Arithmetic of Numerical Semigroups on Compound Sequences

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http://www-rohan.sdsu.edu/~vadim/compound-talk.pdf

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http://www.sci.sdsu.edu/math-reu/index.html

This work was done in Summer 2014, jointly with undergraduate Claire Spychalla, and postdoc Christopher O'Neill.

Background

Numerical Semigroup S:

Additive subsemigroup of $\mathbb{N}_0$

Set $g(S) = \max(\mathbb{N}_0 \setminus S)$, Frobenius number

S has finitely many atoms, # = embedding dimension

Goals:

(1) arithmetical properties

(2) numerical semigroup properties

Geometric Sequences

Geometric sequence: $a, ar, ar^2, \ldots, ar^k$

For these to be the atoms of a numerical semigroup:

(1) each atom is a positive integer

(2) $\gcd(a, ar, \ldots, ar^k) = 1$

(3) $r \notin \mathbb{Z}$, else only one atom

Conditions give: $a^m, a^{m-1}b, a^{m-2}b^2, \ldots, ab^{m-1}, b^m$

(1) $a, b \in \mathbb{Z}$; (2) $b > a \geq 2$; (3) $\gcd(a, b) = 1$

Known: Frobenius number, little else

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Build your own Compound Sequence with materials you have around the house

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Pick $a_1, a_2, \ldots, a_m \geq 2$ freely.

Pick each b_i so that

(1) $b_i > a_i$; and

(2) $\gcd(b_i, a_i a_{i+1} \cdots a_m) = 1$.

Ex: $a_1 = 5, a_2 = 2, b_1 = 7, b_2 = 5$ $S = \langle 10, 14, 35 \rangle$

$\langle a, b, c \rangle$, from $[2, 200]$. 1% compound, 0.6% arithmetic

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Compound Sequence Properties

$$n_0 = a_1 a_2 \cdots a_m$$
$$n_1 = b_1 a_1 \cdots a_m$$
$$\vdots$$
$$n_m = b_1 b_2 \cdots b_m$$

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$n_1 n_2 \cdots n_{m-1} = \gcd(n_0, n_1) \gcd(n_1, n_2) \cdots \gcd(n_{m-1}, n_m)$

3-generated ($m = 2$): $n_1 = \gcd(n_0, n_1) \gcd(n_1, n_2)$

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Main Structural Result

For each $i \in [1, m]$, $a_i n_i = b_i n_{i-1}$.

Thm: These m relations generate the monoid of swaps.

i.e. between any two factorizations of $s \in S$, a chain exists where each step is some $a_i n_i - b_i n_{i-1}$ or its negative.

Note: n_i, n_{i-1} atoms, $a_i, b_i \in \mathbb{N}$.

Catenary Degree

Thm: $c(S) = \max\{b_1, b_2, \ldots, b_m\}$

Pf: One direction, build a chain with $a_i n_i - b_i n_{i-1}$.
Each step has distance of $\max(a_i, b_i) = b_i$.

Other direction, b_m maximal, look at $b_m n_{m-1} (= a_m n_m)$.

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Delta Sets

Thm: Set $N = \{b_1 - a_1, b_2 - a_2, \ldots, b_m - a_m\}$. Then

(1) $\min(\Delta(S)) = \gcd(N) = \gcd(\Delta(S))$,
(2) $N \subseteq \Delta(S)$, and
(3) $\max(\Delta(S)) = \max(N)$

Idea: A step with $a_i n_i - b_i n_{i-1}$, lengths differ by $b_i - a_i$.

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Cor 1: If $|N| = 1$, then $\Delta(S) = N$. (includes geometric)

Cor 2: If $N = \{\alpha, 2\alpha, \ldots, \beta\alpha\}$ or $N = \{2\alpha, 3\alpha, \ldots, \beta\alpha\}$, then $\Delta(S) = \{\alpha, 2\alpha, \ldots, \beta\alpha\}$.

Ex: $a_1 = a_2 = 2, b_1 = 7, b_2 = 9, S = \langle 4, 14, 63 \rangle, N = \{5, 7\}$.
Thm says $\{1, 5, 7\} \subseteq \Delta(S) \subseteq \{1, 2, 3, 4, 5, 6, 7\}$ while actually $\Delta(S) = \{1, 2, 3, 5, 7\}$.

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Restricted Length Sets

Thm: Let $k < \min\{a_1, a_2, \ldots, a_m\}$. Then the product of any k atoms from S has unique factorization.

In particular, $\nu_k(S) = \{k\}$.

Frobenius Number and Genus

Frobenius number: $g(S) = \sum_{j=1}^{m} n_j a_j - \sum_{j=0}^{m} n_j$

Genus of S is defined as $|\mathbb{N} \setminus S|$.

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Apéry Sets

For atom n_i, the *Apéry set* is defined as

$$Ap(S, n_i) = \{n \in S : n - n_i \notin S\}$$

i.e. the smallest element in each equivalence class mod n_i.

Thm: $Ap(S, n_i) = \{\sum_{j=0}^{m} u_j n_j : u_j \in \mathbb{Z}, \star\}$

- $\star$ For $j \in [0, i-1]$, $0 \leq u_j < b_{j+1}$
- $\star$ For $j = i$, $u_i = 0$
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For Further Reading

- C. O'Neill, VP, C. Spychalla
  Numerical Semigroups on Compound Sequences
  (under review)
  http://www.sci.sdsu.edu/math-reu/ops1.pdf
- A. Fry, S. McConnell, C. Spychalla, Z. Stanley, B. Van Over
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  http://www.sci.sdsu.edu/math-reu/2014-1.pdf
- D. Ong, VP
  The Frobenius Number of Geometric Sequences
  INTEGERS (8) 2008, #A33.