A Generalization of Bonse's Inequality

Over 100 years ago, Bonse proved (see [**1**]) his famous inequality, which relates the sequence of primes $p_1 = 2, p_2 = 3, p_4 = 5, \ldots$ as:

$$\text{For all } n \geq 4, \text{ we have } p_1 p_2 \cdots p_n > p_{n+1}^2.$$

We offer the following generalization.

Theorem. *Suppose we have constants μ, λ satisfying $1 < \mu \leq \lambda$. Let a_i be a nondecreasing sequence of real numbers satisfying $a_1 = \mu$ and $a_{i+1} \leq \lambda a_i$ (for each $i \geq 1$). Then, taking $K = 2 + 3\log_\mu \lambda$, we get:*

$$\textit{For all } n \geq K, \textit{ we have } a_1 a_2 \cdots a_n \geq a_{n+1}^2.$$

The inequality is strict except possibly for $n = K$.

Note that Bertrand's postulate gives $p_{n+1} \leq 2p_n$, so we may take $\mu = \lambda = 2$ with $a_n = p_n$, and recover Bonse's inequality (apart from $n = 4, 5$).

Proof. Note that K is chosen so that $n \geq K$ implies $\mu^{n-2} \geq \lambda^3$. We get

$$a_1 a_2 \cdots a_{n-1} \geq \mu^{n-2} a_{n-1} \geq \lambda^3 a_{n-1} \geq \lambda^2 a_n \geq \lambda a_{n+1} \geq \frac{a_{n+1}^2}{a_n}$$

The first inequality follows from $a_i \geq \mu$, and the last three each use $a_{i+1} \leq \lambda a_i$. Note that if $n > K$ then $\mu^{n-2} > \lambda^3$, making the second inequality strict. ■

Note that equality is possible if $n = K \in \mathbb{Z}$, via $a_1 = a_2 = \cdots = a_{K-1} = \mu$, $a_K = \lambda\mu$, and $a_{K+1} = \lambda^2 \mu$. Now $\mu^{n-2} = \lambda^3$, so $a_1 a_2 \cdots a_n = a_{n+1}^2$.

REFERENCES

1. Bonse, H. (1907). Über eine bekannte Eigenschaft der Zahl 30 und ihre Verallgemeinerung. *Archiv der Mathematik und Physik*. 12 (3): 292–295.

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