A Further Self-Improvement of the Cauchy–Schwarz Inequality

(authors suppressed for review)

Abstract. In this note we present a new self-improvement of the well-known Cauchy–Schwarz inequality for expectations of random variables. We compare our new result with another self-improvement of the Cauchy–Schwarz inequality.

Inequalities play an important role in probability theory and mathematical statistics. One of the most widely used inequalities in this regard is the well-known *Cauchy–Schwarz inequality*, which states in its probabilistic form for two random variables X and Y that

$$\mathbb{E}^2(XY) \leq \mathbb{E}(X^2)\mathbb{E}(Y^2), \tag{1}$$

where equality holds if and only if $Y = \alpha X$ a.s. for some constant α.

Many proofs are known for the Cauchy–Schwarz inequality. A standard proof can be found in [**7**], however, it can be obtained as a special case of the Hölder's inequality (see, e.g., [**5**]). In [**4**], it is shown that the Cauchy–Schwarz inequality is a consequence of Jensen's inequality. Many generalizations and improvements have also been proposed for the Cauchy–Schwarz inequality, most of them for its non-probabilistic (or discrete) form i.e. $(\sum_i x_i y_i)^2 \leq (\sum_i x_i)(\sum_i y_i)$ with x_i, y_i are non-random numbers (see, e.g., [**2**], [**3**], and [**6**]). Recently, Walker [**8**] proposed the following self-improvement for the probabilistic Cauchy–Schwarz inequality (1):

$$\mathbb{E}^2\left(XY\right) \leq \mathbb{E}(X^2)\mathbb{E}(Y^2) - \left(|\mathbb{E}(X)|\sqrt{\mathrm{Var}(Y)} - |\mathbb{E}(Y)|\sqrt{\mathrm{Var}(X)}\right)^2. \tag{2}$$

Indeed, the concept of self-improvement here is to improve the Cauchy–Schwarz inequality by using itself. Hence, Walker's inequality (6) is interesting because it shows the ability of probabilistic version of Cauchy–Schwarz inequality to improve itself.

We now present a new self-improvement for the Cauchy–Schwarz inequality (1).

Theorem 1. *For any arbitrary random variables X and Y, we have*

$$\mathbb{E}^2(XY) \leq \mathbb{E}(X^2)\mathbb{E}(Y^2) - \mathbb{E}\big(X\mathbb{E}(Y) - Y\mathbb{E}(X)\big)^2.$$

Proof. Applying the Cauchy–Schwarz inequality (1) to centered random variables as

$$\mathbb{E}^2\left((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\right) \leq \mathbb{E}(X-\mathbb{E}(X))^2\mathbb{E}(Y-\mathbb{E}(Y))^2,$$

and hence

$$\big(\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)\big)^2 \leq \big(\mathbb{E}(X^2) - \mathbb{E}^2(X)\big)\big(\mathbb{E}(Y^2) - \mathbb{E}^2(Y)\big).$$

Expanding and rearranging, the theorem follows. ∎

We also offer a corollary to Theorem 1, which allows us to compare to Walker's self-improvement (6). We show that Theorem 1 always gives a better bound, for nonnegative random variables.

Corollary 2. *Let X and Y be two nonnegative random variables. Then*

$$\mathbb{E}^2(XY) \leq \mathbb{E}(X^2)\mathbb{E}(Y^2) - \left(|\mathbb{E}(X)|\sqrt{Var(Y)} - |\mathbb{E}(Y)|\sqrt{Var(X)}\right)^2$$
$$-2\mathbb{E}(X)\mathbb{E}(Y)\sqrt{Var(X)}\sqrt{Var(Y)}(1-r),$$

where $r \in [-1,1]$ is the Pearson's correlation coefficient.

Proof. We begin with

$$r = \frac{\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}}.$$

We rearrange this to

$$\mathbb{E}(XY) = r\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)} + \mathbb{E}(X)\mathbb{E}(Y).$$

We now calculate

$$\begin{aligned}
\mathbb{E}(X\mathbb{E}(Y) - Y\mathbb{E}(X))^2 &= \mathbb{E}(X^2)\mathbb{E}^2(Y) + \mathbb{E}(Y^2)\mathbb{E}^2(X) - 2\mathbb{E}(XY)\mathbb{E}(X)\mathbb{E}(Y) \\
&= \mathbb{E}(X^2)\mathbb{E}^2(Y) + \mathbb{E}(Y^2)\mathbb{E}^2(X) \\
&\quad -2\big(r\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)} + \mathbb{E}(X)\mathbb{E}(Y)\big)\mathbb{E}(X)\mathbb{E}(Y) \\
&= -2r\mathbb{E}(X)\mathbb{E}(Y)\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)} \\
&\quad +\mathbb{E}(X^2)\mathbb{E}^2(Y) + \mathbb{E}(Y^2)\mathbb{E}^2(X) - 2\mathbb{E}^2(X)\mathbb{E}^2(Y) \\
&= -2r\mathbb{E}(X)\mathbb{E}(Y)\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)} \\
&\quad +\big(\mathbb{E}(X)\sqrt{\text{Var}(Y)} - \mathbb{E}(Y)\sqrt{\text{Var}(X)}\big)^2 \\
&\quad +2\mathbb{E}(X)\mathbb{E}(Y)\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)} \\
&= \big(\mathbb{E}(X)\sqrt{\text{Var}(Y)} - \mathbb{E}(Y)\sqrt{\text{Var}(X)}\big)^2 \\
&\quad +2\mathbb{E}(X)\mathbb{E}(Y)\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}(1-r).
\end{aligned}$$

Applying Theorem 1, the corollary follows. ■

REFERENCES

1. Callebaut, D. K. 1965. Generalization of the Cauchy-Schwarz inequality. *J.Math.Anal.Appl.* 12:491–494.
2. Dragomir, S. S. 2003. A survey on Cauchy-Bunyakosky-Schwarz type discrete inequalities, *J. Inequal. Pure Appl. Math.* 4: Art. 63.

3. Farhadian, R. 2024. A note on the Cauchy-Schwarz inequality for expectations. *Communications in Statistics-Theory and Methods* 53(2):812–813.
4. Gut, A. 2013. *Probability: A graduate course*. New York: Springer.
5. Masjed-Jamei, M., Dragomir, S. S. and Srivastava, H. M. 2009. Some generalizations of the Cauchy-Schwarz and the Cauchy-Bunyakovsky inequalities involving four free parameters and their applications. *Mathematical and Computer Modelling* 49:1960–1968.
6. Mukhopadhyay, N. 2000. *Probability and statistical inference*. New York: Dekker.
7. Walker, S. G. 2017. A self-improvement to the Cauchy-Schwarz inequality. *Statistics & Probability Letters* 122(3):86–89.