More on Euler's limit for e

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The well-known *Euler's limit* is defined as $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e = 2.71828...$ (see, e.g, [1]). Recently, in [2], appeared the following generalisation of Euler's limit.

Theorem 1. *Let A_n be a strictly increasing sequence of positive reals satisfying $A_{n+1} \sim A_n$. Then $\lim_{n\to\infty}\left(\frac{A_{n+1}}{A_n}\right)^{\frac{A_n}{A_{n+1}-A_n}} = e$.*

Note that the symbol "$\sim$" means asymptotic equivalence, i.e., $x_n \sim y_n$ if $\lim_{n\to\infty}\frac{x_n}{y_n} = 1$.

Here, we offer the following generalisation.

Theorem 2. *Let A_n be a strictly monotone sequence of positive reals satisfying $A_{n+1} \sim A_n$. Let B_n be any sequence of reals satisfying $B_n \sim \frac{A_n}{A_{n+1}-A_n}$. Then*

$$\lim_{n\to\infty}\left(\frac{A_{n+1}}{A_n}\right)^{B_n} = e.$$

Proof. First, we consider the case of A_n monotone increasing. Theorem 1 gives

$$\lim_{n\to\infty}\left(\frac{A_{n+1}}{A_n}\right)^{B_n} = \lim_{n\to\infty}\left(\left(\frac{A_{n+1}}{A_n}\right)^{\frac{A_n}{A_{n+1}-A_n}}\right)^{\frac{B_n(A_{n+1}-A_n)}{A_n}} = e^1 = e.$$

Now we consider the other case, of A_n monotone decreasing. We set $A'_n = \frac{1}{A_n}$ and $B'_n = B_n$ to get

$$\lim_{n\to\infty}\left(\frac{A_{n+1}}{A_n}\right)^{B_n} = \lim_{n\to\infty}\left(\frac{A'_n}{A'_{n+1}}\right)^{B'_n}.$$

We conclude by observing that $B_n \sim \frac{A_n}{A_{n+1}-A_n} = -\frac{A'_{n+1}}{A'_{n+1}-A'_n} \sim -\frac{A'_n}{A'_{n+1}-A'_n}$, and applying the first case to B'_n and the monotone increasing A'_n. Theorem 2 is proved.

Theorem 2 allows us to compare the speed of convergence of $(\frac{A_{n+1}}{A_n})^{B_n}$ towards e as n increases by choosing different sequences A_n and B_n. For example, let $A_n = n, B_n = n, n = 100$. This gives $(\frac{A_{n+1}}{A_n})^{B_n} \simeq 2.7048$. If $A_n = n, B_n = n + \frac{1}{2}, n = 100$, then $(\frac{A_{n+1}}{A_n})^{B_n} \simeq 2.7183$, which is a much better estimate. However, for these two examples, it can be seen that when n increases the speed of convergence in the two cases approaches each other.

By changing A_n and B_n, we can further generalise Theorem 2. We take $A_{n+1} = A_n(1+\epsilon_n)$, where $\epsilon_n \to 0$. Our previous assumptions of monotone increasing (decreasing) A_n now correspond to ϵ_n positive (negative). We have $B_n \sim \frac{1}{\epsilon_n}$. Set r_n to be a positive sequence with $r_n \to 1$. Now, Theorem 2 is equivalent to

$$\lim_{n\to\infty} (1+\epsilon_n)^{\frac{r_n}{\epsilon_n}} = e. \tag{1}$$

The sign of ϵ_n do not matter for this limit, so we can generalise the left-hand side of (1). For any constant k and δ_n a sequence with $|\delta_n|$ monotone decreasing to 0, we have

$$\lim_{n\to\infty} (1+\epsilon_n)^{\delta_n+k} = 1. \tag{2}$$

Multiplying (1) by (2) we obtain

$$\lim_{n\to\infty} (1+\epsilon_n)^{\frac{r_n}{\epsilon_n}+\delta_n+k} = e.$$

This allows the reader to choose parameters to optimize convergence.

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References

[1] A. J. Macintyre, Euler's limit for e^x and the exponential series, *Edinburgh Mathematical Notes*, **37** (1949), pp. 26-28.

[2] R. Farhadian, A Generalization of Euler's Limit, *Amer. Math. Monthly.* **129** (2022), p. 384.

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