How to Play the One-Lie Rényi-Ulam Game

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Abstract

The one-lie Rényi-Ulam liar game is a 2-player perfect information zero sum game, lasting q rounds, on the set $[n] := \{1, \ldots, n\}$. In each round Paul chooses a subset $A \subseteq [n]$ and Carole either assigns one lie to each element of A or to each element of $[n] \setminus A$. Paul wins the original (resp. pathological) game if after q rounds there is at most one (resp. at least one) element with one or fewer lies. We exhibit a simple, unified, optimal strategy for Paul to follow in both games, and use this to determine which player can win for all q, n and for both games.

Key words: Rényi-Ulam game, pathological liar game, searching with lies

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1 Introduction

The *Rényi-Ulam liar game* and its many variations have a long and beautiful history, which began in [1,2] and is surveyed in [3]. The players Paul and Carole play a q-round game on a set of n elements, $[n] := \{1, \ldots, n\}$. Each round, Paul splits the set of elements by choosing a *question* set $A \subseteq [n]$; Carole then completes the round by answering "yes" or "no". This assigns one *lie* either to each of the elements of A, or to each of the elements of $[n] \setminus A$. A given element is removed from play if it accumulates more than k lies, for some predetermined k. In choosing the question set A, we may consider the game to be restricted to the *surviving* elements, which have at most k lies. The game starts with each element having no associated lies. If after q rounds at most one element survives, Paul wins the original game; otherwise Carole wins. The dual *pathological liar* game, in which Paul wins whenever at least one element survives, has recently been explored in [4,5]. The original one-lie game corresponds to adaptive 1-error-correcting codes (introduced in [7]), while the pathological one-lie game corresponds to adaptive radius 1 covering codes. The original game with $k = 1$ was solved in [6], which contains a three-page algorithm for Paul's strategy. We give a substantial simplification which not only provides an alternate solution to the original one-lie ($k = 1$) game, but also solves the pathological one-lie game.

We represent a game state as $(q, \mathbf{x})$, where $\mathbf{x} = (x_0, x_1)$, x_0 denotes the number of elements with no lies, and x_1 denotes the number of elements with one lie. We denote Paul's question A by $\mathbf{a} = (a_0, a_1)$, where A contains a_0 elements that currently have no lies and a_1 elements that currently have a lie. Carole may then choose the successor state for the game, between $(q - 1, \mathbf{y}')$ and

$(q-1,\mathbf{y}'')$, where $\mathbf{y}' = (a_0, a_1 - a_0 + x_0)$ (attaching a lie to elements of $[n] \setminus A$) and $\mathbf{y}'' = (x_0 - a_0, x_1 - a_1 + a_0)$ (attaching a lie to elements of A).

Following Berlekamp in [7], the weight function for q questions, $\mathrm{wt}_q(\mathbf{x}) = (q+1)x_0 + x_1$, satisfies the relation $\mathrm{wt}_q(\mathbf{x}) = \mathrm{wt}_{q-1}(\mathbf{y}') + \mathrm{wt}_{q-1}(\mathbf{y}'')$, regardless of A. In the original game, Paul wants to decrease the weight as fast as possible; in the pathological game, Paul wants to keep the weight as high as possible. Since Carole is adversarial, Paul can do no better than choosing questions where the weight will divide in half. Hence, with q questions remaining, Carole has a winning strategy in the original (resp. pathological) game if the weight is greater (resp. less) than 2^q. The converse is not true; since all states and weights must be integral, Paul might not be able to divide the weight in half and Carole would then be able to cross the 2^q threshold.

2 The Splitting Strategy

Let $(q, \mathbf{x})$ be a game state. We call it *Paul-favorable* if $\mathrm{wt}_q(\mathbf{x}) \leq 2^q$ (in the original game), or $\mathrm{wt}_q(\mathbf{x}) \geq 2^q$ (in the pathological game). Carole has a winning strategy from any state that is not Paul-favorable, by simply choosing the higher-weight (in the original game) or lower-weight (in the pathological game) state for her turns.

For $(q, \mathbf{x})$, let the *splitting* question A be $\mathbf{a} = \begin{cases} \left(\frac{x_0}{2}, \lfloor \frac{x_1}{2} \rfloor\right), & 2 | x_0, \\ \left(\frac{x_0+1}{2}, \lceil \frac{x_1-q+1}{2} \rceil\right), & 2 \not| x_0. \end{cases}$

We will show that this is the optimal question for Paul to ask, although it may not be legal because the game rules require $\mathbf{0} \leq \mathbf{a} \leq \mathbf{x}$ (coordinate-wise). Call Paul-favorable state $(q, \mathbf{x})$ *splitting* if the splitting question is a legal question for Paul to ask. For technical reasons, in the original game call $\mathbf{a} = (2, 0)$ the splitting question for the specific state $(5, (3, 2))$, which becomes splitting after this exception.

Lemma 1 *$(q, \mathbf{x})$ is splitting if and only if at least one of the following holds:*

(1) x_0 is even, or

(2) $x_0 - x_1 < \frac{\mathrm{wt}_q(\mathbf{x})+(3-q)(q+2)}{q+1}$ (equivalently $x_1 > q - 3$), or

(3) $(q, \mathbf{x}) = (5, (3, 2))$ (in the original game).

PROOF. $\mathbf{x}$ is always splitting if x_0 is even; otherwise, $\mathbf{x}$ is splitting if and only if $x_1 - q + 1 > -2$, which gives $x_1 > q - 3$. Multiplying by $q + 2$, then adding $x_0(q+1)$, yields $x_0(q+1)+x_1(q+2) > (q-3)(q+2)+x_0(q+1)$. This is rearranged to $x_0(q + 1) + x_1 + (3 - q)(q + 2) > (q + 1)(x_0 - x_1)$, which is equivalent to $x_0 - x_1 < \frac{\mathrm{wt}_q(\mathbf{x})+(3-q)(q+2)}{q+1}$. Condition (3) is the technical special case of the splitting question. □

Example 2 *In the pathological game, consider $(4, \mathbf{x})$ for $\mathbf{x} = (3, 1)$. We see that $\mathrm{wt}_4(\mathbf{x}) = 16 \geq 2^4$, so $(4, \mathbf{x})$ is Paul-favorable. However, it is not splitting since $x_1 = 1 \leq 4 - 3 = q - 3$.*

This shows that Paul cannot always win from all Paul-favorable states. However, we will show that Paul can always win from any splitting state by repeatedly asking the splitting questions. Further, we will subsequently show that 'Paul-favorable but not splitting' states do not arise after the first, optimal, question.

In the original game, an excessive q spoils the splitting strategy. In this case, Paul can play the game as if q were smaller, and will have unused questions at the end. Therefore, in the original game we need not only $\mathrm{wt}_q(\mathbf{x}) \le 2^q$, but also $\mathrm{wt}_{q-1}(\mathbf{x}) > 2^{q-1}$. Reducing q in this way does not change a splitting state to a non-splitting state.

Theorem 3 *Let $(q, \mathbf{x})$ be splitting. In the original game, assume also that $\mathrm{wt}_{q-1}(\mathbf{x}) > 2^{q-1}$. Let $(q-1, \mathbf{y})$ be the state after the splitting question and Carole's response. Then $\mathrm{wt}_{q-1}(\mathbf{y}) = \lfloor \mathrm{wt}_q(\mathbf{x})/2 \rfloor$ or $\lceil \mathrm{wt}_q(\mathbf{x})/2 \rceil$, and the state $(q-1, \mathbf{y})$ must be splitting.*

PROOF. If x_0 is even, then $\mathrm{wt}_{q-1}(\mathbf{y}') = q\frac{x_0}{2} + \frac{x_0}{2} + \lceil \frac{x_1}{2} \rceil = \lceil \frac{x_0(q+1)+x_1}{2} \rceil = \lceil \mathrm{wt}_q(\mathbf{x})/2 \rceil$, and $\mathrm{wt}_{q-1}(\mathbf{y}'') = q\frac{x_0}{2} + \frac{x_0}{2} + \lfloor \frac{x_1}{2} \rfloor = \lfloor \frac{x_0(q+1)+x_1}{2} \rfloor = \lfloor \mathrm{wt}_q(\mathbf{x})/2 \rfloor$. If x_0 is odd, then $\mathrm{wt}_{q-1}(\mathbf{y}') = q\frac{x_0+1}{2} + \frac{x_0-1}{2} + \lceil \frac{x_1-q+1}{2} \rceil = \lceil \frac{x_0(q+1)+x_1}{2} \rceil = \lceil \mathrm{wt}_q(\mathbf{x})/2 \rceil$, and $\mathrm{wt}_{q-1}(\mathbf{y}'') = q\frac{x_0-1}{2} + \frac{x_0+1}{2} + x_1 - \lceil \frac{x_1-q+1}{2} \rceil = \lfloor \frac{x_0(q+1)+x_1}{2} \rfloor = \lfloor \mathrm{wt}_q(\mathbf{x})/2 \rfloor$.

In the pathological game, because $(q, \mathbf{x})$ is Paul-favorable, $\mathrm{wt}_q(\mathbf{x}) \ge 2^q$ and hence $\mathrm{wt}_{q-1}(\mathbf{y}) \ge \lfloor \mathrm{wt}_q(\mathbf{x})/2 \rfloor \ge \lfloor 2^q/2 \rfloor = 2^{q-1}$. In the original game, $\mathrm{wt}_{q-1}(\mathbf{x}) \ge 2^{q-1} + 1$, and hence $\mathrm{wt}_{q-1}(\mathbf{y}) \ge \lfloor \mathrm{wt}_q(\mathbf{x})/2 \rfloor = \lfloor (\mathrm{wt}_{q-1}(\mathbf{x}) + x_0)/2 \rfloor \ge 2^{q-2}$.

To show that $\mathbf{y}$ is splitting, we will show that $y_0 - y_1 < \frac{\mathrm{wt}_{q-1}(\mathbf{y}) + (4-q)(q+1)}{q}$. For the pathological game, $\mathrm{wt}_{q-1}(\mathbf{y}) \ge 2^{q-1}$ and for the original game, $\mathrm{wt}_{q-1}(\mathbf{y}) \ge 2^{q-2}$. Therefore $\frac{\mathrm{wt}_{q-1}(\mathbf{y}) + (4-q)(q+1)}{q}$ is greater than 1 for all q (except in the original game for $q = 4, 5, 6$, when it is greater than 0).

We now calculate $y_0 - y_1$ after the splitting question. If x_0 is even, then either $y_0 - y_1 = -\lfloor \frac{x_1}{2} \rfloor$ or $y_0 - y_1 = -\lceil \frac{x_1}{2} \rceil$; in either case $y_0 - y_1 \le 0$. If x_0 is odd, then $y_0 - y_1 = -1 - x_1 + \lceil \frac{x_1-q+1}{2} \rceil = \lceil \frac{-x_1-q-1}{2} \rceil \le 0$; or $y_0 - y_1 = 1 - \lceil \frac{x_1-q+1}{2} \rceil$. Because $(q, \mathbf{x})$ is splitting, $x_1 - q + 1 > -2$; hence $y_0 - y_1 \le 1$.

Hence $(q-1, \mathbf{y})$ is splitting except possibly in the original game when x_0 and

y_0 are odd, $y_0 - y_1 = 1$, and $4 \le q \le 6$. Since $\mathrm{wt}_{q-1}(\mathbf{y}) = (q+1)y_0 - 1$, $(q-1, y)$ is splitting unless $1 \ge \frac{(q+1)y_0 - 1 + (4-q)(q+1)}{q}$, which holds if and only if $y_0 \le q - 3$. Thus we are only concerned about states $(5, (3, 2))$ and $(q, (1, 0))$. The former is splitting by definition; in the latter, Paul has won. □

We now apply this strategy to the original and pathological one-lie games. The initial states remaining to resolve are those that are Paul-favorable but not splitting. We show that the first question will settle things; either any first question will make the subsequent state not Paul-favorable, or the optimal first question will make the subsequent state splitting.

Corollary 4 *The original one-lie game is a win for Paul if and only if:*

(1) $n \le 2^q/(q+1)$, *for* n *even, or*

(2) $n \le (2^q - q + 1)/(q+1)$, *for* n *odd.*

PROOF. The initial state is $(q, \mathbf{x})$ for $\mathbf{x} = (n, 0)$. If n is even, then the initial state is either splitting or not Paul-favorable, depending on whether Condition (1) holds. If n is odd and (2) fails, then regardless of Paul's question the next state will not be Paul-favorable. If n is odd, (2) holds, and $n+1 \le 2^q/(q+1)$, then Paul adds an imaginary element to the set; he can win with this additional element and therefore can win without it. Otherwise, $n + 1 > 2^q/(q+1)$. Although $(q, \mathbf{x})$ is not splitting Paul can ask $(\frac{n+1}{2}, 0)$; in which case the next state $(q-1, \mathbf{y})$ will have $\mathbf{y} = (\frac{n+1}{2}, \frac{n-1}{2})$ or $\mathbf{y} = (\frac{n-1}{2}, \frac{n+1}{2})$. We have $\mathrm{wt}_{q-1}(\mathbf{y}) \le q\frac{n+1}{2} + \frac{n-1}{2} = \frac{(q+1)n+(q-1)}{2} \le 2^{q-1}$, applying $\mathrm{wt}_q(\mathbf{x}) \le 2^q - (q-1)$. Because $2^q/(q+1) - (2q-5) > 0$ for all $q > 0$ (a simple calculus exercise), in fact $n+1 > 2q-5$ and hence $n \ge 2q-5$ and $\frac{n-1}{2} \ge q-3 > (q-1)-3$. Therefore, $(q-1, \mathbf{y})$ is splitting. □

Corollary 5 *The pathological one-lie game is a win for Paul if and only if:*

(1) $n \geq 2^q/(q+1)$*, for* n *even, or*

(2) $n \geq (2^q + q - 1)/(q+1)$*, for* n *odd.*

PROOF. The initial state is $(q, \mathbf{x})$ for $\mathbf{x} = (n, 0)$. If n is even, then the initial state is either splitting or not Paul-favorable, depending on whether Condition (1) holds. If n is odd and (2) holds, then $(q, \mathbf{x})$ is not splitting; however Paul can ask $(\frac{n+1}{2}, 0)$; in which case the next state $(q-1, \mathbf{y})$ will have $\mathbf{y} = (\frac{n+1}{2}, \frac{n-1}{2})$ or $\mathbf{y} = (\frac{n-1}{2}, \frac{n+1}{2})$. We have $\mathrm{wt}_{q-1}(\mathbf{y}) \geq q\frac{n-1}{2} + \frac{n+1}{2} = \frac{(q+1)n+(1-q)}{2} \geq 2^{q-1}$, applying $\mathrm{wt}_q(\mathbf{x}) \geq 2^q + (q-1)$. Because $(2^q + q - 1)/(q+1) - (2q-7) > 0$ for all $q > 0$ (a simple calculus exercise), in fact $n > 2q - 7$ and hence $\frac{n-1}{2} > (q-1) - 3$. Therefore, $(q-1, \mathbf{y})$ is splitting. If n is odd and (2) fails, then regardless of Paul's question the next state will not be Paul-favorable. □

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