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Complete Linear Fractional Trees

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Double Blind No Name1 and Double Blind No Name2

Abstract. We characterize all complete trees of rationals on real intervals that arise via matrix multiplication.

Keywords: Calkin-Wilf tree, Kepler tree, Stern-Brocot tree, Farey sequence

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1. INTRODUCTION In 1619, Kepler published his third law of planetary motion [1], and in the same work described a binary tree with root $1/2$ and branches specified by $a/b \mapsto \{a/(a+b), b/(a+b)\}$. This turns out to produce only reduced fractions and, further, contains every rational in the interval $(0,1)$. In the four centuries since, many other trees of rationals have been proposed and studied.

Another early tree was independently discovered by both Stern and Brocot [2, 3] in the late 19th century. They built their tree using repeated mediants $a/b \oplus c/d = (a+c)/(b+d)$. It too contains all reduced fractions, this time in $(0,\infty)$. This has had many applications, such as the Minkowski question mark function [4]. This tree continues to draw considerable attention (e.g., [5, 6, 7]).

Another famous tree is the Calkin-Wilf tree, appearing famously in this Monthly [8]. It is built with root $1/1$ and branches specified by $a/b \mapsto \{a/(a+b), (a+b)/b\}$. Like the Stern-Brocot tree, it contains all reduced fractions in $(0,\infty)$. This too has many applications, such as Stern's diatomic sequence [9]. It too continues to draw considerable attention (e.g., [10, 11, 12]).

The recent [13] presented the new Parsanian-Shaheen tree. Its root is $1/2$ and branches specified by $a/b \mapsto \{a/(a+b), b/(2b-a)\}$. It was proved to contain all reduced fractions in $(0,1)$. This motivated us to discover a new tree, with root $1/2$ and branches specified by $a/b \mapsto \{(b-a)/(2b-a), b/(a+b)\}$. We proved this contains all reduced fractions in $(0,1)$ but motivated a more general question: What are all the possible complete trees of rationals, of this type?

To be specific about "this type," consider an integer matrix $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. We work with extended rationals $\mathbb{Q}_\infty = \mathbb{Q} \cup \{+\infty, -\infty\}$. For each, we fix a normal form u/v where $u, v \in \mathbb{Z}$, $\gcd(u,v) = 1$, $v \geq 0$. Here we have $+\infty = \frac{1}{0}$ and $-\infty = \frac{-1}{0}$. We will use $[u/v]$ to denote the column vector $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$, for u/v in normal form, a vector in the upper half-plane of $\mathbb{Z}^2$. We consider trees whose branches are built by the action of multiplication by M. For example, the Kepler tree has branches specified by $\{\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]\}$.

If $|ad - bc| = 1$, then this matrix action $M\,[u/v]$ induces a natural linear fractional transformation on $\mathbb{Q}_\infty$, provided u/v is in normal form, via

$$M(u/v) = \frac{a(u/v)+b}{c(u/v)+d} = \frac{au+bv}{cu+dv} = \frac{-(au+bv)}{-(cu+dv)}.$$

Note that if $\frac{u}{v}$ is in normal form, then so is either $\frac{au+bv}{cu+dv}$ or $\frac{-(au+bv)}{-(cu+dv)}$, by Lemma

1, below. Linear fractional transformations were known to Ptolemy in 150 CE, and continue to be popular, with many applications (e.g., [14, 15, 16]).

Suppose we fix $x, r, y \in \mathbb{Q}_\infty$ with $x < r < y$, and we have two linear fractional transformations L, R satisfying $L : [x, y] \to [x, r]$ and $R : [x, y] \to [r, y]$, surjective on $[x, r]$ and $[r, y]$ respectively. Then $\{L, R\}$ induces a linear fractional tree on the interval (x, y), whose root is r and each node has two children, reached via L and R. We say this induced linear fractional rational tree is complete if its nodes include each rational in (x, y) exactly once. Our goal in this paper is to find all such complete (linear fractional) trees.

We restrict to linear fractional transformations because they preserve primitivity (Lemma 1) and removing the determinant restriction cannot yield complete trees (Lemma 2).

Lemma 1. *Let* $a, b, c, d, u, v \in \mathbb{Z}$ *with* $|ad - bc| = 1$. *Then* $\gcd(u, v) = \gcd(au + bv, cu + dv)$.

Proof. First we note that $\gcd(u, v) | \gcd(au + bv, cu + dv)$, as it divides each term. Then, apply this to the inverse transformation $(a', b', c', d') = e(d, -b, -c, a)$, where $e = \frac{1}{ad-bc} = \pm 1$, to get $\gcd(au + bv, cu + dv) | \gcd(a'(au + bv) + b'(cu + dv), c'(au + bv) + d'(cu + dv)) = \gcd(u, v)$. ∎

Lemma 2. *Let* $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ *be an integer matrix with determinant* δ *satisfying* $\delta \neq \pm 1$. *Let* $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right], y = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] \in \mathbb{Z}^2$, *satisfying* $\gcd(x_1, x_2) = 1 = \gcd(y_1, y_2)$ *and* $\frac{x_1}{x_2} < \frac{y_1}{y_2}$. *Set* $x' = Mx = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$, $y' = My = \begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix}$. *Suppose* $\frac{x'_1}{x'_2} \neq \frac{y'_1}{y'_2}$. *Then there is an integer vector* $z = \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$ *where* $\gcd(z_1, z_2) = 1$, $\frac{z_1}{z_2}$ *is strictly between* $\frac{x'_1}{x'_2}$ *and* $\frac{y'_1}{y'_2}$, *and* $M^{-1}z \notin \mathbb{Z}^2$.

Proof. First note that if $\delta = 0$ then $\frac{x'_1}{x'_2} = \frac{y'_1}{y'_2}$, so we may assume that $|\delta| \geq 2$. If each of $d/\delta, -c/\delta, -b/\delta, a/\delta$ were integers, then $ad - bc$ would be a multiple of δ^2, hence $|\delta| = 1$, which has been excluded. So choose either $r = M^{-1}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = \begin{bmatrix} d/\delta \\ -c/\delta \end{bmatrix}$ or $r = M^{-1}\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] = \begin{bmatrix} -b/\delta \\ a/\delta \end{bmatrix}$, to ensure $r \notin \mathbb{Z}^2$. Now, set $s = K(x' + y') + r = \left[\begin{smallmatrix} s_1 \\ s_2 \end{smallmatrix}\right]$, where K is a large enough integer to make $\frac{s_1}{s_2}$ strictly between $\frac{x'_1}{x'_2}$ and $\frac{y'_1}{y'_2}$. Now $M^{-1}s = KM^{-1}(x' + y') + M^{-1}r = Kx + Ky + M^{-1}r \notin \mathbb{Z}^2$. Lastly, the desired $z = \begin{bmatrix} s_1/\gcd(s_1,s_2) \\ s_2/\gcd(s_1,s_2) \end{bmatrix}$. ∎

The Kepler tree, Calkin-Wilf tree, Parsanian-Shaheen tree, and the new tree we found, are each complete linear fractional trees. The Stern-Brocot tree is not, since it is built using a different operation. Others (e.g., [17]) have looked at similar trees without the determinant condition; these are not complete due to Lemma 2.

In this paper, we will characterize all complete linear fractional trees, including all four of the above examples, as belonging to one specific family (Farey Neighbor Trees) which we will fully describe. We will also describe which intervals (x, y) – such as $(0, 1)$ and $(0, \infty)$ – admit a complete linear fractional tree. Further, for each such interval, we will prove there are exactly four complete linear fractional trees.

2. FAREY NEIGHBOR TREES

The Farey sequence of rationals first appeared in Farey's [18], received the attention of Cauchy, and has been popular ever since (e.g., [19, 20]). They are related to the Stern-Brocot tree, but our interest is specifically in

Farey neighbors, namely adjacent rationals in the Farey sequence. These are studied in [21, 22], and are characterized as reduced fractions $a/b, c/d$ where $|ad-bc|=1$.

Let $x,y\in\mathbb{Q}_\infty$ have normal forms $x=\frac{x_1}{x_2}, y=\frac{y_1}{y_2}$. Suppose that $x<y$ and x,y are Farey neighbors, i.e., $|x_1y_2-x_2y_1|=1$. We rearrange $\frac{x_1}{x_2}<\frac{y_1}{y_2}$ to $x_1y_2-x_2y_1<0$, so in fact $x_1y_2-x_2y_1=-1$. Note that x,∞ are Farey neighbors if and only if $x\in\mathbb{Z}$ (similarly for $-\infty,x$). Define $[xy]=\left[\begin{smallmatrix}x_1&y_1\\x_2&y_2\end{smallmatrix}\right]$. Note that $\det[xy]=-1$, so $[xy]^{-1}=\left[\begin{smallmatrix}-y_2&y_1\\x_2&-x_1\end{smallmatrix}\right]$. In this context, we define four matrices, as follows.

$$L^+(x,y)=[xy]\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right][xy]^{-1}=\left[\begin{smallmatrix}x_1x_2+1&-x_1^2\\x_2^2&1-x_1x_2\end{smallmatrix}\right]$$

$$L^-(x,y)=[xy]\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right][xy]^{-1}=\left[\begin{smallmatrix}x_1x_2-x_1y_2-y_1y_2&y_1^2+x_1y_1-x_1^2\\x_2^2-x_2y_2-y_2^2&y_1y_2+x_2y_1-x_1x_2\end{smallmatrix}\right]$$

$$R^+(x,y)=[xy]\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right][xy]^{-1}=\left[\begin{smallmatrix}1-y_1y_2&y_1^2\\-y_2^2&1+y_1y_2\end{smallmatrix}\right]$$

$$R^-(x,y)=[xy]\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right][xy]^{-1}=\left[\begin{smallmatrix}x_1x_2+x_2y_1-y_1y_2&y_1^2-x_1y_1-x_1^2\\x_2^2+x_2y_2-y_2^2&y_1y_2-x_1y_2-x_1x_2\end{smallmatrix}\right]$$

Each of these matrices induces a natural linear fractional transformation on $\mathbb{Q}_\infty$. We recall the mediant $x\oplus y=\frac{x_1+y_1}{x_2+y_2}$. Provided x,y are Farey neighbors, this is in normal form and $x<x\oplus y<y$.

We now collect several basic properties of these matrices.

Proposition 3. *Let $x,y\in\mathbb{Q}_\infty$, where $x<y$ and x,y are Farey neighbors. Then*

1. $|L^+(x,y)|=|R^+(x,y)|=1$*; while* $|L^-(x,y)|=|R^-(x,y)|=-1$.
2. $L^\pm(x,y):(x,y)\to(x,x\oplus y)$*, as surjective linear fractional transformations.*
3. $R^\pm(x,y):(x,y)\to(x\oplus y,y)$*, as surjective linear fractional transformations.*
4. *Linear fractional transformations* $L^+(x,y),R^+(x,y)$ *preserve orientation while* $L^-(x,y),R^-(x,y)$ *reverse it.*

Proof. (1): immediate, since $\det[xy]=-1$.
(2), (3), (4): Note that since $[xy]^{-1}[xy]=\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$, we have $[xy]^{-1}[x]=\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $[xy]^{-1}[y]=\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Hence $L^+(x,y)[x]=[xy]\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]=[x]$, $L^+(x,y)[y]=[xy]\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]=[x\oplus y]$, $L^-(x,y)[x]=[xy]\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]=[x\oplus y]$, $L^-(x,y)[y]=[xy]\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]=[x]$. Similarly, $R^+(x,y)[x]=[xy]\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]=[x\oplus y]$, $R^+(x,y)[y]=[xy]\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]=[y]$, $R^-(x,y)[x]=[xy]\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]=[y]$, $R^-(x,y)[y]=[xy]\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]=[x\oplus y]$. Surjectivity follows from continuity. ■

We now present our main theorem, that each pair of Farey neighbors induces four complete linear fractional trees. We observe that this includes the Kepler tree $\{L^+(0,1),R^-(0,1)\}$, the Calkin-Wilf tree $\{L^+(0,\infty),R^+(0,\infty)\}$, the Parsanian-Shaheen tree $\{L^+(0,1),R^+(0,1)\}$, and our new tree $\{L^-(0,1),R^-(0,1)\}$. For completeness, the fourth Farey neighbor tree on $(0,1)$ is $\{L^-(0,1),R^+(0,1)\}$.

Theorem 4. *Let $x,y\in\mathbb{Q}_\infty$, where $x<y$ and x,y are Farey neighbors. Choose $L\in\{L^+(x,y),L^-(x,y)\}$ and $R\in\{R^+(x,y),R^-(x,y)\}$. Then $\{L,R\}$ induces a complete linear fractional tree on $[x,y]$ with root $r=x\oplus y$.*

Proof. Write $[xy]=\left[\begin{smallmatrix}x_1&y_1\\x_2&y_2\end{smallmatrix}\right]$. Since $\det([xy])=-1$, $[xy]^{-1}$ is an integer matrix.

We will use this fact to establish a bijection between pairs of coprime positive integers and $\mathbb{Q}_\infty\cap(x,y)$. For $z\in(x,y)$ with normal form $z=z_1/z_2$, we define its corresponding vector $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ as $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = [xy]^{-1}[z]$. Hence $a = y_1 z_2 - y_2 z_1$ and $b = x_2 z_1 - x_1 z_2$. Because $x < z$, $x_1 z_2 - x_2 z_1 < 0$, so $b > 0$. Similarly, since $z < y$, $z_1 y_2 - z_2 y_1 < 0$, so $a > 0$. Hence a, b are each positive integers. Since $[xy]^{-1}$ is unimodular, $\gcd(a, b) = \gcd(z_1, z_2) = 1$. For the reverse direction, $[z] = [xy]\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$. Since x, y are in normal form, $x_2, y_2 \geq 0$. The determinant condition ensures they are not both zero, so $z_2 = x_2 a + y_2 b$ is strictly positive. Also, since $[xy]$ is unimodular, $\gcd(z_1, z_2) = \gcd(a, b) = 1$, so $[xy]\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ produces the normal form of z.

Under this bijection, the root r corresponds to $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, since $[xy]\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_1+y_1 \\ x_2+y_2 \end{smallmatrix}\right] = [x] + [y]$. The transformations L, R act on the $\{[x], [y]\}$ basis through their inner matrices. Let M_L, M_R be the corresponding inner matrices in $GL(2, \mathbb{Z})$, where $M_L \in \{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]\}$ and $M_R \in \{\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]\}$. Applying L or R to a fraction z is equivalent to left-multiplying its coordinate vector $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ by M_L or M_R.

The theorem is now reduced to proving that the binary tree rooted at $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, generated by M_L and M_R, forms a bijection with the set of all coprime positive integer pairs. We prove this by strong induction on the sum $a + b$. The base case $a + b = 2$ corresponds to $a = b = 1$, which is the root of the tree. Now suppose that (a, b) are a coprime pair of positive integers with $a + b > 2$. Note that $a \neq b$ since they are coprime, so we have two cases.

If $a > b$, the pair must be a left child, and we find $M_L^{-1}\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \{\left[\begin{smallmatrix} a-b \\ b \end{smallmatrix}\right], \left[\begin{smallmatrix} b \\ a-b \end{smallmatrix}\right]\}$. For both options, since $a > b > 0$, the coordinates are positive with sum less than $a + b$. They are coprime since M_L^{-1} is unimodular, so by the inductive hypothesis appears (uniquely) in the binary tree.

If instead $a < b$, the pair must be a right child, and we find $M_R^{-1}\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \{\left[\begin{smallmatrix} a \\ b-a \end{smallmatrix}\right], \left[\begin{smallmatrix} b-a \\ a \end{smallmatrix}\right]\}$. Since $b > a > 0$, the coordinates are positive, and have sum less than $a + b$. As with $a > b$, the coordinates are coprime and we again apply the inductive hypothesis.

Since for each (a, b) coprime, positive, with $a + b > 2$, exactly one of $\{a < b, b < a\}$ must hold, we have a unique path from $(1, 1)$ to (a, b), and hence a unique path from $[r]$ to $[z]$, for any $z \in (x, y) \cap \mathbb{Q}_\infty$. ■

3. COMPLETE LINEAR FRACTIONAL TREES We now offer a converse to Theorem 4. Suppose $x, y \in \mathbb{Q}_\infty$ and we have a complete (binary) tree on the interval (x, y) with root $r \in (x, y)$ via some L, R expressible as matrix multiplication. Lemma 2 requires L, R to be unimodular. Our converse will prove that $r = x \oplus y$, that x, y are Farey neighbors, and that our tree is one of our Farey neighbor trees.

Theorem 5. *Let $x, r, y \in \mathbb{Q}_\infty$ with $x < r < y$. Suppose we have $L : (x, y) \to (x, r)$ and $R : (x, y) \to (r, y)$, surjective maps expressible as 2×2 unimodular integer matrices. Then $r = x \oplus y$ and $\det([xy]) = -1$; i.e., x, y are Farey neighbors. Also, $L \in \{L^+(x, y), L^-(x, y)\}$ and $R \in \{R^+(x, y), R^-(x, y)\}$.*

Proof. Set $D = \det([xy]), D_1 = \det([xr]), D_2 = \det([ry])$. Since $x < r < y$, each of D, D_1, D_2 are negative integers. Since $D \neq 0$, vectors $[x], [y]$ form a basis for $\mathbb{R}^2$, and we can uniquely express $r = ax + by$. Using determinant properties, we find $D_1 = \det([xr]) = \det([x\ ax + by]) = b\det([xy]) = bD$. Since $D, D_1 < 0$, we find $b > 0$. Similarly, $D_2 = \det([ry]) = \det([ax + by\ y]) = a\det([xy]) = aD$ shows that $a > 0$.

We now compute $\det(L)D = \det(L[xy]) \in \{\det([xr]), \det([rx])\} = \{D_1, -D_1\} = \pm bD$. Since L is unimodular and $b > 0$, this forces $b = 1 = \pm\det(L)$. Similarly, $\det(R)D = \det(R[xy]) \in \{\det([ry]), \det([yr])\} = \{D_2, -D_2\} = \pm aD$, so $a = 1 = \pm\det(R)$. Combining $r = ax + by = 1x + 1y = x \oplus y$.

Since linear fractional transformation L maps (x, y) to (x, r) continuously without passing through infinity, it must map the positive cone generated by $[x]$ and $[y]$ to either the positive or negative cone generated by $[x]$ and $[r]$. Replacing L by $-L$ if necessary (which induces the same linear fractional transformation), we assume L maps the positive cone to the positive cone.

If L is orientation-preserving, it maps the ray of $[x]$ to the ray of $[x]$, and the ray of $[y]$ to the ray of $[r]$. Hence $L[x] = c_1[x]$ and $L[y] = c_2[r] = c_2([x] + [y])$ for some $c_1, c_2 > 0$. The matrix of L relative to the basis $\{[x], [y]\}$ is $\left[\begin{smallmatrix} c_1 & c_2 \\ 0 & c_2 \end{smallmatrix}\right]$, which has determinant c_1c_2. Because L and the change-of-basis matrix $[xy]$ are both unimodular integer matrices, this relative matrix must also be a unimodular integer matrix. Since $c_1, c_2 > 0$, in fact $c_1 = c_2 = 1$. Thus $L[x] = [x]$ and $L[y] = [x] + [y]$. Changing back to the standard basis gives $L = [xy] \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] [xy]^{-1} = L^+(x, y)$.

If L is orientation-reversing, $L[x] = c_1([x] + [y])$ and $L[y] = c_2[x]$ with $c_1, c_2 > 0$. The matrix of L in the $\{[x], [y]\}$ basis is $\left[\begin{smallmatrix} c_1 & c_2 \\ c_1 & 0 \end{smallmatrix}\right]$, which has determinant $-c_1c_2$. Since $c_1, c_2 > 0$, in fact $c_1 = c_2 = 1$. Hence $L = [xy] \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right] [xy]^{-1} = L^-(x, y)$. By similar reasoning, if R is orientation-preserving, its relative matrix is $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$, yielding $R = R^+(x, y)$; if instead R is orientation-reversing, its relative matrix is $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, yielding $R = R^-(x, y)$.

As shown, if L is orientation-preserving, $L[x] = [x]$ and $L[y] = [x] + [y]$. Setting $A = L - I$, we have $A[x] = 0, A[y] = [x]$. We compute A explicitly using $[xy]^{-1}$ as

$$A = [0\ x] \frac{1}{D} \left[\begin{smallmatrix} y_2 & -y_1 \\ -x_2 & x_1 \end{smallmatrix}\right] = \frac{1}{D} \left[\begin{smallmatrix} -x_1x_2 & x_1^2 \\ -x_2^2 & x_1x_2 \end{smallmatrix}\right].$$

Since A is an integer matrix, D must divide all entries, including the coprime $x_1^2, -x_2^2$. Hence $|D| = 1$.

Similarly, if R is orientation-preserving, $R[x] = [x] + [y]$ and $R[y] = [y]$. Setting $B = R - I$, we have $B[x] = [y], B[y] = 0$. We compute B explicitly using $[xy]^{-1}$ as

$$B = [y\ 0] \frac{1}{D} \left[\begin{smallmatrix} y_2 & -y_1 \\ -x_2 & x_1 \end{smallmatrix}\right] = \frac{1}{D} \left[\begin{smallmatrix} -y_1y_2 & -y_1^2 \\ -y_2^2 & -y_1y_2 \end{smallmatrix}\right].$$

Since B is an integer matrix, D must divide all entries, including the coprime $-y_1^2, -y_2^2$. Hence $|D| = 1$.

Lastly we consider the case where L, R are both orientation-reversing, i.e., $L[x] = [x] + [y], L[y] = [x], R[x] = [y], R[y] = [x] + [y]$. Now, set $C = LR - I$; we have $C[x] = 0, C[y] = 2[x]$. We compute C explicitly using $[xy]^{-1}$ as

$$C = [0\ 2x] \frac{1}{D} \left[\begin{smallmatrix} y_2 & -y_1 \\ -x_2 & x_1 \end{smallmatrix}\right] = \frac{2}{D} \left[\begin{smallmatrix} -x_1x_2 & x_1^2 \\ -x_2^2 & -x_1x_2 \end{smallmatrix}\right].$$

Since C is an integer matrix, D must divide $2x_1^2$ and $-2x_2^2$; hence $|D| \in \{1, 2\}$. Suppose, by way of contradiction, that $|D| = 2$. We look at entries of the integer matrix $L = [x + y\ x]M^{-1}$. We have $L_{12} = \frac{x_1^2 - x_1y_1 - y_1^2}{D}$ and $L_{21} = \frac{x_2y_2 + y_2^2 - x_2^2}{D}$. Since $L_{21} \in \mathbb{Z}$, we must have x_2, y_2 both even. Since x, y are in normal form, this means that x_1, y_1 must both be odd. But now $L_{12} \notin \mathbb{Z}$, a contradiction.

Hence $|D| \neq 2$, so $|D| = 1$. ■

Note in particular that Theorem 5 implies (by checking the determinant condition) that it is not possible to have a complete tree on interval (x, y) where $x < 0 < y$, including one on $(-\infty, +\infty)$.

We conjecture that no complete ternary (or n-ary) tree of this type is possible either. The question of complete trees not built from this type of matrix multiplication is left open.

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