Continued Fractions in the 21st Century

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Abstract. We present useful (though largely not original) continued fraction tools, that deserve to be more widely known to a broad mathematical audience.

Many mathematicians, myself included, learned about continued fractions in a first course on elementary number theory (from a book like [**1**], [**2**], or [**3**]). They are typically presented briefly, as a curiosity with various technical properties, useful for solving Pell's equation and in other special contexts, but otherwise of niche interest.

The intent of this paper is to reframe continued fractions as a more natural and usually superior alternative to decimals as a representation of real numbers, much like radians are a more natural and usually superior alternative to degrees as a representation of angles. Towards this ambitious goal, we will create new notation conducive to better intuition, and describe "folklore" arithmetic results, familiar to experts.

We first review the basics. We only consider so-called simple continued fractions, whose numerators are all 1 and whose denominators are built from a sequence (finite or infinite) of integers, all positive (apart from, possibly, the first term). See Figure 1 for examples.

Because $n = (n-1) + \frac{1}{1}$, any finite continued fraction has two representations, e.g. $[2; 4] = [2; 3, 1]$ and $[2; 4, 6] = [2; 4, 5, 1]$ in the traditional notation. Historically, the shorter representation has been canonical. Instead, we choose the representation that gives an even number of terms. This allows us to use a new notation, pairing off the terms, with semicolons separating pairs. In our proposed new notation we would canonically write $[2, 4]$ and $[2, 4; 5, 1]$.

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}}} \qquad\qquad \pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{\ddots}}}}}$$

Old: $[a_0; a_1, a_2, a_3]$
New: $[a_0, a_1; a_2, a_3]$

Old: $\pi \approx [3; 7, 15, 1, 292]$
New: $\pi \approx [3, 7; 15, 1; 291, 1]$

Figure 1. Examples of finite and infinite continued fractions

1. CONTINUED FRACTION MATRICES

To better enjoy the merits of continued fractions, we first take a brief detour into vectors and matrices. Our vectors $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ henceforth are assumed to have real[1] entries, not both zero. We build an equivalence relation $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] \equiv \left[\begin{smallmatrix}\alpha\gamma\\ \beta\gamma\end{smallmatrix}\right]$, for all positive $\gamma \in \mathbb{R}$. We choose as canonical equivalence class representative $\left[\begin{smallmatrix}\alpha/\beta\\ 1\end{smallmatrix}\right]$, where for convenience (abusing notation) call this vector just $\frac{\alpha}{\beta}$. If $\beta = 0$, we all this vector ∞; if this is troubling, consider instead a limit, leading to the same outcome.

[1] More precisely, from the real projective line $\mathbb{R} \cup \{\infty\}$.

For any $a, b \in \mathbb{Z}$, with b positive, we define the continued fraction matrix

$$C(a,b) = \left[\begin{smallmatrix} ab+1 & a \\ b & 1 \end{smallmatrix}\right].$$

We now introduce a nonnegative uncertainty vector $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] = x$. A priori that uncertainty is complete: $0 \le x \le \infty$. Note that $C(a,b)x = \left[\begin{smallmatrix} ab+1 & a \\ b & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} (ab+1)x+a \\ bx+1 \end{smallmatrix}\right] \equiv \left[\begin{smallmatrix} \frac{(ab+1)x+a}{bx+1} \\ 1 \end{smallmatrix}\right] = \frac{(ab+1)x+a}{bx+1}$. As x varies from 0 to ∞, this fraction varies from $\frac{a}{1} = a$ to $\frac{ab+1}{b} = a + \frac{1}{b}$. Hence, multiplication by $C(a,b)$ has reduced our uncertainty from $[0,\infty]$ to $[a, a+\frac{1}{b}]$.

From an intuitive standpoint, we consider the fraction built from the first column of $C(a,b)$, namely $\frac{ab+1}{b} = a + \frac{1}{b}$, as our best estimate. The fraction from the second column, namely $\frac{a}{1} = a$, identifies the maximal error in that estimate, by giving a lower bound. We know $a + \frac{1}{b}$ is an upper bound, with an error at most $(a+\frac{1}{b}) - a = \frac{1}{b}$.

We can refine that estimate, and shrink the error, by taking additional continued fraction matrices. If x did not vary through the entirety of $[0,\infty]$, then the range of $C(a,b)x$ would be smaller as well. $C(a,b)C(c,d)x$ means that in

$$C(a,b)C(c,d)x = \frac{(ab+1)C(c,d)x + a}{bC(c,d)x+1},$$

$C(c,d)x$ varies from c to $c + \frac{1}{d}$, reducing the uncertainty of $C(a,b)C(c,d)x$. Fortunately, we can use matrix multiplication here, calculating

$$\left[\begin{smallmatrix} ab+1 & a \\ b & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} cd+1 & c \\ d & 1 \end{smallmatrix}\right] x = \left[\begin{smallmatrix} (ab+1)(cd+1)+ad & (ab+1)c+a \\ b(cd+1)+d & bc+1 \end{smallmatrix}\right] x,$$

where we take our best estimate as $\frac{(ab+1)(cd+1)+ad}{b(cd+1)+d}$, and our lower bound as $\frac{(ab+1)c+a}{bc+1}$. The best estimates from these continued fraction matrices translate directly into the new continued fraction notation as $C(a,b) = [a,b]$ and $C(a,b)C(c,d) = [a,b;c,d]$.

We can now pause and compare with decimal notation. When we say $\pi \approx 3.1$, our best guess is that $\pi = 3 + \frac{1}{10}$. The true value is either in $[3 + \frac{1}{10} - \frac{1}{20}, 3 + \frac{1}{20})$, if 3.1 was rounded[2], or $[3+\frac{1}{10}, 3+\frac{1}{10}+\frac{1}{10})$, if 3.1 was truncated. If we don't know if it is rounded or truncated, to be safe we must say the true value is in $[3+\frac{1}{10}-\frac{1}{20}, 3+\frac{1}{10}+\frac{1}{10})$. This is an interval of size $\frac{3}{20}$. What a confusing mess! Instead consider the continued fraction notation $\pi \approx [3,7]$. Our best guess is that $\pi = 3 + \frac{1}{7}$, with the true value in $(3, 3+\frac{1}{7}]$. This is simpler, known to be an overestimate, and with a (slightly) smaller error interval than $\frac{3}{20}$.

Taking twice as many terms, $\pi \approx 3.141$ gives our best guess of $\pi = 3 + \frac{141}{1000}$, with an error interval of size $\frac{3}{2000}$. Instead taking $\pi \approx [3,7;15,1] = \left[\begin{smallmatrix} 22 & 3 \\ 7 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 16 & 15 \\ 1 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 355 & 333 \\ 113 & 106 \end{smallmatrix}\right]$. This gives a best guess of $\pi = \frac{355}{113}$, with an error interval of size $\frac{355}{113} - \frac{333}{106} = \frac{1}{11978}$. We get a nicer fraction, and a much smaller error. Indeed, any large term (like 15) in a continued fraction makes a particularly small error interval. Decimal notation always gives the same error interval, with no hope for better.

Suppose now we have two reals x, y and we wish to resolve trichotomy: are they equal, or is one of them greater? With decimal notation, we start by finding the first place where they disagree. Sometimes this is enough, but sometimes our work is just beginning – if $x \approx 1.24$ and $y \approx 1.23$, we still don't know if $x = y$ or $x > y$ (or

[2] If we agreed that 3.05 would be rounded up to 3.1. This adds even more complexity!

even $x < y$ if x was rounded up from 1.236 while y was truncated down from 1.238). We must continue, possibly forever, looking at more terms. However, with continued fraction notation our work is done once we find the first term of disagreement. Suppose $x \approx [\ldots; c, d]$ and $y \approx [\ldots; c', d']$. Comparing $c + \frac{1}{d}$ with $c' + \frac{1}{d'}$, we see that if $c > c'$ then x is bigger; if instead $c = c'$ and $d > d'$ then x is smaller. This remains true regardless of any subsequent terms. Equality can only hold if all terms agree.

Note that if we momentarily allowed $b = 0$, then $C(0,0) = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$, so $C(a,b) = C(a,b)C(0,0) = C(a,b)C(0,0)C(0,0)$. Hence, just as with decimal representation, we can interpret any finite continued fraction $[a_1, b_1; a_2, b_2; \ldots; a_k, b_k]$ as the infinite continued fraction $[a_1, b_1; a_2, b_2; \ldots; a_k, b_k; 0, 0; 0, 0; \ldots]$. Apart from this application we resume insisting that $b > 0$ in $C(a,b)$.

For one more comparison with decimal notation, to prove Cantor's famous diagonalization theorem requires several annoying technicalities, partly due to the fact that $1.\overline{0} = 0.\overline{9}$. Many real numbers have two decimal representations, so we need to pick one when we place them in a putative bijection with $\mathbb{N}$. Then, when we build our new real number digit-by-digit, we can't just add one since there are only ten digits available; this is usually handled with awkward cases. Lastly, when we are done, we need to be careful that our new number doesn't have two decimal representations, to ensure it is not on the list. Compare with the following proof using continued fractions, avoiding all of this mess.

Proof of Cantor's theorem. Suppose $f : \mathbb{N} \to \mathbb{R}$ via $n \mapsto [a_{n,1}, a_{n,2}; \ldots]$ were a bijection, choosing infinite representations. We see $x = [a_{1,1} + 1, a_{2,2} + 1; \ldots]$ is not in the image of f, with its n-th entry differing from $f(n)$. ■

To calculate $C(a,b)$ to estimate a known real x, we can iteratively use the classical algorithm: $a = \lfloor x \rfloor$, $b = \lfloor \frac{1}{x-a} \rfloor$. This suffices, with some care, to prove that continued fraction representation is unique (up to, possibly, the last nonzero term), but we offer a different, novel, uniqueness result. Consider generic 2×2 integer matrices $M = \left[\begin{smallmatrix}u_1 & u_2\\ v_1 & v_2\end{smallmatrix}\right]$, not necessarily built as a product of continued fraction matrices, with $v_1 \geq 0$ and u_1, v_1 not both zero. Considering Mx, we can take $\frac{u_1}{v_1}$ as our best estimate for x, and $\frac{u_2}{v_2}$ as providing the error. The restrictions maintains generality, since $\frac{u_1}{v_1} = \frac{-u_1}{-v_1}$, and $u_1 = v_1 = 0$ is no estimate. If $u_2 = v_2 = 0$ then the error is unknown.

For matrices M, C, we say that C nicely left divides M to mean there is some matrix $N = \left[\begin{smallmatrix}u_1' & u_2'\\ v_1' & v_2'\end{smallmatrix}\right]$ with $CN = M$, $u_1' \geq 0$, and $v_1' \geq 0$. Theorem 1, below, tells us that for any M meeting our two restrictions, there is a unique sequence of continued fraction nice left divisors $M = C(a_1, b_1)C(a_2, b_2)\cdots C(a_k, b_k)N$, where terminal matrix N has lower-left entry 0. Further, each a_i, b_i are positive, except possibly a_1. Hence the continued fraction corresponding to M is $[a_1, b_1; a_2, b_2; \ldots, a_k, b_k]$, with the terminal N just ignored as its best estimate is ∞. This allows us to consider any rational estimate and rational error, not just ones arising from a product of continued fraction matrices.

Theorem 1. *Let $M = \left[\begin{smallmatrix}u_1 & u_2\\ v_1 & v_2\end{smallmatrix}\right]$, where $u_1, u_2, v_1, v_2 \in \mathbb{Z}$, $v_1 \geq 0$, and u_1, v_1 are not both zero. If $v_1 = 0$, then there are no continued fraction matrices that nicely left divide M.*

If, instead, $v_1 \neq 0$, then there is exactly one continued fraction matrix $C(a,b)$ that nicely left divides M. Taking $N = \left[\begin{smallmatrix}u_1' & u_2'\\ v_1' & v_2'\end{smallmatrix}\right]$ with $C(a,b)N = M$, we must have $u_1' > v_1' \geq 0$. We have $a > 0$ if and only if $u_1 > v_1$. Lastly, we also must have $v_1 > v_1'$.

Proof. We have $C(a,b)^{-1} = \begin{bmatrix} 1 & -a \\ -b & ab+1 \end{bmatrix}$, so we have $N = \begin{bmatrix} u_1 - av_1 & u_2 - av_2 \\ v_1(ab+1) - bu_1 & v_2(ab+1) - bu_2 \end{bmatrix}$. If $v_1 = 0$, then $u_1' = u_1$ and $v_1' = -bu_1$. Since $b > 0$ and $u_1 \neq 0$, either $u_1' < 0$ or $v_1' < 0$. Hence $C(a,b)$ fails to nicely left divide M.

If instead $v_1 > 0$, then choose $a = \lceil \frac{u_1}{v_1} \rceil - 1$. Note that this is positive if and only if $u_1 > v_1$. This choice makes $\frac{u_1}{v_1} - 1 \leq a < \frac{u_1}{v_1}$, which rearranges to $0 < u_1 - av_1 \leq v_1$. Now we have $u_1' = u_1 - av_1$ and $0 < u_1' \leq v_1$. This is a type of division algorithm, but with remainder strictly greater than zero. Now choose $b = \lfloor \frac{v_1}{u_1'} \rfloor$. This choice makes $\frac{v_1}{u_1'} - 1 < b \leq \frac{v_1}{u_1'}$, which rearranges to $0 \leq v_1 - bu_1' < u_1'$. This is the usual division algorithm. Now we have $v_1' = v_1 - bu_1'$, with $0 \leq v_1' < u_1'$. Note that $v_1' = v_1 - bu_1' = v_1 - b(u_1 - av_1) = v_1(ab+1) - bu_1$. Combining $v_1' < u_1'$ with $u_1' \leq v_1$, we see that $v_1' < v_1$. ■

Note also that Theorem 1 gives an upper bound for the number of continued fraction terms nicely left dividing M, since the lower left entry strictly decreases with each step. Better bounds are known, using the details of the division algorithm.

2. CONTINUED FRACTION ARITHMETIC We turn our attention now to the four basic operations of addition, subtraction, multiplication, and division. Using decimal representation, addition and subtraction have an unknown but potentially large amount of annoying carrying and borrowing. At least we can do these operations on n digits in $O(n)$ operations. Multiplication and division (at least in the traditional way) with decimal representation is horrible. Still plenty of carrying and borrowing, and now we must do $O(n^2)$ operations for n-digit inputs, multiplying each digit of each input by each digit of the other. These four operations have four completely different algorithms, and the error of the result is tricky to estimate.

It turns out that if we use continued fraction representation, there is a single, simple, $O(n)$ algorithm on inputs of n terms, for all four operations, with no carrying or borrowing, that also gives an error bound. This was first discovered by Bill Gosper in the 1972 HAKMEM technical report [**4**], as well as its difficult-to-find appendix [**5**]. This "folklore" is well-known to experts in computing (see [**6, 7, 8, 9, 10, 11, 12**]), but has only rarely appeared in the mathematical literature (see [**13, 14, 15**]) and never (to the best of our knowledge) in a mathematical introduction to continued fractions.

In fact, this algorithm lets us calculate the more general function

$$f(x,y) = \frac{\alpha_1 xy + \alpha_2 x + \alpha_3 y + \alpha_4}{\beta_1 xy + \beta_2 x + \beta_3 y + \beta_4},$$

for our choice of parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$. Addition corresponds to $\alpha_2 = \alpha_3 = \beta_4 = 1$ (all others zero), subtraction corresponds to $\alpha_2 = \beta_4 = 1, \alpha_3 = -1$, multiplication corresponds to $\alpha_1 = \beta_4 = 1$, while division corresponds to $\alpha_2 = \beta_3 = 1$. We can even do other operations in one step, like $\frac{x-y}{xy+2}$.

Formally, this is done with $2 \times 2 \times 2$ tensors, but it is simpler to think in terms of two 2×2 matrices. We calculate $\begin{bmatrix} y \\ 1 \end{bmatrix}^T \begin{bmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = [\alpha_1 xy + \alpha_2 x + \alpha_3 y + \alpha_4]$, and similarly $\begin{bmatrix} y \\ 1 \end{bmatrix}^T \begin{bmatrix} \beta_1 & \beta_3 \\ \beta_2 & \beta_4 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = [\beta_1 xy + \beta_2 x + \beta_3 y + \beta_4]$. Stacking these on top of each other gives $\begin{bmatrix} \alpha_1 xy + \alpha_2 x + \alpha_3 y + \alpha_4 \\ \beta_1 xy + \beta_2 x + \beta_3 y + \beta_4 \end{bmatrix}$. As x, y each vary in $[0, \infty]$, the fraction $\frac{\alpha_1 xy + \alpha_2 x + \alpha_3 y + \alpha_4}{\beta_1 xy + \beta_2 x + \beta_3 y + \beta_4}$ varies in the convex hull of the four fractions $\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \frac{\alpha_3}{\beta_3}, \frac{\alpha_4}{\beta_4}$. We can build a 2×2 matrix M by putting the terms of the largest of these on the left, the smallest of these on the right, and interpret (or write in continued fraction notation) using Theorem 1.

At first glance, this might seem silly, since we picked all eight parameters, so the four fractions are fixed. However this is only for generic uncertainty vectors x, y. Suppose we have approximations for each: $C(a_1, b_1)C(a_2, b_2)\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ and $C(a_3, b_3)C(a_4, b_4)\left[\begin{smallmatrix} y \\ 1 \end{smallmatrix}\right]$. Substituting into the first, we get

$$\left(C(a_3,b_3)C(a_4,b_4)\left[\begin{smallmatrix} y \\ 1 \end{smallmatrix}\right]\right)^T \left[\begin{smallmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{smallmatrix}\right] \left(C(a_1,b_1)C(a_2,b_2)\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\right) =$$

$$\left[\begin{smallmatrix} y \\ 1 \end{smallmatrix}\right]^T \underbrace{\left(C(a_3,b_3)C(a_4,b_4)\right)^T \left[\begin{smallmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{smallmatrix}\right] C(a_1,b_1)C(a_2,b_2)} \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] =$$

$$\left[\begin{smallmatrix} y \\ 1 \end{smallmatrix}\right]^T \begin{bmatrix} \alpha_1' & \alpha_3' \\ \alpha_2' & \alpha_4' \end{bmatrix} \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$$

If M_y is the product of continued fractions representing y, and M_x is the product of continued fractions representing x, then the inner term (giving our new $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ parameters) is just $\begin{bmatrix} \alpha_1' & \alpha_3' \\ \alpha_2' & \alpha_4' \end{bmatrix} = M_y^T \left[\begin{smallmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{smallmatrix}\right] M_x$. Similarly, the second matrix is $\begin{bmatrix} \beta_1' & \beta_3' \\ \beta_2' & \beta_4' \end{bmatrix} = M_y^T \begin{bmatrix} \beta_1 & \beta_3 \\ \beta_2 & \beta_4 \end{bmatrix} M_x$. If we do this calculation and find we want more accuracy, we can calculate additional continued fraction matrices for x, y. We multiply on the right by the former, and multiply on the left by the transpose of the latter. Hence, if desired, we can compute $f(x, y)$ iteratively, improving the accuracy one step at a time.

Example 2. *We have* $\sqrt{2} \approx [1, 2; 2, 2] = \left[\begin{smallmatrix} 17 & 7 \\ 12 & 5 \end{smallmatrix}\right]$, *and* $e \approx [2, 1; 2, 1] = \left[\begin{smallmatrix} 11 & 8 \\ 4 & 3 \end{smallmatrix}\right]$. *To find* $\sqrt{2} + e$ *we calculate* $\left[\begin{smallmatrix} 11 & 8 \\ 4 & 3 \end{smallmatrix}\right]^T \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} 17 & 7 \\ 12 & 5 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 200 & 83 \\ 147 & 61 \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} 11 & 8 \\ 4 & 3 \end{smallmatrix}\right]^T \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 17 & 7 \\ 12 & 5 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 48 & 20 \\ 36 & 15 \end{smallmatrix}\right]$. *Of the fractions* $\frac{200}{48}, \frac{83}{20}, \frac{147}{36}, \frac{61}{15}$, *the first is the largest and the last is the smallest, so* $\sqrt{2} + e \approx \left[\begin{smallmatrix} 200 & 61 \\ 48 & 15 \end{smallmatrix}\right] \approx [4, 6]$.

The maximum error here is fairly high $\left(\frac{200}{48} - \frac{61}{15} = \frac{1}{10}\right)$. *Suppose we wish to improve, by improving our estimate for* e. *So, consider* $e \approx [2, 1; 2, 1; 1, 4] = \left[\begin{smallmatrix} 11 & 8 \\ 4 & 3 \end{smallmatrix}\right] \left[\begin{smallmatrix} 5 & 1 \\ 4 & 1 \end{smallmatrix}\right]$. *We update* $\left[\begin{smallmatrix} 5 & 4 \\ 1 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 200 & 83 \\ 147 & 61 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1588 & 659 \\ 347 & 144 \end{smallmatrix}\right]$, *and* $\left[\begin{smallmatrix} 5 & 4 \\ 1 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 48 & 20 \\ 36 & 14 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 384 & 160 \\ 84 & 35 \end{smallmatrix}\right]$. *Now we get* $\sqrt{2} + e \approx \left[\begin{smallmatrix} 1588 & 144 \\ 384 & 35 \end{smallmatrix}\right]$, *which is* $[4, 7; 2, 1; 1, 2]$, *with maximum error less than* $\frac{1}{47}$.

Now let's calculate $\sqrt{2} \times e$. *We have* $\left[\begin{smallmatrix} 5 & 1 \\ 4 & 1 \end{smallmatrix}\right]^T \left[\begin{smallmatrix} 11 & 8 \\ 4 & 3 \end{smallmatrix}\right]^T \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} 17 & 7 \\ 12 & 5 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1479 & 609 \\ 323 & 133 \end{smallmatrix}\right]$, *while the second matrix is* $\left[\begin{smallmatrix} 384 & 160 \\ 84 & 35 \end{smallmatrix}\right]$ *as before. We find* $\sqrt{2} \times e \approx \left[\begin{smallmatrix} 1479 & 133 \\ 384 & 35 \end{smallmatrix}\right] \approx [3, 1; 5, 1; 2, 1; 3, 1]$, *with maximum error less than* $\frac{1}{19}$.

By comparison, suppose we instead tried with decimals. $\sqrt{2} \approx 1.4$ *and* $e \approx 2.7$, *so we would calculate* $\sqrt{2} \times e \approx 1.4 \times 2.7 = 3.78$. *How many of those digits can we trust? Is the first digit 3 or 4 or even something else? Is the second digit 7 or 8 or completely unknown? Is the third digit 8?*

To be fair, continued fraction computations do have limitations. Rather than a single digit, the terms can grow arbitrarily large, which creates its own problems and reduces computational efficiency. These terms can be kept smaller through computational tricks (see, e.g., [**5**]). In addition, the matrix tools needed are too sophisticated for most 10 year olds, so decimal notation has its place pedagogically. Nevertheless, we hope to see greater adoption of continued fraction notation in the future.

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