Introduction to Factorization Theory

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http://vadim.sdsu.edu/intro-factorization.pdf

Semigroups

Let S be a set of “numbers”, and $\star$ a binary operation on S.

$\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\mathbb{Q}$, $\mathbb{C}$, words, multisets

$\star$: $\times$, $+$, concatenation, multiset union

We require some properties:

$a \star b = b \star a$ (commutativity)

$a \star (b \star c) = (a \star b) \star c$ (associativity)

$I \star a = a$, for all a (identity, optional)

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Divisibility

Let $(S, \star)$ be a semigroup, with $a, c \in S$.
We say that a divides c, writing $a|c$, to mean:
There exists $b \in S$ with $a \star b = c$.

Ex1: $(\mathbb{N}, \times)$, does $3|6$? $6|3$? $3|5$?
Ex2: $(\mathbb{N}_0, +)$, does $3|5$? $6|3$?

If there is an identity I, and $x|I$, we call x a unit.
The good stuff happens with non-units!

If everything is a unit, this is called a group.

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Irreducibles/Atoms

Let $(S, \star)$ be a semigroup, with $a \in S$ a nonunit ($a \nmid I$).
If there are nonunits b, c with $a = b \star c$, we call a reducible.
Otherwise, we call a irreducible, or an atom.

Ex1: $(\mathbb{N}, \times)$, consider $6, 5, 1$.
Ex2: $(\mathbb{N}_0, +)$, consider $0, 1, 2$.

If every nonunit in S can be factored into atoms in at least one way, we call $(S, \star)$ atomic.

Our main interest is *multiple* factorizations into atoms.

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Two simple examples

Ex1: $(\mathbb{N}, \times)$, factorization is unique. FTA

Ex2: $(S, \times)$, for $S = \{1\} \cup 2\mathbb{N} = \{1, 2, 4, 6, 8, \ldots\}$.

Atoms: $2(2k + 1)$, for $k \in \mathbb{N}_0$.

$60 = (2 \cdot 3) \times (2 \cdot 5) = (2) \times (2 \cdot 15)$

Not unique factorization! Half-factorial (same length).

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Arithmetic Congruence Monoids

Let $a, b \in \mathbb{N}$ with $a \leq b$ and $a^2 \equiv a \pmod{b}$.
$S = \{1\} \cup \{n \in \mathbb{N} : n \equiv a \pmod{b}\}$. Write $M_{a,b}$.

Operation $\times$, identity 1, atoms?

Ex0: $M_{2,2} = \{1, 2, 4, 6, 8, \ldots\}$
Ex1: $M_{1,4}$ has $441 = 9 \times 49 = 21 \times 21$. "Hilbert monoid"
Ex2: $M_{4,6}$ has $154 \times 154 \times 154 = 1732 \times 2662$
"Meyerson monoid"

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Integers in Algebraic Number Field

Squarefree $d \in \mathbb{Z}$, take $S = \{a + b\sqrt{d} : a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$.

Operation $\times$, identity $1 = 1 + 0\sqrt{d}$, atoms?

Ex: $d = -5$, $S = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$,
$6 = 2 \times 3 = (1 + \sqrt{-5}) \times (1 - \sqrt{-5})$

Each d gives a class group (hard to compute) $\mathbb{Z}_2$

Factorization here is the same as in a Block Monoid over that class group

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Block Monoids

Let G be an abelian group with operation $+$.

Ex1: $G = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$

Ex2: $G = \mathbb{Z}_2 \times \mathbb{Z}_{10} = \{(a, b) : a \in \mathbb{Z}_2, b \in \mathbb{Z}_{10}\}$

Ex3: $G = \mathbb{Z}$

Block is multiset from G which sums to zero. "sequence"

Ex1: $G = \mathbb{Z}_5$ $2^5, 2^{10}, 3^5, 2^1 3^1, 2^3 4^1,$
$2^5 3^5 = (2^5)^1 (3^5)^1 = (2^1 3^1)^5$

Operation multiset union (concat), identity empty set

$G_0 \subseteq G$, block monoid $(G_0, \cup)$ is $\mathcal{B}(G, G_0)$.

Often $G_0 = G$, block monoid is $\mathcal{B}(G)$.

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Numerical Semigroups

Choose some naturals $a_1, a_2, \ldots, a_k$ with $\gcd$ 1.

$S = \langle a_1, a_2, \ldots, a_k \rangle = \{\#a_1 + \#a_2 + \cdots + \#a_k : \# \in \mathbb{N}_0\}$

Ex1: $\langle 3, 5 \rangle = \{0, 3, 5, 6, 8, 9, 10, \rightarrow\}$

Ex2: $\langle 3, 5, 6 \rangle = \langle 3, 5 \rangle$

Ex3: $\langle 4, 6, 9 \rangle = \{4, 6, 8, 9, 10, 12, 13, 14, \rightarrow\}$

Operation $+$, identity 0, atoms are among a_i

In $\langle 3, 5 \rangle$, we have $18 = 6 \cdot 3 + 0 \cdot 5 = 1 \cdot 3 + 3 \cdot 5$

six atoms, and four atoms

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Puiseux Monoids

Choose some positive rationals $a_1, a_2, \ldots, a_k$.

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Operation $+$, identity 0, atoms are among a_i

If finitely many a_i, isomorphic to a numerical semigroup!

Ex1: $S = \left\langle \frac{1}{p} : p \text{ prime} \right\rangle$

Ex2: $S = \left\langle p + \frac{1}{p} : p \text{ prime} \right\rangle$

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Puiseux Monoids

Choose some positive rationals $a_1, a_2, \ldots, a_k$.

$S = \langle a_1, a_2, \ldots, a_k \rangle = \{\#a_1 + \#a_2 + \cdots + \#a_k : \# \in \mathbb{N}_0\}$

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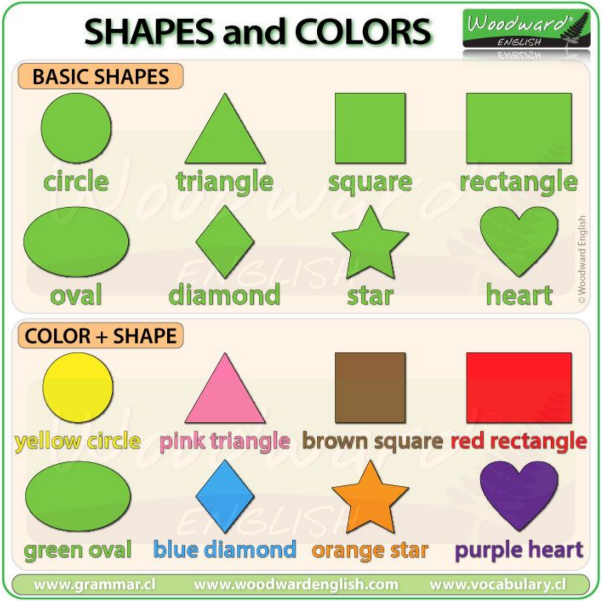
Invariants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Elasticity (Local)

Atomic semigroup $(S, \star)$, $x \in S$

x has some factorizations into atoms. Each factorization has a length (# of atoms). $\mathcal{L}(x)$ is set of lengths.

$L(x) = \max \mathcal{L}(x)$. $l(x) = \min \mathcal{L}(x)$. elasticity $\rho(x) = \frac{L(x)}{l(x)}$.

Ex1: $S = \langle 3, 5 \rangle$ numerical semigroup. $\mathcal{L}(18) = \{4, 6\}$, $\rho(18) = \frac{6}{4} = 1.5$.

Ex2: $\mathcal{B}(\mathbb{Z}_5)$ block monoid. $\mathcal{L}(2^5 3^5) = \{2, 5\}$, $\rho(2^5 3^5) = 2.5$.

Ex3: $(S, \star)$ half-factorial. $x \in S$ must have $\rho(x) = 1$.

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Elasticity (Global)

Atomic semigroup $(S, \star)$

We define the elasticity $\rho(S) = \sup_{x \in S} \rho(x)$

We say the elasticity is accepted if there is some $x \in S$ with $\rho(x) = \rho(S)$.

We say the elasticity is full if every rational $t \in [1, \rho(S))$ has some $x \in S$ with $\rho(x) = t$.

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Delta Sets

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The Delta set $\Delta(x)$ is the set of gaps in $\mathcal{L}(x)$.

Ex1: $S = \langle 3, 5 \rangle$ numerical semigroup. $\mathcal{L}(18) = \{4, 6\}$, $\Delta(18) = \{2\}$.

Ex2: $\mathcal{B}(\mathbb{Z}_5)$ block monoid. $\mathcal{L}(2^{10}3^{10}) = \{4, 7, 10\}$, $\Delta(2^{10}3^{10}) = \{3\}$.

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