Odds Inversion Problem With Replacement

In the recent piece [**1**] in this MONTHLY, Moniot worked toward determining which probabilities $\frac{p}{q}$ were achievable when drawing two balls from a jar of x red balls and y blue balls, without replacement, where a successful trial has the two drawn balls of different colors. This was done by reducing the problem to solving the Pell-like equation $u^2 - Dv^2 = p^2$, where $D = q(q - 2p)$. Solutions to this, however, sometimes give only extraneous x, y: e.g., $p = 4, q = 9, D = 9$, has no possible nonnegative integers x, y with $x + y \geq 2$ giving probability $\frac{4}{9}$.

We completely answer the simpler question where the draws are with replacement, by reducing the problem to finding nontrivial solutions to the similar Diophantine equation $u^2 - Dv^2 = 0$.

Theorem. *Probability $\frac{p}{q}$ is achievable as the probability of two drawn balls (with replacement, from x red and y blue balls) being different, if and only if $D = q(q - 2p)$ is a perfect square.*

Proof. Probability $\frac{p}{q}$ is achievable, if and only if there are nonnegative integers x, y, not both zero, with $\frac{p}{q} = \frac{2xy}{(x+y)^2}$. This rearranges to $px^2 - 2(q-p)xy + py^2 = 0$. Taking the substitution $v = y + x, t = y - x$, this rearranges to $(2p - q)v^2 + qt^2 = 0$, which in turn rearranges to $(qt)^2 - Dv^2 = 0$. Lastly, taking $u = qt$, this becomes $u^2 - Dv^2 = 0$ (where u, v are not both zero).

If $\frac{p}{q}$ is achievable, then D must be a perfect square (by considering unique factorization of integers u, v, D into primes). Suppose now that $D = m^2$. We take $u = 2qm, v = 2q$, which satisfy $u^2 - Dv^2 = 0$. These u, v correspond to $t = 2m, x = q - m, y = q + m$, where $\frac{2xy}{(x+y)^2} = \frac{p}{q}$. Each of x, y are nonnegative integers, since $m^2 = D = q^2 - 2qp \leq q^2$. Hence $\frac{p}{q}$ is achievable. ■

In particular, no probabilities greater that $\frac{1}{2}$ (i.e. $D < 0$) are achievable. Compare to the "elliptical case" in [**1**], where infinitely many such probabilities are achieved without replacement. Note also that achievable probabilities are dense in $[0, \frac{1}{2}]$, because a brief calculation shows that achievable $\frac{p}{q}$ are exactly those rationals equal to $\frac{1}{2}(1 - r^2)$ for some rational $r \in [0, 1]$.

REFERENCES

1. Moniot, R. K. (2021). Solution of an Odds Inversion Problem. *Amer. Math. Monthly*. 128(2): 140–149.

—Submitted by Kieran Hilmer, San Diego State University and
Ron Lycan, San Diego State University and
Vadim Ponomarenko, San Diego State University

doi.org/10.XXXX/amer.math.monthly.122.XX.XXX
MSC: Primary 11D09