The Indifference Graph Numerical Monoid Conjecture

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Abstract - A numerical monoid is a subset of the nonnegative integers, containing zero, that is closed under addition. We conjecture that every indifference graph whose vertex degrees all appear in a numerical monoid, must have its order appear in that same numerical monoid. We offer several results in the direction of this conjecture.

Keywords : indifference graph; interval graph; intersection graph; numerical monoid; numerical semigroup

Mathematics Subject Classification (2020) **:** 05C10

1 Introduction

An indifference graph is an undirected graph whose vertices are a finite (multi)set of real numbers, and whose edges are those pairs of vertices which are within distance one of each other, as real numbers. Since each vertex is within one of itself, we take a loop at each vertex. Indifference graphs are an important class of graphs with many applications such as to order theory and algebra, and are the subject of considerable study (see, e.g., [3, 4, 5, 6, 9, 10, 12, 14]).

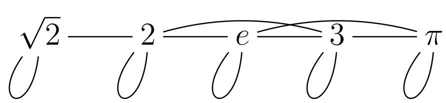

Figure 1: An indifference graph

A numerical monoid is a subset of the nonnegative integers $\mathbb{N}_0$, containing 0, closed under addition. A numerical monoid that is also cofinite is called a numerical semigroup. Both of these are important objects (particularly numerical semigroups), themselves well-studied (see, e.g., [1, 2]).

For graph G, we denote by $|G|$ the *order* of $|G|$, i.e. the number of vertices of G. For vertex $v \in G$, we denote by $\deg(v)$ the *degree* of v, i.e. the number of edges incident to v. We consider a loop at v to contribute 1 to $\deg(v)$; hence the example in Figure 1 has $\deg(\pi) = 3$. Given a graph G and a numerical monoid S, we say that G *respects* S if:

$$(\forall v \in G, \deg(v) \in S) \to |G| \in S$$

Note that G can respect S vacuously, if the degree of some vertex of G does not appear in S. If instead all vertex degrees of G appear in S, then the order of G must also appear in S, for G to respect S. By $[\![a, b]\!]$ we mean the integers between a and b, inclusive; similarly, by $[\![a, \infty)\!)$ we mean all integers greater than or equal to a. For any set $S \subseteq \mathbb{N}_0$, by kS we mean $\{kn : n \in S\}$. Note that the example in Figure 1 respects $2\mathbb{N}_0$ (vacuously), respects $[\![2, \infty)\!)$ and $[\![2, 5]\!]$, but does not respect $[\![2, 4]\!]$.

Inspired by a recent result (Theorem 2.1 below), we propose several conjectures. Although we are not able to prove any of them, we do have a variety of partial results.

Conjecture 1.1 Every indifference graph respects every numerical monoid.

Conjecture 1.2 Every indifference graph respects every numerical semigroup.

Conjecture 1.3 Every indifference graph respects $k\mathbb{N}_0$, for all $k \in \mathbb{N}_0$.

Because $k\mathbb{N}_0$ and numerical semigroups are each numerical monoids, Conjecture 1 implies the others. Perhaps surprisingly, Conjecture 2 is equivalent to Conjecture 1.

Theorem 1.4 *If Conjecture 2 holds, then Conjecture 1 holds.*

Proof. Let G be an indifference graph and S a numerical monoid such that $\forall v \in G, \deg(v) \in S$. Set $S' = S \cup [\![|G| + 1, \infty)\!)$. Note that S' is a numerical semigroup that agrees with S on $[\![0, |G|]\!]$. Hence, $\forall v \in G, \deg(v) \in S'$. Applying Conjecture 2, G respects S', so $|G| \in S'$. Hence $|G| \in S$, so G respects S. $\square$

In the remainder, we approach these conjectures from several directions. In Section 2, we consider fixed numerical monoids/semigroups respected by all indifference graphs. In Section 3, we instead consider fixed indifference graphs that respect every numerical monoid or every $k\mathbb{N}_0$.

2 Fixed Numerical Monoids

In this section, we consider specific numerical monoids that are respected by many indifference graphs.

We begin with the motivating theorem for these conjectures. It appears in [8] in contrapositive form. For completeness, we provide a proof.

Theorem 2.1 (Balof/Pinchasi) *Every indifference graph respects* $2\mathbb{N}_0$.

Proof. Let G be an indifference graph with adjacency matrix M. Suppose that $\deg(v)$ is even for every $v \in G$. Then every row sum of M is even, hence the sum of all the entries of M is even. Now consider $M - I$. This is symmetric with 0 along the diagonal, so the sum of all its entries is even. Combining these results, we conclude that the sum of all entries of I (namely, $|G|$) is even. $\square$

We collect some basic properties in the following theorem.

Theorem 2.2 *Let G be a graph, S, S' be numerical monoids, and $a \in \mathbb{N}_0$.*

1. *Every indifference graph respects $0\mathbb{N}_0 = \{0\}$.*
2. *Every indifference graph respects $1\mathbb{N}_0 = \mathbb{N}_0$.*
3. *If G respects S, then G also respects $S \cup [\![a, \infty)\!)$.*
4. *If G respects S and S', then G also respects $S \cap S'$.*
5. *If G respects S, then G also respects $S \setminus [\![1, a+1]\!]$.*

Proof. (1) Every vertex in an indifference graph has degree at least one, since we assume it has a loop. Hence, all nonempty graphs respect $0\mathbb{N}_0$ vacuously, and the empty graph respects it nonvacuously.
(2) clear
(3) Suppose all vertices of G have their degree in $S \cup [\![a, \infty)\!)$. If any vertex v has $\deg(v) \notin S$, then $\deg(v) \geq a$; in this case, $|G| \geq a$ (looking at the neighbors of v alone), and so $|G| \in S \cup [\![a, \infty)\!)$. Otherwise, all vertices $v \in G$ have $\deg(v) \in S$. Since G respects S by hypothesis, we have $|G| \in S \subseteq S \cup [\![a, \infty)\!)$.
(4) Suppose all vertices of G have their degrees in $S \cap S'$. Then, in particular, all vertices of G have their degrees in S. Since G respects S, $|G| \in S$. Repeating for S' we find $|G| \in S'$; hence $|G| \in S \cap S'$.
(5) By (1) and (3), G respects $S' = \{0\} \cup [\![a+2, \infty)\!)$; now apply (4). □

We combine Theorems 2.1 and 2.2 to get the following result.

Corollary 2.3 *Let $t \in \mathbb{N}$ be odd. Every indifference graph respects the numerical semigroup $\langle 2, t\rangle = \{2x + ty : x, y \in \mathbb{N}_0\}$.*

Proof. $\langle 2, t\rangle = 2\mathbb{N}_0 \cup [\![t, \infty)\!)$. □

We can move from one numerical monoid to another with the following result.

Theorem 2.4 *Let S be a numerical monoid and $k \in \mathbb{N}_0$. Suppose that every indifference graph respects kS. Then every indifference graph respects S.*

Proof. Let G be an indifference graph with $\deg(v) \in S$ for all $v \in G$. We produce a new indifference graph G' which has the same vertices as G, only each repeated k times. For every $v \in G$, there are k copies $v_1, v_2, \ldots, v_k \in G'$ with $k \deg(v) = \deg(v_1) = \deg(v_2) = \cdots = \deg(v_k)$. Hence $\deg(v_i) \in kS$ for all $v_i \in G'$. Since G' respects kS, we have $|G'| \in kS$. Hence there is some $n \in S$ with $kn = |G'| = k|G|$, so $|G| \in S$. □

In particular, Theorem 2.4 tells us that if a numerical monoid T is respected by all indifference graphs, we can set $k = \gcd(T)$. Then $T = kS$ for a numerical monoid S with $\gcd(S) = 1$, and now all indifference graphs respect S. It is well-known (e.g. [1]) that a numerical monoid with no common factor is a numerical semigroup, so in fact S is a numerical semigroup.

For our remaining results, it will be useful to develop some machinery. Let G be an indifference graph with vertices $v_1 \leq v_2 \leq \cdots \leq v_n$. We relabel these real values by their integer indices $1, 2, \ldots n$. The neighborhood of each vertex of G will then form some integer interval. For vertex $i \in [\![1, n]\!]$, define $L(i)$ and $R(i)$ to be the left and right endpoints of that interval. That is, i is connected to each element of $[\![L(i), R(i)]\!]$, and nothing else. Note that $L(i) \le i \le R(i)$; hence $L(1) = 1$ and $R(n) = n$. We also have the key property that $\deg(i) = R(i) - L(i) + 1$. By symmetry, $j \le R(i)$ if and only if $i \ge L(j)$. Also, L and R are each nondecreasing functions; i.e. $L(i) \le L(i+1)$ and $R(i) \le R(i+1)$.

For example, the indifference graph of Figure 1 can be relabeled as shown in Figure 2. Here $1 = L(1) = L(2)$, $2 = R(1) = L(3) = L(4)$, $3 = L(5)$, $4 = R(2)$, $5 = R(3) = R(4) = R(5)$. We see $\deg(3) = R(3) - L(3) + 1 = 5 - 2 + 1 = 4$.

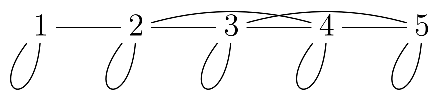

Figure 2: An indifference graph, relabeled

We turn briefly to numerical semigroups. Given a numerical semigroup S, its *multiplicity* $m(S)$ is the smallest positive integer contained in S. Its *Frobenius number* $F(S)$ is the largest integer not contained in S. Let $\mathfrak{S}$ denote the set of numerical semigroups S such that $2m(S) > F(S)$. This set $\mathfrak{S}$ is independently interesting (see, e.g. [7, 11]). Further, $\mathfrak{S}$ has been shown (in [13]) to be asymptotically a strictly positive fraction of all numerical semigroups.

Theorem 2.5 *Let $S \in \mathfrak{S}$. Then every indifference graph respects S.*

Proof. Let G be an indifference graph with $\deg(v) \in S$ for all $v \in G$. Set $n = |G|$. Since every vertex degree must be at least $m(S)$, we have $R(1) \ge m(S)$ and $L(n) \le n - m(S) + 1$. We have two cases: if $L(n) > R(1)$, then $n - m(S) + 1 > m(S)$ and hence $n \ge 2m(S) > F(S)$. Since $n > F(S)$ and $F(S)$ is the last integer missing from S, we must have $n \in S$. If instead $L(n) \le R(1)$, then $R(1)$ is connected to all vertices, i.e. $L(R(1)) = 1$ and $R(R(1)) = n$. We calculate $\deg(R(1)) = R(R(1)) - L(R(1)) + 1 = n$, and hence in this case also $n \in S$. $\square$

3 Fixed Indifference Graphs

We now approach the conjectures from the other direction. Our remaining results are for specific indifference graphs which respect all numerical monoids or all $k\mathbb{N}_0$.

Theorem 3.1 *Let G be an indifference graph whose diameter is 1. Then G respects every numerical monoid S.*

Proof. Here $R(1) = |G|$, so $\deg(1) = R(1) - L(1) + 1 = |G| - 1 + 1 = |G|$. If $\deg(1) \in S$, then $|G| \in S$. $\square$

Theorem 3.2 *Let G be an indifference graph whose diameter is 2. Then G respects every numerical monoid S.*

Proof. Here $R(R(1)) = |G|$, so $\deg(R(1)) = R(R(1)) - L(R(1)) + 1 = |G| - 1 + 1 = |G|$. If $\deg(R(1)) \in S$, then $|G| \in S$. $\square$

Theorem 3.3 *Let G be an indifference graph whose diameter is 3. Then G respects $k\mathbb{N}_0$ for every $k \in \mathbb{N}_0$.*

Proof. Let G be an indifference graph of diameter 3 with $\deg(v) \in k\mathbb{N}_0$ for all $v \in G$. Set $a = R(1), b = R(a)$. Since the diameter of G is 3, we must have $|G| = R(b)$. Note that $R(1) - L(1) + 1 = a$ and $R(a) - L(a) + 1 = b$, so by hypothesis $a, b \in k\mathbb{N}_0$. If $R(b) - b \in k\mathbb{N}_0$, we are done (since $b, |G| - b \in k\mathbb{N}_0$, so is their sum). We turn to the remaining case of $R(b) - b \notin k\mathbb{N}_0$.

Consider the function $R(x) - b$. We have $R(a) - b = 0 \in k\mathbb{N}_0$, while $R(b) - b \notin k\mathbb{N}_0$. Hence there is some $e \in [\![a+1, b]\!]$ such that $R(e-1) - b \in k\mathbb{N}_0$ while $R(e) - b \notin k\mathbb{N}_0$. Subtracting, we find $R(e) - R(e-1) \notin k\mathbb{N}_0$ and in particular we must have $R(e) > R(e-1)$, and hence $L(R(e)) = e$. Since $e \geq a$, $R(e) \geq R(a) = b$, so $R(R(e)) = |G|$. So, we find that $R(R(e)) - L(R(e)) + 1 = |G| - e + 1 \in k\mathbb{N}_0$.

We now prove that $L(e-1) \neq L(e)$; supposing to the contrary that $L(e) = L(e-1)$, we find $R(e) - L(e) + 1, R(e-1) - L(e) + 1 \in k\mathbb{N}_0$. Subtracting, we find $R(e) - R(e-1) \in k\mathbb{N}_0$, a contradiction. Hence $L(e-1) < L(e)$ and so $R(L(e-1)) = e - 1$. Also, $e - 1 \leq b$, so $L(e-1) \leq L(b) \leq a$, so $L(L(e-1)) = 1$. So, we find that $R(L(e-1)) - L(L(e-1)) + 1 = e - 1 \in k\mathbb{N}_0$. Combining with $|G| - e + 1 \in k\mathbb{N}_0$ we find $|G| \in k\mathbb{N}_0$. $\square$

Extending these results substantially beyond diameter 3 seems difficult with these methods.

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