Arithmetic of Congruence Monoids

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http://www-rohan.sdsu.edu/~vadim/cm.pdf

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This work was done in Summer 2012, jointly with undergraduates Arielle Fujiwara, Joseph Gibson, Matthew Jenssen, Daniel Montealegre, Ari Tenzer.

Standard Notation

We consider arithmetic in certain (multiplicative) submonoids of $\mathbb{N}$.

As a tool, we also consider multiplication in $\mathbb{Z}_n$.

For any set S, we write:

$S^\times$ for the units of S

$S^\bullet$ for the non-units of S

irreducibles, elasticity ρ, valuation $\nu_p(x)$, etc.

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Nonstandard Notation

Let $\Gamma \subseteq \mathbb{N}$, and let $n \in \mathbb{N}$.

We let $[\ ]_n : \mathbb{Z} \to \mathbb{Z}_n$ be the natural epimorphism.

$$[\Gamma]_n = \{[x]_n \in \mathbb{Z}_n : x \in \Gamma^\bullet\} \subseteq \mathbb{Z}_n$$

$$\langle \Gamma \rangle_n = \{x \in \mathbb{N} : [x]_n \in [\Gamma]_n\} \cup \{1\} \subseteq \mathbb{N}$$

$$\Gamma_n = \{\gcd(x, n) : x \in \Gamma^\bullet\} \subseteq [1, n]$$

$$= \{\gcd(x, n) : [x] \in [\Gamma]_n\}$$

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Monoids Defined

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Trivial: $\Gamma \subseteq \langle \Gamma \rangle_n \subseteq \langle \Gamma \rangle_k$, for any $k|n$.

If $\langle \Gamma \rangle_n$ is closed, we call $\langle \Gamma \rangle_n$ a *congruence monoid*.
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For ACM: $|[\Gamma]_n| = |\Gamma_n| = 1$. $\Gamma_n = \{d\}$, $d = \gcd(m,n)$
$[\Gamma]_n = \{[m]\}$, and $[m][m] = [m]$. (in $\mathbb{Z}_n$)

1. $[\Gamma]_n = [\Gamma]_n^\times$. "regular" Note: must have $[m] = [1]$.
2. $[\Gamma]_n = [\Gamma]_n^\bullet$. "singular"

2.1 $[m] = [0]$. "$M_{a,a}$". Here $\langle\Gamma\rangle_n = (n\mathbb{N}) \cup \{1\}$

2.2 $d = p^\alpha$ for $\alpha, p \in \mathbb{N}$, p prime. "local"

2.3 d is not a prime power. "global"

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Regular ACM: $[\Gamma]_n = [\Gamma]_n^\times = \{[1]\}$
e.g. Hilbert monoid $1 + 4\mathbb{N}_0$.

Very nice behavior: $\langle \Gamma \rangle_n$ is saturated in $\mathbb{N}$, and hence Krull.
There exists a transfer homomorphism $\phi : \langle \Gamma \rangle_n \to \mathcal{B}(\mathbb{Z}_n^\times)$,
the block monoid on the (multiplicative) group $\mathbb{Z}_n^\times$.
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$M_{a,a}$ local: $m = n = d = p^\alpha$

$\nu_p : \langle\Gamma\rangle_n \to \langle \alpha, \alpha+1, \ldots, 2\alpha - 1\rangle$ is a transfer homomorphism into $\mathbb{N}_0$ under addition.

$\rho = \frac{2\alpha-1}{\alpha}$, accepted, no primes

If $\alpha = 1$, half-factorial

If $\alpha > 1$, not half-factorial, not fully elastic, $\Delta = \{1\}$.

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There is a natural transfer homomorphism $\phi : \langle \Gamma \rangle_n \to (\alpha_1, \ldots, \alpha_k) + \mathbb{N}_0^k$, a submonoid of $\mathbb{N}_0^k$ under addition.

Each element is product of two irreducibles “bifurcus”.
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Theorem: $\langle\Gamma\rangle_n^\bullet = (d\mathbb{N}) \cap \langle\Gamma\rangle_{n/d}^\bullet$ where

$\langle d\mathbb{N}\rangle_d$ is a singular $M_{a,a}$ ACM
$\langle\Gamma\rangle_{n/d}$ is a regular ACM

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Singular Local ACM: $[\Gamma]_n = \{[m]\}$, $\Gamma_n = \{d\}$ where $d = p^\alpha$

There exists minimal $\beta \geq \alpha$ such that $p^\beta \in \langle\Gamma\rangle_n$.

$\rho = \frac{\alpha+\beta-1}{\alpha}$, half-factorial if $\alpha = \beta = 1$, Δ a known interval. Accepted? sometimes. Full? sometimes.

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Each element is the product of at most λ irreducibles.
no primes, $\rho = \infty$, not fully elastic, $\Delta \subseteq [1, \lambda - 2]$.

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Set $d = \operatorname{lcm}(\Gamma_n)$, $\delta = \gcd(\Gamma_n)$

1. If $d = \delta$ then $|\Gamma_n| = 1$. “J-monoid”
2. $[\Gamma]_n = [\Gamma]_n^\times$. “regular” i.e. $\Gamma_n = \{1\}$
3. $[\Gamma]_n = [\Gamma]_n^\bullet$. “singular” i.e. $1 \notin \Gamma_n$

   $d = p^\alpha$ “local” $d = p^\alpha r$ “global”
4. $[\Gamma]_n^\times, [\Gamma]_n^\bullet$ each nonempty. “semi-singular”

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Congruence Monoid General Result

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$d = \operatorname{lcm}(\Gamma_n)$, $\delta = \gcd(\Gamma_n)$

Thm: $\langle\Gamma\rangle_{n/d}$ is a regular CM and $[\Gamma]_{n/d} \leq \mathbb{Z}_{n/d}^\times$

CM Results: Regular

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Regular CM: $[\Gamma]_n = [\Gamma]_n^\times$ $\quad d = 1$ $\quad [\Gamma]_n \leq \mathbb{Z}_n^\times$

Lemma: $\langle\Gamma\rangle_n$ is saturated in $\mathbb{N}$ (hence Krull)

Pf: Let $x, y \in \mathbb{N}^\bullet$ with $x, xy \in \langle\Gamma\rangle_n$. Then $[x]_n \in [\Gamma]_n \leq \mathbb{Z}_n^\times$.
Let $z \in \mathbb{N}$ with $[z]_n[x]_n = [1]_n$. $zxy \in \langle\Gamma\rangle_n$, so
$[zxy]_n = [z]_n[x]_n[y]_n = [y]_n \in [\Gamma]_n$. Hence $y \in \langle\Gamma\rangle_n$.

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$d = \operatorname{lcm}(\Gamma_n)$, $\delta = \gcd(\Gamma_n)$

Theorem: $(d\mathbb{N}) \cap \langle \Gamma \rangle^\bullet_{n/d} \subseteq \langle \Gamma \rangle^\bullet_n \subseteq (\delta\mathbb{N}) \cap \langle \Gamma \rangle^\bullet_{n/d}$

Note 1: Recall that $\langle \Gamma \rangle_{n/d}$ is a regular CM

Note 2: If $d = \delta$ "J-monoid" $(d\mathbb{N}) \cap \langle \Gamma \rangle^\bullet_{n/d} = \langle \Gamma \rangle^\bullet_n$

J-monoids Group Structure

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J-monoid: $\Gamma_n = \{d\}$ $\quad \langle\Gamma\rangle_n^\bullet = (d\mathbb{N}) \cap \langle\Gamma\rangle_{n/d}^\bullet$

Theorem: $[\Gamma]_n$ has a group structure under multiplication

Example: $\Gamma = \{4, 16, 24, 36, 44, 56, 64, 76, 84, 96\}$,
$n = 100, d = 4$. $[\Gamma]_{100} \cong \mathbb{Z}_{10}$. The identity is. . . 76.
The $\phi(10) = 4$ generators are. . . 4, 44, 64, and 84.

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Thm: Suppose $[\Gamma]_n^\times \neq \emptyset$. Then $\langle \Gamma \rangle_n$ has ∞ many primes.

Pf: $[1] \in [\Gamma]_n$, Dirichlet's theorem on primes.

Note: If a CM is singular, then it has no primes.

Thm: Suppose $\langle \Gamma \rangle_n$ is semi-singular, and $[\Gamma]_n^\bullet$ is a global singular ACM. Then $\rho = \infty$ and the elasticity is *full*.

e.g. $\Gamma = \{1, 6\}, n = 6$; $\{1\}$ $\rho = 1$, $\{6\}$ $\rho = \infty$ not full

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Recall that $\Gamma \subseteq \mathbb{N}$, $n \in \mathbb{N}$, $[\Gamma]_n = \{[x]_n \in \mathbb{Z}_n : x \in \Gamma^\bullet\}$,
$\langle\Gamma\rangle_n = \{x \in \mathbb{N} : [x]_n \in [\Gamma]_n\} \cup \{1\}$, $\Gamma_n = \{\gcd(x, n) : x \in \Gamma^\bullet\}$

Thm: Suppose $[\Gamma]_n^\times \neq \emptyset$. Then $\langle\Gamma\rangle_n$ has ∞ many primes.
Pf: $[1] \in [\Gamma]_n$, Dirichlet's theorem on primes.

Note: If a CM is singular, then it has no primes.

Thm: Suppose $\langle\Gamma\rangle_n$ is semi-singular, and $[\Gamma]_n^\bullet$ is a global singular ACM. Then $\rho = \infty$ and the elasticity is *full*.

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Singular Local CM: $d = p^\alpha$, $\delta = p^\gamma$

There exists minimal $\beta \geq \gamma$ such that $p^\beta \in \langle\Gamma\rangle_n$.

Thm: $\frac{\alpha+\beta-1}{c\gamma} \leq \rho \leq \frac{\alpha+\beta-1}{\gamma}$, for $c = \lceil(\alpha + \beta - 1 - \gamma)/\beta\rceil$

Note 1: If J-monoid, then $c = 1$, $\alpha = \gamma$, and equality
Note 2: Half-factorial if $\alpha = \beta = \gamma = 1$
Accepted? sometimes. Full? sometimes.

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Thm: Suppose d, δ share the same prime factors. Then each element is the product of at most λ irreducibles.

Note 1: If J-monoid, hypothesis is met

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For Further Reading

P. Baginski, S. Chapman
Arithmetic Congruence Monoids: A Survey (under review)

L. Crawford, VP, J. Steinberg, M. Wlliams
Accepted Elasticity in Local ACMs (under review)

M. Jenssen, D. Montealegre, VP
Irreducible Factorization Lengths and the Elasticity Problem Within $\mathbb{N}$ (to appear in *American Math Monthly*)

A. Fujiwara, J. Gibson, M. Jenssen, D. Montealegre, VP, Ari Tenzer
Arithmetic of Congruence Monoids (in preparation)

C. Allen, VP, W. Radil, R. Rankin, H. Williams
Full Elasticity in Local ACMs (in preparation)