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REDUCTION OF JUMP SYSTEMS

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Abstract. A jump system is a set of integer lattice points satisfying an exchange axiom. We discuss an operation on lattice points, called reduction, that preserves the jump system two-step axiom. We use reduction to prove a weakened version of a matroid conjecture by Rota[7].

1. Introduction

Matroids have long been an important structure in pure combinatorics. We recall that a matroid $(E, \mathcal{M})$ consists of a finite set of edges, E, together with $\mathcal{M}$, a collection of subsets of E, satisfying an exchange axiom.

Axiom 1 (Matroid)**.** *For all $A, B \in \mathcal{M}$, and for all $e \in A \setminus B$, there is some $f \in B \setminus A$ with $A\Delta\{e, f\} \in \mathcal{M}$. Here Δ denotes symmetric difference. The elements of $\mathcal{M}$ are the* bases *of the matroid.*

Delta-matroids are a generalization of matroids introduced in 1987 by Bouchet [1]. They are closely related to the concepts of pseudomatroids [3], metroids [5], and universal polymatroids [10]. A delta-matroid $(E, \mathcal{D})$ consists of a finite set of edges, E, together with $\mathcal{D}$, a collection of subsets of E, satisfying an exchange axiom.

Axiom 2 (Delta-Matroid)**.** *For all $A, B \in \mathcal{D}$, and for all $e \in A\Delta B$, there is some $f \in B\Delta A$ with $A\Delta\{e, f\} \in \mathcal{D}$. The elements of $\mathcal{D}$ are the* feasible sets *of the delta-matroid.*

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It is easy to see that a matroid is precisely a delta-matroid with equicardinal feasible sets. Delta-matroids satisfy a variety of nice properties, including a greedy algorithm and polyhedral description.

Jump systems were introduced in 1995 by Bouchet and Cunningham [2] as a generalization of delta-matroids. Let E be a finite set. We now fix $\mathbf{Z}^E$ as our ground set. Fix the l^1 metric, where for $x, y \in \mathbf{Z}^E$, $d(x,y) = \sum_{i\in E} |(x)_i - (y)_i|$. Let $x \xrightarrow{y} z$ denote that $d(x,z) = 1$ and $d(x,y) > d(z,y)$. We say that z is a *step* from x toward y. A jump system J is a subset of $\mathbf{Z}^E$ satisfying a two-step axiom.

Axiom 3 (Jump System)**.** *For all $x, y \in J$ and for all $z_1 \in \mathbf{Z}^E$ with $x \xrightarrow{y} z_1$, then either $z_1 \in J$, or there is some $z_2 \in J$ with $x \xrightarrow{y} z_1 \xrightarrow{y} z_2$.*

There is a simple bijection between delta-matroids $(E, \mathcal{D})$ and jump systems J on $\{0,1\}^E \subseteq \mathbf{Z}^E$. The elements of E correspond to the coordinates of $\mathbf{Z}^E$, and feasible sets correspond to characteristic vectors.

We also recall the following theorem, which gives (under a special condition) a test for membership in a jump system.

Theorem 1.1 (Lovász [9])**.** *Let J be a jump system on $\mathbf{Z}^E$. Suppose that for all $x \in J$, $\|x\|_1 = \alpha$ for some fixed α. Let $v \in \mathbf{Z}^E$ with $\|v\|_1 = \alpha$. Then $v \in J$ if and only if for all $A \subseteq E$, there is some $x \in J$ with $\sum_{i\in A} (x)_i \geq \sum_{i \in A} (v)_i$.*

If the jump system is on $\{0,1\}^E$ (and hence represents a delta-matroid), then the condition of Theorem 1.1 is equivalent to the corresponding delta-matroid actually being a matroid.

We present several new operations on jump systems. In Section 2, we define reduction and show that reduction preserves Axiom 1.3. Finally, in Section 3, we use reduction to prove a weakened version of a matroid conjecture by Rota[7].

2. Reduction

A reduction of an integer lattice is a lower-dimensional integer lattice, related in a natural way to the original lattice.

Definition 1. Fix $\mathbf{Z}^E$. Let $F \subseteq E$. Set $E' = \big(E \setminus F\big) \cup \{f\}$, where f is a new element ($f \notin E$). Let R denote the *reduction* map from $\mathbf{Z}^E$ to $\mathbf{Z}^{E'}$ defined by

$$\big(R(x)\big)_i = \left\{ \begin{matrix} (x)_i & i \in E \setminus F \\ \sum_{j\in F} (x)_j & i = f \end{matrix} \right\}.$$

Let J be a jump system on $\mathbf{Z}^E$. We have $R(J) = \{R(x) \mid x \in J\} \subseteq \mathbf{Z}^{E'}$. We say that $R(J)$ is a *reduction* of J formed by *reducing* F.

Reduction satisfies a number of nice properties. It is quite easy to see that $R(v) + R(w) = R(v + w)$, and that $\|R(v)\| = \|v\|$. Further, the composition of two reductions is a reduction. A reduction of a jump system is, in turn, a jump system[1]. This can be used to immediately prove that the coordinatewise sum of two jump systems is a jump system (by reducing their direct product).

Theorem 2.1. *$R(J)$ is a jump system.*

Proof using composition of jump systems. We recall from [2] the operation of composition, which preserves the jump system axiom. Let J_E be a jump system on E, and let $J_{E'}$ be a jump system on E'. We recall the composition $J_E \triangle J_{E'} := \{z \in \mathbf{Z}^{(E\setminus E')\cup(E'\setminus E)} : x \in J_E, y \in J_{E'}, (x)_{E\cap E'} = (y)_{E\cap E'}, (z)_{E\setminus E'} = (x)_{E\setminus E'}, (z)_{E'\setminus E} = (y)_{E'\setminus E}\}$.

Consider $J' \subseteq \mathbf{Z}^{F\cup\{f\}}$, defined by $J' = \{x \in \mathbf{Z}^{F\cup\{f\}} : \sum_{j\in F}(x)_j = (x)_f\}$. We can see that this infinite set is a jump system – any first step will always be outside of J' but there must always be a second step (in the right direction) back into J'. Now, $R(J) = J \triangle J'$, which proves the theorem. □

Elementary Proof. Suppose $E = \{1, 2, \dots, n\}$. It is sufficient to prove the result for $F = \{1, 2\}$, since (as noted above) the composition of two reductions is a reduction, so reductions of $|F| > 2$ can be achieved by iterating reductions of size 2. Let the unit vectors of $\mathbf{Z}^E$ be $\{e_1, e_2, \dots, e_n\}$. Let the unit vectors of $\mathbf{Z}^{E'}$ be $\{f_0, f_3, f_4, \dots, f_n\}$, with $R(e_1) = R(e_2) = f_0, R(e_3) = f_3, \dots, R(e_n) = f_n$. Let $x, y \in R(J)$, and let $x \xrightarrow{y} x + a$, for any $a \in \mathbf{Z}^{E'}$. If $x + a \in R(J)$, then Axiom 1.3 is satisfied (proving the theorem), so henceforth we make the assumption that $x + a \notin R(J)$. We would like to produce some $z \in R(J)$ with $x + a \xrightarrow{y} z$. Let $u, v \in J$ with $R(u) = x, R(v) = y$. Let $b \in \mathbf{Z}^E$ with $R(b) = a$, and with $u \xrightarrow{v} u + b$ (note that if $(x)_0 < (y)_0$, then either $(u)_1 < (v)_1$ or $(u)_2 < (v)_2$ so some such b always exists). If $u + b \in J$, then $R(u + b) = x + a \in R(J)$, contradicting our assumption. Therefore, we can apply Axiom 1.3 to get $u \xrightarrow{v} u + b \xrightarrow{v} u + b + c$, with $u + b + c \in J$. Now $R(u + b + c) = x + a + R(c) \in R(J)$ and $x + a + R(c)$ is a step from $x + a$, since $\|R(c)\| = 1$. If it is in the direction of y, then Axiom 1.3 holds and the theorem follows. If $c = \pm e_i (3 \le i \le n)$, then $R(c)$ is in the direction of y and the theorem follows. Otherwise, without loss of generality, we assume that $c = e_1$ and $(u + b)_1 < (v)_1$. There are three cases to consider:

[1] A similar result, independently discovered by Kabadi and Sridhar, can be found in their unpublished manuscript [8].

Case 1. $(u+b)_1 + (u+b)_2 < (v)_1 + (v)_2$ $\big($*i.e.* $(x+a)_0 < (y)_0\big)$.
This is the easy case, as we have $x + a \xrightarrow{y} x + a + R(c)$.

Case 2. $(u+b)_1 + (u+b)_2 > (v)_1 + (v)_2$ $\big($*i.e.* $(x+a)_0 > (y)_0\big)$.
We have $(u+b)_1 < (v)_1$ *and* $(u+b)_2 > (v)_2$. *This case proceeds by a sequence of steps. In each step, either the process terminates and Axiom 1.3 holds, or a* $w_i \in J$ *is produced, and the process continues. However, each step gets closer to* v *(i.e.* $\|w_1 - v\| > \|w_2 - v\| > \dots$*), so the process must terminate.*

> **Step 1.** *Set* $w_1 = u + b + e_1 \in J$. *We have* $w_1 \xrightarrow{v} w_1 - e_2$. *If* $w_1 - e_2 \in J$, *then* $R(w_1 - e_2) = x + a \in R(J)$, *contradicting our assumption. We can therefore apply Axiom 1.3 to get* $w_1 \xrightarrow{v} w_1 - e_2 \xrightarrow{v} w_1 - e_2 + d_1 \in J$. *If* $d_1 = \pm e_i (3 \le i \le n)$, *then* $x + a \xrightarrow{y} x + a + R(d_1)$ *and Axiom 1.3 holds. Suppose that either* $d_1 = e_1$ *or* $d_1 = -e_2$. *If* $d_1 = -e_2$, *then Axiom 1.3 holds, as* $(x+a)_0 > (y)_0$ *and* $x + a \xrightarrow{y} x + a + R(d_1)$. *Hence, either Axiom 1.3 holds, or we must have* $d_1 = e_1$ *and we continue to step 2.*
>
> **Step 2.** *Set* $w_2 = u + b + e_1 - e_2 + e_1 \in J$. *We have* $R(w_2) = x + a + e_0$ *(note that* $w_2 = w_1 - e_2 + e_1$*) and* $w_2 \xrightarrow{v} w_2 - e_2$. *If* $w_2 - e_2 \in J$, *then* $R(w_2 - e_2) = x + a \in R(J)$, *contradicting our assumption. As before, we apply Axiom 1.3 to get* $w_2 \xrightarrow{v} w_2 - e_2 \xrightarrow{v} w_2 - e_2 + d_2 \in J$. *If* $d_2 = \pm e_i (3 \le i \le n)$, *then Axiom 1.3 must hold. If* $d_2 = -e_2$, *then again Axiom 1.3 must hold. Hence either Axiom 1.3 holds, or we must have* $d_2 = e_1$, *and the process continues.*

Case 3. $(u+b)_1 + (u+b)_2 = (v)_1 + (v)_2$ $\big($*i.e.* $(x+a)_0 = (y)_0\big)$.
We have $(u+b)_1 < (v)_1$ *and* $(u+b)_2 > (v)_2$. *This case proceeds by a sequence of steps. In each step, either Axiom 1.3 holds, or a* $w_i \in J$ *is produced, and the process continues. However, each step gets closer to* v *(i.e.* $\|w_1 - v\| > \|w_2 - v\| > \dots$*), so it must terminate.*

> **Step 1.** *Set* $w_1 = u + b + e_1 \in J$. *We have* $w_1 \xrightarrow{v} w_1 - e_2$. *If* $w_1 - e_2 \in J$, *then* $R(w_1 - e_2) = x + a \in R(J)$, *which contradicts our assumption. We can therefore apply Axiom 1.3 to get* $w_1 \xrightarrow{v} w_1 - e_2 \xrightarrow{v} w_1 - e_2 + d_1 \in J$. *If* $d_1 = \pm e_i (3 \le i \le n)$, *then* $x + a \xrightarrow{y} x + a + R(d_1)$ *and Axiom 1.3 holds. Otherwise, we must*

have $d_1 = e_1$ or $d_1 = -e_2$. Set $d_1^\star$ so that $d_1 + d_1^\star = e_1 - e_2$ (note that $R(d_1 + d_1^\star) = R(e_1 - e_2) = 0$). We have $u + b + e_1 \xrightarrow{v} u + b + e_1 - e_2 \xrightarrow{v} u + b + e_1 - e_2 + d_1$, where $u + b + e_1 \in J$ and $u + b + e_1 - e_2 + d_1 \in J$.

Step 2. *Set $w_2 = u + b + e_1 - e_2 + d_1 \in J$. If $d_1 = e_1$, then we have $(u + b + e_1 - e_2)_1 < (v)_1$, $(u + b + e_1 - e_2)_2 > (v)_2$, and hence $w_2 \xrightarrow{v} w_2 + d_1^\star$. If $d_1 = -e_2$, then a similar argument gives $w_2 \xrightarrow{v} w_2 + d_1^\star$. If $w_2 + d_1^\star \in J$, then $R(w_2 + d_1^\star) = x + a \in R(J)$, which contradicts our assumption. We therefore apply Axiom 1.3 to get $w_2 \xrightarrow{v} w_2 + d_1^\star \xrightarrow{v} w_2 + d_1^\star + d_2$. If $d_2 = \pm e_i (3 \leq i \leq n)$, then $x + a \xrightarrow{y} x + a + R(d_2)$, so Axiom 1.3 holds. Otherwise, we must have $d_2 = e_1$ or $d_2 = -e_2$, and the process continues.*

□

3. Matroid Consequences

Let $B_1, B_2, \ldots, B_n$ be pairwise nonintersecting bases of a rank n matroid $(E, \mathcal{M})$. Rota has conjectured in [7] that there always exists an $n \times n$ matrix A, whose jth column consists of the elements of B_j, ordered in such a way that the rows of A are bases as well. Recent results toward this conjecture can be found in [4, 14, 6].

We confirm a weaker version of this conjecture, namely that for any i with $1 \leq i \leq n$, there always exists an $n \times n$ matrix A whose jth column consists of the elements of B_j, and that the first i rows are a disjoint union of i bases. This result has recently been shown [12] through completely different methods involving mimeomatroids.

First, we recall some notions about matroid union. If $(E, \mathcal{M})$ is a matroid, then the union $\mathcal{M} \bigvee \mathcal{M}$ is a matroid on E, each of whose bases is a disjoint union of two bases of $\mathcal{M}$. Similarly, the union $\overset{i}{\bigvee} \mathcal{M}$ is a matroid on E each of whose bases is a disjoint union of i bases of $\mathcal{M}$ (provided one such disjoint union exists, which must be true in the context of Rota's conjecture). For more information about matroid unions, see [13] or [11].

Theorem 3.1. *Let $B_1, B_2, \ldots, B_n$ be any bases of a rank n matroid $(E, \mathcal{M})$. If $1 \leq i \leq n$, then there is a basis B of $\overset{i}{\bigvee} \mathcal{M}$ with $|B \cap B_j| = i$ for $1 \leq j \leq n$.*

Proof. Without loss of generality, we assume that $E = B_1 \cup B_2 \cup \cdots \cup B_n$. If any element $e \in E$ appears in both B_1 and B_2, we may clone e, as follows. Consider the similar matroid on $E \cup \{e'\}$, where each basis that contains e also appears containing e' instead. Because e and e' are now interchangeable, we may assume that $e \in B_1$ and $e' \in B_2$. Therefore, without loss of generality we may assume that the B_i are pairwise disjoint.

Let J be the jump system on $\mathbf{Z}^{n^2}$ corresponding to the matroid $\bigvee^{i} \mathcal{M}$. For $1 \leq j \leq n$, let $F_j = \{$coordinates corresponding to the elements of $B_j\}$. We now reduce each F_j to produce a jump system on $\mathbf{Z}^n$. $R(J)$ is a jump system by Theorem 2.1. If $i \cdot \vec{1} = (\underbrace{i, i, \ldots, i}_{n}) \in R(J)$, then the theorem follows.

By Theorem 1.1, $i \cdot \vec{1} \in R(J)$ if for all $A \subseteq \{1, 2, \ldots, n\}$, there is some $x \in R(J)$ with $\sum_{j \in A} (x)_j \geq i|A|$. Fix $A \subseteq \{1, 2, \ldots, n\}$. We define $C \subseteq \{1, 2, \ldots, n\}$ as follows:

(1) If $|A| < i$, set $C = A \cup$ any other $(i - |A|)$ elements of $\{1, 2, \ldots, n\}$.
(2) If $|A| \geq i$, let $C =$ any i elements of A.

By construction, $|C| = i$. Let $y \in \mathbf{Z}^E$ be the incidence vector of $\bigcup_{j \in C} B_j$. Because y corresponds to i disjoint bases, we must have $y \in J$. Set $x = R(y) \in R(J)$. Observe that for $i \notin C$, $(x)_i = 0$. There are two cases to consider:

(1) If $|A| < i$, then $\sum_{j \in A} (x)_j = |A|n \geq i|A|$.
(2) If $|A| \geq i$, then $\sum_{j \in A} (x)_j = \sum_{j \in C} (x)_j = in \geq i|A|$.

Hence, in either case, we have constructed $x \in R(J)$ with $\sum_{j \in A} (x)_j \geq i|A|$, as desired. □

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