A Further Self-Improvement of the Cauchy–Schwarz Inequality

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Abstract. In this note, we present a new self-improvement of the well-known Cauchy–Schwarz inequality for expectations of random variables. We compare our new result with another self-improvement of the Cauchy–Schwarz inequality.

Inequalities play an important role in mathematical analysis and many areas of applied mathematics, such as probability theory and statistics. One of the most widely used inequalities in this regard is the well-known *Cauchy–Schwarz inequality*. In its probabilistic version, for two random variables X and Y with expectations $\mathbb{E}(X)$ and $\mathbb{E}(Y)$, respectively, the Cauchy–Schwarz inequality states that

$$\mathbb{E}^2(XY) \leq \mathbb{E}(X^2)\mathbb{E}(Y^2), \tag{1}$$

where equality holds if and only if $Y = \alpha X$ a.s. for some constant α.

Many proofs are known for the Cauchy–Schwarz inequality. A standard proof can be found in [**6**]. However, it can be obtained as a special case of Hölder's inequality (see, e.g., [**4**]). In [**3**], it is shown that the Cauchy–Schwarz inequality is a consequence of Jensen's inequality. Many generalizations and improvements have also been proposed for the Cauchy–Schwarz inequality, most of them are for its non-probabilistic (or discrete) form, i.e., $(\sum_i x_i y_i)^2 \leq (\sum_i x_i)(\sum_i y_i)$, where x_i, y_i are non-random numbers (see, e.g., [**1**], [**2**], and [**5**]). Recently, S.G. Walker [**7**] proved the following self-improvement for the probabilistic Cauchy–Schwarz inequality (1):

$$\mathbb{E}^2\left(XY\right) \leq \mathbb{E}(X^2)\mathbb{E}(Y^2) - \left(|\mathbb{E}(X)|\sqrt{\mathrm{Var}(Y)} - |\mathbb{E}(Y)|\sqrt{\mathrm{Var}(X)}\right)^2, \tag{2}$$

where $\mathrm{Var}(X) = E\big(X - E(X)\big)^2 = E(X^2) - E^2(X)$.

Indeed, the concept of self-improvement here is to improve the Cauchy–Schwarz inequality by using itself. This idea that shows that the ability of the Cauchy–Schwarz inequality to improve itself is interesting and important if it is developed to produce more and better self-improvements. In this note, we present a new self-improvement for the Cauchy–Schwarz inequality (2). In general, we prove the following theorem.

Theorem 1. *For any arbitrary random variables X and Y, we have*

$$\mathbb{E}^2(XY) \leq \mathbb{E}(X^2)\mathbb{E}(Y^2) - \mathbb{E}\big(X\mathbb{E}(Y) - Y\mathbb{E}(X)\big)^2.$$

Proof. Applying the Cauchy–Schwarz inequality (1) to centered random variables as

$$\mathbb{E}^2\left((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\right) \leq \mathbb{E}(X - \mathbb{E}(X))^2\mathbb{E}(Y - \mathbb{E}(Y))^2,$$

and hence

$$\big(\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)\big)^2 \leq \big(\mathbb{E}(X^2) - \mathbb{E}^2(X)\big)\big(\mathbb{E}(Y^2) - \mathbb{E}^2(Y)\big).$$

Expanding and rearranging, the theorem follows. ∎

Next, we offer a corollary to Theorem 1, which allows us to compare it to Walker's self-improvement (2). Our corollary of Theorem 1 always gives a better bound for nonnegative random variables.

Corollary 2. *Let X and Y be two nonnegative random variables. Then*

$$\mathbb{E}^2\left(XY\right) \leq \mathbb{E}(X^2)\mathbb{E}(Y^2) - \left(\mathbb{E}(X)\sqrt{\mathit{Var}(Y)} - \mathbb{E}(Y)\sqrt{\mathit{Var}(X)}\right)^2$$
$$-2\mathbb{E}(X)\mathbb{E}(Y)\sqrt{\mathit{Var}(X)}\sqrt{\mathit{Var}(Y)}(1-r),$$

where $r \in [-1,1]$ is the Pearson correlation coefficient.

Proof. We begin with

$$r = \frac{\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}},$$

that is, the Pearson correlation coefficient.
We rearrange this to

$$\mathbb{E}(XY) = r\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)} + \mathbb{E}(X)\mathbb{E}(Y). \tag{3}$$

We now calculate

$$\begin{aligned}
\mathbb{E}(X\mathbb{E}(Y) - Y\mathbb{E}(X))^2 &= \mathbb{E}(X^2)\mathbb{E}^2(Y) + \mathbb{E}(Y^2)\mathbb{E}^2(X) - 2\mathbb{E}(XY)\mathbb{E}(X)\mathbb{E}(Y) \\
&= \mathbb{E}(X^2)\mathbb{E}^2(Y) + \mathbb{E}(Y^2)\mathbb{E}^2(X) \\
&\quad -2\big(r\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)} + \mathbb{E}(X)\mathbb{E}(Y)\big)\mathbb{E}(X)\mathbb{E}(Y) \text{ (by (3))} \\
&= \mathbb{E}(X^2)\mathbb{E}^2(Y) + \mathbb{E}(Y^2)\mathbb{E}^2(X) \\
&\quad -2r\mathbb{E}(X)\mathbb{E}(Y)\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)} - 2\mathbb{E}^2(X)\mathbb{E}^2(Y) \\
&= -2r\mathbb{E}(X)\mathbb{E}(Y)\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)} \\
&\quad +\big(\mathbb{E}(X)\sqrt{\mathrm{Var}(Y)} - \mathbb{E}(Y)\sqrt{\mathrm{Var}(X)}\big)^2 \\
&\quad +2\mathbb{E}(X)\mathbb{E}(Y)\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)} \\
&= \big(\mathbb{E}(X)\sqrt{\mathrm{Var}(Y)} - \mathbb{E}(Y)\sqrt{\mathrm{Var}(X)}\big)^2 \\
&\quad +2\mathbb{E}(X)\mathbb{E}(Y)\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}(1-r).
\end{aligned}$$

Applying Theorem 1, the corollary follows. ■

Obviously, Corollary 2 shows that the right-hand side of Theorem 1 is always less than or equal to the right-hand side of Walker's inequality (2) when X and Y are nonnegative random variables.

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