A SIMPLE SELF-IMPROVEMENT OF THE CAUCHY-SCHWARZ INEQUALITY

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Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be arbitrary real numbers. In 1821, Augustin-Louis Cauchy proved (in [3]) that

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2. \tag{1}$$

After Cauchy, Viktor Bunyakovsky (a former student of Cauchy) published [1] a generalized version in 1859 and Hermann Schwarz (unaware of Bunyakovsky's work) also published [9] a generalized version in 1888. That is why it is sometimes known in the literature as *Cauchy-Schwarz inequality* or *Cauchy-Bunyakovsky-Schwarz inequlity*. The Cauchy-Schwarz inequality is one of the most famous and important inequalities in mathematics today. It is used in algebra, linear algebra, probability theory, and analysis. Many proofs, generalizations, and refinements are known, see, for example, [2, 4, 5, 6, 7, 8, 10].

We offer a simple self-improvement of the Cauchy-Schwarz inequality. The concept of self-improvement is to improve the Cauchy-Schwarz inequality by using itself (see [2, 11] for such self-improvements of the Cauchy-Schwarz inequality). We prove the following theorem.

Theorem 1. *Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be arbitrary real numbers. We have*

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - n\sum_{i=1}^{n} (\bar{y}x_i - \bar{x}y_i)^2. \tag{2}$$

Proof. Let $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$. The algebraic transformation of Cauchy-Schwarz inequality (1) for centered sequences $x_i - \bar{x}$ and $y_i - \bar{y}$ is

$$\left(\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})\right)^2 \leq \sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2. \tag{3}$$

By simple caculations we have $\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y}) = (\sum_{i=1}^{n} x_i y_i) - n\bar{x}\bar{y}$, $\sum_{i=1}^{n}(x_i-\bar{x})^2 = (\sum_{i=1}^{n} x_i^2) - n\bar{x}^2$, and $\sum_{i=1}^{n}(y_i-\bar{y})^2 = (\sum_{i=1}^{n} y_i^2) - n\bar{y}^2$. Therefore

$$\left(\left(\sum_{i=1}^{n} x_i y_i\right) - n\,\overline{x}\,\overline{y}\right)^2 \leq \left(\left(\sum_{i=1}^{n} x_i^2\right) - n\,\overline{x}^2\right)\left(\left(\sum_{i=1}^{n} y_i^2\right) - n\,\overline{y}^2\right).$$

Expanding and cancelling terms, gives

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - n\left(\overline{y}^2 \sum_{i=1}^{n} x_i^2 + \overline{x}^2 \sum_{i=1}^{n} y_i^2 - 2\,\overline{x}\,\overline{y} \sum_{i=1}^{n} x_i y_i\right). \tag{4}$$

This simplifies to

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - n\sum_{i=1}^{n}(\overline{y}x_i - \overline{x}y_i)^2.$$

The proof is complete. □

Note that the equality in (2) holds when $y_i - \bar{y} = \alpha(x_i - \bar{x})$ for some $\alpha \in \mathbb{R}$ (this is clear from (3)).

From (4), we can also provide an improvement related to a consequence of Jensen's inequality. For a real convex function $\varphi(x)$ with numbers $x_1, x_2, \ldots, x_n$ in its domain, Jensen's inequality states that $\varphi\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \leq \frac{\sum_{i=1}^{n} \varphi(x_i)}{n}$. Thus, for two arbitrary nonzero real vectors $(\mathbf{x}) = (x_1, x_2, \ldots, x_n)$ and $(\mathbf{y}) = (y_1, y_2, \ldots, y_n)$, Jensen's inequality implies that

$$\frac{\bar{x}^2}{\sum_{i=1}^{n} x_i^2} + \frac{\bar{y}^2}{\sum_{i=1}^{n} y_i^2} \leq \frac{2}{n}. \tag{5}$$

Under certain properties for the vectors $(\mathbf{x})$ and $(\mathbf{y})$, we improve inequality (5) in the following theorem.

Theorem 2. *Let us consider two nonzero real vectors* $(\boldsymbol{x}) = (x_1, x_2, \ldots, x_n)$ *and* $(\boldsymbol{y}) = (y_1, y_2, \ldots, y_n)$. *If* $(\boldsymbol{x})$ *and* $(\boldsymbol{y})$ *are orthogonal, then*

$$\frac{\bar{x}^2}{\sum_{i=1}^{n} x_i^2} + \frac{\bar{y}^2}{\sum_{i=1}^{n} y_i^2} \leq \frac{1}{n}.$$

Proof. Since $(\mathbf{x})$ and $(\mathbf{y})$ are orthogonal, therefore $\sum_{i=1}^{n} x_i y_i = 0$. Using this fact in inequality (4), the desired result is obtained. The theorem is proved. □

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