}^2 = F_p F_{p-2n}$ for all $n \in \mathbb{N}$. This implies that the first two terms of (1), as well as the first term of the right-hand side of (2), are congruent to $0 \pmod{F_p}$. Therefore $L_k^2 + L_{p-k}^2 \equiv 2((-1)^k + (-1)^{p-k}) \equiv 2(-1)^k(1 + (-1)^p) = 0 \pmod{F_p}$.

■

REFERENCES

1. B. He, Y. L. Mao & A. Togbé. (2023). A Fibonacci Version of Wolstenholme's Harmonic Series Congruence, Amer. Math. Monthly, 130:1, 83-85, DOI: 10.1080/00029890.2022.2128165

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