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On numerical semigroup elements and the ℓ_0 and ℓ_∞ norms of their factorizations

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Abstract. A numerical semigroup S is a cofinite, additively-closed subset of $\mathbb{Z}_{\geq 0}$ that contains 0, and a factorization of $x \in S$ is a k-tuple $z = (z_1, \ldots, z_k)$ where $x = z_1 a_1 + \cdots + z_k a_k$ expresses x as a sum of generators of the semigroup $S = \langle a_1, \ldots, a_k \rangle$. Much of the study of non-unique factorization centers on factorization length $z_1 + \cdots + z_k$, which coincies with the ℓ_1-norm of z as the k-tuple. In this paper, we study the ℓ_∞-norm and ℓ_0-norm of factorizations, viewed as alternative notions of length, with particular focus on the generalizations $\Delta_\infty(x)$ and $\Delta_0(x)$ of the delta set $\Delta(x)$ from classical factorization length. We prove that the ∞-delta set $\Delta_\infty(x)$ is eventually periodic as a function of $x \in S$, classify $\Delta_\infty(S)$ and the 0-delta set $\Delta_0(S)$ for several well-studied families of numerical semigroups, and identify families of numerical semigroups demonstrating $\Delta_\infty(S)$ and $\Delta_0(S)$ can be arbitrarily long intervals and can avoid arbitrarily long subintervals.

1. Introduction

A *numerical semigroup* is a cofinite, additively closed set $S \subseteq \mathbb{Z}_{\geq 0}$ containing 0. We often specify a numerical semigroup via a list of generators, i.e.,

$$S = \langle a_1, \ldots, a_k \rangle = \{z_1 a_1 + \cdots + z_k a_k : z_i \in \mathbb{Z}_{\geq 0}\}.$$

As ubiquitous mathematical objects, numerical semigroups arise in countless settings across the mathematics spectrum; see [**3, 21**] for a thorough introduction. Most notably for this manuscript, numerical semigroups arise frequently in factorization theory [**13**] and discrete optimization [**22**].

A *factorization* of an element $x \in S$ is an expression

$$x = z_1 a_1 + \cdots + z_k a_k$$

of x with each $z_i \in \mathbb{Z}_{\geq 0}$. The *support* and *length* of a factorization z are

$$\mathrm{supp}(z) = \{i : z_i > 0\}, \qquad \text{and} \qquad \ell_1(z) = z_1 + \cdots + z_k,$$

respectively. We denote by

$$\mathsf{Z}(x) = \{z \in \mathbb{Z}_{\geq 0}^k : x = z_1 a_1 + \cdots + z_k a_k\} \qquad \text{and} \qquad \mathcal{L}(x) = \{\ell_1(z) : z \in \mathsf{Z}(x)\}$$

the *set of factorizations* and *length set* of x, respectively. Factorization lengths are a cornerstone of factorization theory, and numerous combinatorial invariants derived

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from length sets used to quantify and compare the non-uniqueness of factorizations across rings and semigroups [**12**]. One of the more popular such invariants is the *delta set*, which is defined on semigroup elements as

$$\Delta(x) = \{c_i - c_{i-1} : i = 2, \ldots, r\} \qquad \text{where} \qquad \mathcal{L}(x) = \{c_1 < \cdots < c_r\},$$

and defined on semigroups as $\Delta(S) = \bigcup_{x \in S} \Delta(x)$. For numerical semigroups, $\Delta(x)$ is known to be eventually periodic as a function of x [**8**], and $\Delta(S)$ is more varied than for some other well-studied families of semigroups [**4**], such as Krull monoids [**7**].

In this paper, we study the *0-length* and ∞*-length* of factorizations z, which are

$$\ell_0(z) = |\operatorname{supp}(z)| \qquad \text{and} \qquad \ell_\infty(z) = \max(z_1, \ldots, z_k),$$

respectively. For each $p \in \{0, 1, \infty\}$, we define the *p-length set* of x as

$$\mathcal{L}_p(x) = \{\ell_p(z) : z \in \mathsf{Z}(x)\}.$$

(when $p = 1$, we recover the classical definitions). In discrete optimization, factorizations achieving minimal 0-length are known as *sparse solutions* and have been studied in the context of numerical semigroups [**1**] as well as for more general semigroups [**5, 15**]. Additionally, the asymptotic behavior of ∞-length was recently studied in [**9**], along with the extremal ℓ_p-norms of factorizations for $p \in [1, \infty) \cap \mathbb{Z}$.

In this paper, we study the *p-delta set* of x, defined as

$$\Delta_p(x) = \{c_i - c_{i-1} : i = 2, \ldots, r\} \qquad \text{where} \qquad \mathcal{L}_p(x) = \{c_1 < \cdots < c_r\},$$

and the *p-delta set* of S, defined as $\Delta_p(S) = \bigcup_{x \in S} \Delta_p(x)$.

The contributions of this manuscript are two-fold. First, we prove several structural results about the set $\mathcal{L}_\infty(x)$ for large elements $x \in S$. Our results are reminiscent of the *structure theorem for sets of length*, which drives much of the study of factorization theory [**11, 12**] and a specialized version of which was recently proven for numerical semigroups [**16**]. We derive as a consequence that $\Delta_\infty(x)$ is an eventually periodic function of $x \in S$ (Theorem 2.6), a result that is also known for the classical delta set [**8**] and joins a vast literature of eventual-peridicity results for large numerical semigroup elements [**18**].

Second, we characterize $\Delta_\infty(S)$ and $\Delta_0(S)$ for several well-studied families of numerical semigroups, and demonstrate via explicit families of numerical semigroups that $\Delta_\infty(S)$ and $\Delta_0(S)$ can each be arbitrarily long intervals and in general can contain arbitrarily long "gaps". Our results lead us to make the following conjecture.

Conjecture 1.1. For every finite set $D \subset \mathbb{Z}_{\geq 1}$ with $1 \in D$, there exists numerical semigroups S and S' with $\Delta_0(S) = D$ and $\Delta_\infty(S') = D$.

This part of our work is motivated by the *delta set realization problem* [**10**], which makes an analogous claim for the classical delta set $\Delta(S)$. The delta set realization problem is known to be difficult, in part because proving a given integer lies outside of $\Delta(S)$ necessitates a large amount of control over the factorization structure of S; see [**4**] for examples. Given this, and the technical nature of our arguments in Sections 3 and 4, we suspect Conjecture 1.1 to be difficult in general.

2. A structure theorem for sets of ∞-length

NOTATION 2.1. Throughout this paper, $S = \langle a_1, \ldots, a_k \rangle$ denotes a numerical semigroup with minimal generators $a_1 < a_2 < \cdots < a_k$. Additionally, throughout this section,

$$A = a_1 + \cdots + a_k, \qquad g_i = \gcd(\{a_j : i \neq j\}), \qquad \text{and} \qquad S_i = \langle \tfrac{1}{g_i} a_j : j \neq i \rangle$$

for each i. Additionally, for each i, fix $a_i' \in \mathbb{Z}$ with $a_i' a_i \equiv 1 \bmod g_i$, let

$$\mathsf{Z}(x, i) = \{z \in \mathsf{Z}(x) : z_i = \ell_\infty(z)\} \qquad \text{and} \qquad \mathcal{L}_\infty(x, i) = \{\ell_\infty(z) : z \in \mathsf{Z}(x, i)\},$$

and let

$$L_\infty(x, i) = \max \mathcal{L}_\infty(x, i) \qquad \text{and} \qquad l_\infty(x, i) = \min \mathcal{L}_\infty(x, i).$$

This section contains several structural results concerning the sets $\mathcal{L}_\infty(x)$, $\mathcal{L}_\infty(x, i)$, and $\Delta_\infty(x)$ for large $x \in S$. We briefly outline these results here.

- We prove in Theorem 2.3 that each $\mathcal{L}_\infty(x, i)$ forms what is known as an *almost arithmetic sequence (AAP)* (i.e., an arithmetic sequence with some missing values near either end), a central ingredient to the classical *structure theorem for sets of length* [**11**].
- We prove that in the AAP description of $\mathcal{L}_\infty(x, i)$, the "missing values" near either end depend only on the equivalence class of x modulo cetain products of the a_i's and g_i's (Theorem 2.4). This result is reminiscent of [**16**, Theorem 4.2], a more detailed version of the structure theorem for sets of length recently proven for numerical semigroups.
- Proposition 2.5 and Theorem 2.6 are the culmination of these results, collecting the conclusions drawn about $\Delta_\infty(x)$ for large x and $\Delta_\infty(S)$.

The depiction in Figure 1 illustrates how the structure of each $\mathcal{L}_\infty(x, i)$ for large x contributes to that of $\mathcal{L}_\infty(x)$ and $\Delta_\infty(x)$.

Recall that the *Frobenius number* of S is $\mathsf{F}(S) = \max(\mathbb{Z}_{\geq 0} \setminus S)$, and the *Apéry set* of S with respect to a nonzero element $m \in S$ is

$$\mathrm{Ap}(S; m) = \{n \in S : n - m \notin S\}.$$

It is known $\mathrm{Ap}(S; m) = \{0, w_1, \ldots, w_{m-1}\}$, where each $w_i \equiv i \bmod m$ is the smallest element of S in its equivalence class modulo m.

LEMMA 2.2. *For ever $x \in S$, the following inequalities hold:*

(a) $\frac{1}{A}x \leq l_\infty(x) \leq \frac{1}{A}x + a_k$*; and*
(b) for each i, $\frac{1}{a_i}x - ka_k \leq L_\infty(x, i) \leq \frac{1}{a_i}x$.

PROOF. Letting $z \in \mathsf{Z}(x)$ with $\ell_\infty(z) = l_\infty(x)$, we see

$$x = z_1 a_1 + \cdots + z_k a_k \leq \ell_\infty(z) a_1 + \cdots + \ell_\infty(z) a_k = l_\infty(x) A.$$

Next, write $x = a + qA$ for $a \in \mathrm{Ap}(S; A)$. We claim $l_\infty(a) \leq a_k$. Indeed, by way of contradiction, fix a factorization $z \in \mathsf{Z}(a)$ with $\ell_\infty(z) = l_\infty(a)$, and assume some $z_j > a_k$. Some $z_i' = 0$ since $a \in \mathrm{Ap}(S; A)$, so trading a_i copies of a_j for a_j copies of a_i yields a factorization $z' \in \mathsf{Z}(a)$ with strictly fewer copies of a_j and no new coordinates larger than a_k. After applying such a trade to each maximal entry in z, we obtain a factorization z' with $\ell_\infty(z') < \ell_\infty(z) = l_\infty(a)$, which is a contradiction.

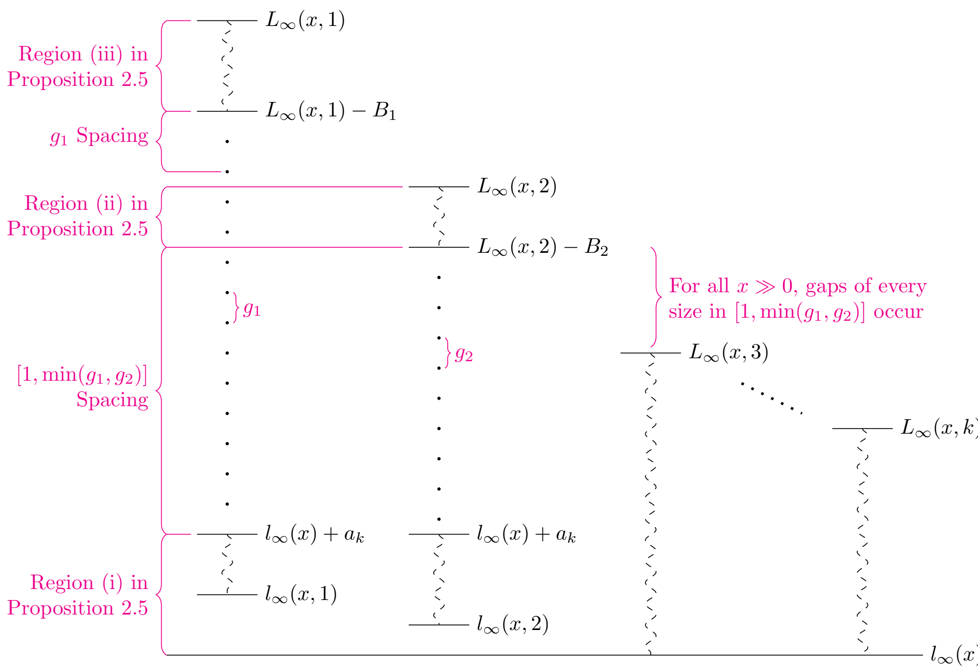

FIGURE 1. Diagram of ∞-length set elements for large $x \in S$, where the B_i are defined in Theorem 2.3

Letting $z \in \mathsf{Z}(a)$ with $\ell_\infty(z) = l_\infty(a)$, [**9**, Theorem 2.6] implies the factorization $z'' = (z_1 + q, \ldots, z_k + q)$ of $x = a + qA$ has $l_\infty(x) = \ell_\infty(z'') \le q + a_k$. Thus,

$$l_\infty(x) \le q + a_k = \tfrac{1}{A}(x - a) + a_k \le \tfrac{1}{A}x + a_k,$$

Proceeding to part (b), suppose $z \in \mathsf{Z}(x)$ satisfies $\ell_\infty(z) = z_i = L_\infty(x, i)$. Then

$$L_\infty(x,i)a_i \le L_\infty(x,i) + \sum_{j \ne i} z_j a_j = x.$$

Additionally, we must have $z_j < a_i$ for $j \ne i$, as otherwise one could trade a_i copies of a_j for a_j copies of a_i and constradict the maximality of z_i. As such,

$$x \le L_\infty(x,i)a_i + \sum_{j \ne i} z_j a_j < L_\infty(x,i)a_i + \sum_{j \ne i} a_i a_k \le L_\infty(x,i)a_i + k a_i a_k$$

from which the last remaining inequality is immediately obtained. $\square$

THEOREM 2.3. *For each $i = 1, \ldots, k$, there exists $B_i \in \mathbb{Z}$ such that*

$$[\tfrac{1}{A}x + a_k, \tfrac{1}{a_i}x - B_i] \cap (g_i\mathbb{Z} + a_i'x) \subseteq \mathcal{L}_\infty(x,i) \subseteq g_i\mathbb{Z} + a_i'x.$$

PROOF. The second containment holds since $\ell \in \mathcal{L}_\infty(x,i)$ implies $x - \ell a_i \in g_i S_i$. Define

$$B_i = \tfrac{1}{a_i} g_i(F(S_i) + 1)$$

and fix $\ell \in g_i\mathbb{Z} + a_i'x$ with $\frac{1}{A}x + a_k \le \ell \le \frac{1}{a_i}x - B_i$. Since

$$x - \ell a_i = a_i(\tfrac{1}{a_i}x - \ell) \ge g_i(F(S_i) + 1),$$

we have $x - \ell a_i \in g_i S_i$. Moreover, we claim $x - \ell a_i$ has a factorization in $g_i S_i$ of ∞-length at most ℓ. Indeed, notice that

$$x - \ell a_i + a_k(A - a_i) \le x + a_k A - \ell a_i = (\tfrac{1}{A}x + a_k)A - \ell a_i \le \ell(A - a_i),$$

from which we obtain

$$\tfrac{1}{A-a_i}(x - \ell a_i) + \max(\{a_j : j \ne i\}) \le \tfrac{1}{A-a_i}(x - \ell a_i) + a_k \le \ell.$$

Now applying Lemma 2.2 to $g_i S_i$ implies $x - \ell a_i$ has a factorization in $g_i S_i$ of ∞-length at most ℓ, which completes the proof. □

Theorem 2.4. *Fix $B, B' > 0$. For all $x \gg 0$, we have*

$$\mathcal{L}_\infty(x + a_i, i) \cap [\tfrac{1}{a_i}(x + a_i) - B, \infty) = 1 + \big(\mathcal{L}_\infty(x, i) \cap [\tfrac{1}{a_i}x - B, \infty)\big)$$

for each i, as well as

$$\mathcal{L}_\infty(x + A) \cap [0, \tfrac{1}{A}(x + A) + B'] = 1 + \big(\mathcal{L}_\infty(x) \cap [0, \tfrac{1}{A}x + B']\big)$$

In particular, these hold whenever $x > a_i^2 C + a_i B$ and $x > \frac{1}{a_i}A(A - a_i)B'$, respectively, where $C = \lceil \frac{1}{a_j}(B + 1)\rceil$.

Proof. One can readily check $z \in \mathsf{Z}(x, i)$ implies $z + e_i \in \mathsf{Z}(x + a_i, i)$, which shows one containment in the first equality. For the converse direction, we first claim any factorization $z \in \mathsf{Z}(x, i)$ with $\ell_\infty(z) = z_i \ge \frac{1}{a_i}(x + a_i) - B$ has $z_i > z_j$ for all $j \ne i$. Indeed, if $z_j = z_i$ for some j, then

$$z_j \ge \tfrac{1}{a_i}x - B > a_i C + B - B \ge a_i C,$$

so trading $a_i C$ copies of a_j in z for $a_j C$ copies of a_i yields a factorization in $\mathsf{Z}(x, i)$ with i-th coordinate

$$z_i + a_j C = z_i + a_j \lceil \tfrac{1}{a_j}(B + 1)\rceil > \tfrac{1}{a_i}x - B + B = \tfrac{1}{a_i}x$$

which is impossible by Lemma 2.2(b). Having now proven the claim, we conclude any $z' \in \mathsf{Z}(x + a_i, i)$ has $z' - e_i \in \mathsf{Z}(x, i)$, and the first equality is proven.

For the second equality, any $z \in \mathsf{Z}(x)$ has $z' = z + e_1 + \cdots + e_k \in \mathsf{Z}(x + A)$. For the reverse containment, we claim any $z \in \mathsf{Z}(x)$ with $\ell_\infty(z) \le \frac{1}{A}x + B'$ has no zero entries. Indeed, if some $z_i = 0$, then

$$\begin{aligned} x = \sum_{j \ne i} z_j a_j \le \sum_{j \ne i} (\tfrac{1}{A}x + B')a_j = (\tfrac{1}{A}x + B')(A - a_i) = x - \tfrac{1}{A}a_i x + (A - a_i)B' \\ < x - (A - a_i)B' + (A - a_i)B' = x \end{aligned}$$

a contradiction. As such, any factorization $z' \in \mathsf{Z}(x + A)$ with $\ell_\infty(z') \le \frac{1}{A}x + B' + 1$ has $z' - e_1 - \cdots - e_k \in \mathsf{Z}(x)$, thereby completing the proof. □

Proposition 2.5. For all $x \gg 0$, we have $[1, \min(g_1, g_2)] \cup \{g_1\} \subseteq \Delta_\infty(x)$. Moreover, if $\ell < \ell'$ are successive elements of $\mathcal{L}_\infty(x)$ with

$$\ell' - \ell \notin [1, \min(g_1, g_2)] \cup \{g_1\},$$

then at least one of ℓ and ℓ' lies in one of the following intervals:

(i) $[\frac{1}{A}x, \frac{1}{A}x + a_k]$; (ii) $[\frac{1}{a_2}x - B_2, \frac{1}{a_2}x]$; or (iii) $[\frac{1}{a_1}x - B_1, \frac{1}{a_1}x]$.

PROOF. First, if x is large enough that $(\frac{1}{a_1}x - B_1) - \frac{1}{a_2}x > 3g_1$, then by Lemma 2.2(b),

$$\mathcal{L}_\infty(x) \cap (\tfrac{1}{a_2}x, \tfrac{1}{a_1}x - B_1) = \mathcal{L}_\infty(x, 1) \cap (\tfrac{1}{a_2}x, \tfrac{1}{a_1}x - B_1)$$

contains at least 2 lengths, and any two consecutive lengths therein must have difference $g_1 \in \Delta_\infty(x)$ by Theorem 2.3.

Analogously, if x is large enough that $(\frac{1}{a_2}x - B_2) - \frac{1}{a_3}x > 2g_1g_2$, then by Lemma 2.2(b) and Theorem 2.3,

$$\begin{aligned}\mathcal{L}_\infty(x) \cap (\tfrac{1}{a_3}x, \tfrac{1}{a_2}x - B_2) &= \big(\mathcal{L}_\infty(x, 1) \cup \mathcal{L}_\infty(x, 2)\big) \cap (\tfrac{1}{a_3}x, \tfrac{1}{a_2}x - B_2) \\ &= \big((g_1\mathbb{Z} + a_1'x) \cup (g_2\mathbb{Z} + a_2'x)\big) \cap (\tfrac{1}{a_3}x, \tfrac{1}{a_2}x - B_2),\end{aligned}$$

within which successive elements achieve each difference in $[1, \min(g_1, g_2)]$ by the Chinese Remainder Theorem since $\gcd(g_1, g_2) = \gcd(a_1, \ldots, a_k) = 1$.

For the final claim, by Theorem 2.3, aside from the three claimed intervals, the only subinterval of $[\frac{1}{A}x, \frac{1}{a_1}x]$ not containing an arithmetic sequences of step size $\min(g_1, g_2)$ is $[\frac{1}{a_2}x, \frac{1}{a_1}x - B_1]$, whose lengths form an arithmetic sequence of step size g_1. □

THEOREM 2.6. *If* $x \gg 0$*, then* $\Delta_\infty(x+p) = \Delta_\infty(x)$*, where* $p = \operatorname{lcm}(a_1, g_1a_2, A)$.

PROOF. We begin by considering the intervals (i), (ii), and (iii) in Proposition 2.5. Let

$$\begin{aligned}R_1(x) &= [\tfrac{1}{A}x, \tfrac{1}{A}x + (a_k + g_1)] \cap \mathcal{L}_\infty(x), \\ R_2(x) &= [\tfrac{1}{a_2}x - (B_2 + g_1), \tfrac{1}{a_2}x + g_1] \cap \mathcal{L}_\infty(x), \\ R_3(x) &= [\tfrac{1}{a_1}x - (B_1 + g_1), \tfrac{1}{a_1}x] \cap \mathcal{L}_\infty(x).\end{aligned}$$

Applying Theorem 2.4 with $B' = a_k + g_1$ gives $R_3(x+p) = R_3(x) + \frac{1}{A}p$, and letting $B = B_1 + g_1$, we have

$$R_1(x) = [\tfrac{1}{a_1}x - B, \tfrac{1}{a_1}x] \cap \mathcal{L}_\infty(x, 1)$$

by Lemma 2.2(b), so $R_1(x + p) = R_1(x) + \frac{1}{a_1}p$. Additionally, by Theorem 2.3,

$$R_2(x) = \big([\tfrac{1}{a_2}x - B, \tfrac{1}{a_2}x] \cap \mathcal{L}_\infty(x, 2)\big) \cup \big([\tfrac{1}{a_2}x - B, \tfrac{1}{a_2}x + g_1] \cap (g_1\mathbb{Z} + a_1'x)\big)$$

for $B = B_2 + g_1$. Clearly $g_1\mathbb{Z} + a_1'(x + p) = g_1\mathbb{Z} + a_1'x$, and since $g_1 \mid \frac{1}{a_2}p$,

$$\big(g_1\mathbb{Z} + a_1'(x + p)\big) + \tfrac{1}{a_2}p = g_1\mathbb{Z} + a_1'x$$

as well. As such, $R_2(x + p) = R_2(x) + \frac{1}{a_2}p$ once again by Theorem 2.4.

Lastly, by Proposition 2.5 any successive lengths in $\mathcal{L}_\infty(x)$ or $\mathcal{L}_\infty(x + p)$ not residing in one of the above intervals must have difference in $[1, \min(g_1, g_2)] \cup \{g_1\}$, which is a subset of both $\mathcal{L}_\infty(x)$ and $\mathcal{L}_\infty(x + p)$ by Proposition 2.5. □

COROLLARY 2.7. Fix $B > 0$. For all $x \gg 0$, we have

$$\ell \in \mathcal{L}_\infty(x) \cap [\tfrac{1}{a_1}x - B, \tfrac{1}{a_1}x] \qquad \text{if and only if} \qquad x - \ell a_1 \in g_1S_1 \cap [0, a_1B].$$

In particular, $\Delta(g_1S_1 \cap (a_1\mathbb{Z} + j)) \subseteq \Delta_\infty(S)$ for each j.

PROOF. As in the proof of Theorem 2.3, we have $\ell \in \mathcal{L}_\infty(x,1)$ if and only if (i) $x - \ell a_1 \in g_1 S_1$ and (ii) there exists a factorization of $x - \ell a_1$ in $g_1 S_1$ with ∞-length at most ℓ. Tracing through the proof of Theorem 2.3, so long as condition (i) holds, condition (ii) holds whenever $\ell > \frac{1}{A}x + a_k$, which is certainly the case if $\ell > \frac{1}{a_1}x - B$ for $x \gg 0$. Moreover,

$$x - \ell a_1 < x - (\tfrac{1}{a_1}x - B)a_1 = a_1 B.$$

As such, Lemma 2.2(b) implies the frst claim, and the second claim then follows upon unraveling definitions. □

3. Some families of ∞-delta sets

In this section, we examine the set $\Delta_\infty(S)$ for several families of numerical semigroups. We characterize the ∞-delta set for supersymmetric numerical semigroups [**6**], and numerical semigroups whose generators form an arithmetic sequence [**17**] or a geometric sequence [**19, 23**]. We also demonstrate that $\Delta_\infty(S)$ can be an arbitrarily long interval (Theorem 3.2) and have arbitrarily long gaps (Theorem 3.3).

Many of our arguments in this section and the next utilize trades, presentations, and Betti elements. We briefly review the relevant concepts here, though the reader is encouraged to see [**3**, Chapter 5] and [**20**] for a thorough introduction.

Define an equivalence relation $\sim$ on $\mathbb{Z}^k_{\geq 0}$ that sets $z \sim z'$ whenever $z, z' \in \mathsf{Z}(x)$ are factorizations of the same element $x \in S$. We call each relation $z \sim z'$ between facatorizations of disjoint support a *trade* of S, and sometimes identify the difference $z - z' \in \mathbb{Z}^k$ with the trade $z \sim z'$. A *presentation* of S is a collection ρ of trades with the property that for any $x \in S$ and any $z, z' \in S$, there exists a chain of factorizations

$$z = y_1 \sim y_2 \sim \cdots \sim y_r = z'$$

wherein $y_i - y_{i-1} \in \rho$ or $y_{i-1} - y_i \in \rho$ for each pair of sequential factorizations y_{i-1} and y_i. A presentation is *minimal* if it is minimal with respect to containment among all presentations for S. It is known that any two minimal presentations ρ and ρ' for S have the same number of trades, and in fact the set

$$\mathrm{Betti}(S) = \{z_1 a_1 + \cdots + z_k a_k : z - z' \in \rho\}$$

of *Betti elements* is independent of the choice of minimal presentation ρ.

THEOREM 3.1. *Let $S = \langle a_1, \ldots, a_k \rangle$ with $a_1 < \cdots < a_k$.*

(a) If $a < b$ are coprime and each $a_i = a^{k-1-i} b^{i-1}$, then $\Delta_\infty(S) = \{1, 2, \ldots, b\}$.
(b) If $p_1, \ldots, p_k \in \mathbb{Z}_{\geq 1}$ are pairwise coprime with $p_1 > \cdots > p_k$, $T = p_1 \cdots p_k$, and each $a_i = \frac{1}{p_i}T$, then $\Delta_\infty(S) = \{1, 2, \ldots, p_1\}$.

PROOF. For part (a), the trades $be_i \sim ae_{i+1}$ for $i = 1, \ldots, k-1$ form a minimal presentation for S by [**14**, Theorem 8], so $\max \Delta_\infty(S) \leq b$. Now, if $1 \leq c \leq a$, then

$$\mathsf{Z}((b + a - c)a_1) = \{(b + a - c)e_1, (a - c)e_1 + ae_2\}$$

so $c \in \Delta_\infty(S)$. Moreover, if $a < c \leq b$, then

$$\mathsf{Z}(ca_2) = \{(be_1 + (c - a)e_2, ce_2\},$$

so again $c \in \Delta_\infty(S)$. Thus, $\Delta_\infty(S) = \{1, 2, \ldots, b\}$.

For part (b), the trades $p_{i+1} e_i \sim p_i e_{i+1}$ for $i = 1, \ldots, k-1$ form a minimal presentation for S by [**6**], so $\max \Delta_\infty(S) \leq p_1$. Using a similar argument to part (a), we have $\Delta_\infty((p_1 + p_2 - c)a_1) = \{c\}$ whenever $1 \le c \le a$ and $\Delta_\infty(ca_2) = \{c\}$ whenever $p_2 < c \le p_1$. Thus, $\Delta_\infty(S) = \{1, 2, \ldots, p_1\}$. □

THEOREM 3.2. *Let $S = \langle a, a+d, \ldots, a+kd\rangle$ with $2 \le k < a$ and $\gcd(a,d) = 1$, and write $a - 1 = qk + r$ for $q, r \in \mathbb{Z}_{\ge 0}$ with $0 \le r < k$. Then*

$$\Delta_\infty(S) = \{1, 2, \ldots, q + d + 1\}.$$

PROOF. In what follows, write $a_i = a + ik$ for $i \in [0, k]$, and for $x \in S$, write factorizations $z \in \mathsf{Z}(x)$ as $z = (z_0, \ldots, z_k) = z_0e_0 + \cdots + z_ke_k$. Before beginning the proof, we recall some facts about arithmetical numerical semigroups; see [**2, 17**]. Each $x \in S$ has $\ell \in \mathcal{L}_1(x)$ if and only if

$$x = \ell a + bd \qquad \text{with} \qquad 0 \le b \le k\ell,$$

as for any factorization $z \in \mathsf{Z}(x)$ with $\ell_1(z) = \ell$, we can write

$$x = (z_0 + \cdots + z_k)a + (z_1 + 2z_2 + \cdots + kz_k)d$$

with $b = z_1 + 2z_2 + \cdots + kz_k$. Moreover,

$$\ell - \lceil \tfrac{1}{k} b \rceil \ge z_0 = \ell - (z_1 + \cdots + z_k) \ge \ell - b,$$

with equality on the right if $z_2 = \cdots = z_k = 0$.

We now proceed with the proof. First, suppose $1 \le G \le d$. We see

$$x = (a + d)a + (d - G)(a + d) = (a + d - G)(a + d)$$

are factorizations $z, z' \in \mathsf{Z}(x)$, respectively, with ∞-lengths $\ell_\infty(z) = a + d$ and $\ell_\infty(z') = a + d - G$. Now, since $\ell_\infty(z') = \ell_1(z') = z_1'$, any factorization $z'' \in \mathsf{Z}(x)$ with $\ell_\infty(z'') > \ell_\infty(z')$ must have $\ell_1(z'') > \ell_1(z')$. This means $\ell_1(z'') \ge \ell_1(z)$, and letting $b = z_1'' + 2z_2'' + \cdots + kz_k''$, any such factorization must have

$$z_0'' \ge \ell_1(z'') - b \ge (a + 2d - G) - (d - G) = a + d$$

by the first paragraph above. As such, $G \in \Delta_\infty(x)$.

Next, suppose $d \le G \le d + q + 1$. We see

$$x = (a + G)a = (G - d)a + a(a + d)$$

are factorizations $z, z' \in \mathsf{Z}(x)$, respectively, with $\ell_\infty(z) = a+G$ and $\ell_\infty(z') = a$. Fix a factorization $z'' \in \mathsf{Z}(x)$ with $\ell_\infty(z'') < \ell_\infty(z)$, and let $b = z_1'' + 2z_2'' + \cdots + kz_k''$. Since $z = (a + G)e_1$, we must have $\ell_1(z'') < \ell_1(z) = a + G$. As such, we have $\ell_1(z'') \le a + G - d = \ell_1(z')$ and $b \ge a$, meaning

$$z_0'' \le \ell_1(z'') - \lceil \tfrac{1}{k} b \rceil \le (a + G - d) - \lceil \tfrac{1}{k} a \rceil \le a + G - d - q - 1 \le a.$$

Additionally, if $z_j'' \ge a$ for some $j \ge 1$, then

$$\sum_{i \ne j} z_i'' a_i = x - z_j'' a_j < x - a a_j \le x - a(a + d) = (G - d)a \le (q + 1)a,$$

and all factorizations of such an element have equal 1-length. As such, since

$$x = (G - jd)a + a(a + jd),$$

we must have $\ell_1(z'') = \ell_1(z') - jd$, meaning z'' coincides with the above factorization. Thus $\ell_\infty(z'') \le a$, thereby ensuring $G \in \Delta_\infty(x)$. □

THEOREM 3.3. *Fix $m \ge 3$. If $S = \langle 3, 3m + 1, 3m + 2\rangle$, then*

$$\Delta_\infty(S) = \{1, 2, \ldots, m + 1\} \cup \{2m, 2m + 1\}.$$

PROOF. Since S has max embedding dimension (see [**21**, Chapter 3]):

(i) the trades

$$(2m+1)e_1 \sim e_2 + e_3, \qquad me_1 + e_3 \sim 2e_2, \qquad \text{and} \qquad (m+1)e_1 + e_2 \sim 2e_3$$

comprise a minimal presentation for S;

(ii) for each $x \in S$, the unique factorization $z = (a,b,c) \in \mathsf{Z}(x)$ with $\ell_\infty(z)$ maximal in $\mathcal{L}_\infty(x)$ is also the unique factorization with $b + c \le 1$; and

(iii) for each $a \ge 0$, the factorizations $(a,1,1)$ and $(a+2m-1,0,0)$ of the element $x = 3(a+2m-1)$ have the two highest ∞-lengths in $\mathcal{L}_\infty(x)$.

Fix $x \in S$ and $z = (a,b,c) \in \mathsf{Z}(x)$. Suppose z does not have maximal ∞-length, and let G be minimal with $\ell_\infty(z) + G \in \mathcal{L}_\infty(x)$. By (iii), if $b = c = 1$, then $G \ge 2m$. Otherwise, by (ii) either $b \ge 2$ or $c \ge 2$. If $b \ge 2$, then fixing $q \in \mathbb{Z}$ with $b - 2q \in \{0,1\}$ and performing the second trade in (i) q times yields a chain of factorizations

$$(a,b,c) \sim (a+m, b-2, c+1) \sim \cdots \sim (a+qm, b-2q, c+q),$$

wherein each factorization differs in ∞-length from the previous factorization by at most m, and the final factorization in which has strictly larger ∞-length than z. As such, we have $G \le m$. By an analogous argument, if $c \ge 2$, then $G \le m+1$. This proves $m+2, \ldots, 2m-1 \notin \Delta_\infty(S)$.

Now, by (i), we have $\max \Delta_\infty(S) \le 2m+1$. We can see by inspection that

$$\mathsf{Z}(6m+3) = \{(2m+1,0,0),(0,1,1)\} \quad \text{and} \quad \mathsf{Z}(6m+6) = \{(2m+2,0,0),(1,1,1)\},$$

so $2m, 2m+1 \in \Delta_\infty(S)$. Also by inspection,

$$\mathsf{Z}(6m+8) = \{(m+2,0,1),(2,2,0)\} \quad \text{and} \quad \mathsf{Z}(6m+10) = \{(m+3,1,0),(2,0,2)\},$$

so $m, m+1 \in \Delta_\infty(S)$. Lastly, for each $G \in \{1, \ldots, m-1\}$, we have

$$(0,0,G+1),(m+1,1,G-1) \in \mathsf{Z}(x)$$

for $x = (G+1)(3m+2)$. Fix a factorization $z = (a,b,c) \in \mathsf{Z}(x)$. Since

$$3a = x - (3m+1)b + (3m+2)c = (G+1-b-c)(3m+2) + b,$$

we must have $b+c \le G+1$. If $b+c = G+1$, then $a = \frac{1}{3}b < G+1$, so $\ell_\infty(z) \le G+1$. If $b+c \le G$, then

$$3a \ge (G+1-b-c)(3m+2) \ge 3m+2$$

so $\ell_\infty(z) \ge a \ge m+1$. This proves $m - G \in \Delta_\infty(S)$. □

4. Some families of 0-delta sets

In a similar vein to the prior section, in Theorems 4.2 and 4.3 we characterize $\Delta_0(S)$ for numerical semigroups S residing in several well-studied families, including maximal embedding dimension numerical semigroups [**21**, Chapter 3], supersymmetric numerical semigroups [**6**], 3-generated numerical semigroups [**21**, Chapter 10], and numerical semigroups generated by generalized arithmetic sequences [**17**]. We also identify two families of numerical semigroups achieving notable extremal behavior (Theorems 4.4 and 4.5). First, we demonstrate that the structure of $\mathcal{L}_0(x)$ for large $x \in S$ differs substantially from that of $\mathcal{L}_\infty(x)$ detailed in Section 2.

THEOREM 4.1. *For all $x \gg 0$, we have $\Delta_0(x) = \{1\}$. In particular, $1 \in \Delta_0(S)$.*

PROOF. For fixed $I \subseteq \{1, \ldots k\}$ nonempty, and letting $d = \gcd(a_i : i \in I)$, $x \in S$ has a factorization $z \in \mathsf{Z}(x)$ with $\operatorname{supp}(z) = I$ if and only if $d \mid x$ and

$$\tfrac{1}{d}(x - \textstyle\sum_{i \in I} a_i) > \mathsf{F}(\langle \tfrac{1}{d} a_i : i \in I \rangle).$$

As such, for $x \gg 0$, if x has a factorization with support I, then x also has a factorization with support J for every $J \supseteq I$. Thus, $\mathcal{L}_0(x)$ is an interval and $\Delta_0(x) = \{1\}$. □

THEOREM 4.2. *The following hold.*

(a) If S has a minimal presentation ρ in which every trade if between factorizations with singleton support, then $\Delta_0(S) = 1$. In particular, this occurs whenever S is supersymmetric or generated by a geometric sequence.
(b) If $S = \langle m, a_1, \ldots, a_{m-1} \rangle$ with $m \geq 3$ and each $a_i \equiv i \bmod m$ (i.e., S is maximal embedding dimension), then $\Delta_0(S) = \{1, 2\}$.
(c) If $S = \langle a, ah+d, ah+2d, \ldots, ah+kd \rangle$ with $h \geq 1$, $2 \leq k < a$, and $\gcd(a, d) = 1$ (i.e., S is generated by a generalized arithmetic sequence), then $\Delta_0(S) = \{1, 2\}$.

PROOF. Part (a) follows from the fact that any two factorizations of an element $x \in S$ are connected by a chain of factorizations in which successive factorizations z, z' differ by a trade in ρ, and thus satisfy $|\ell_0(z) - \ell_0(z')| \leq 1$. As such, $\Delta_0(x) = \{1\}$. The claims about supersymmetric numerical semigroups and semigroups generated by geometric sequences immediately follow [**6, 14**].

For part (b), by [**21**, Theorem 8.30] S has a minimal presentation in which each trade has the form

$$e_i + e_j \sim e_k + ce_0 \qquad \text{with} \qquad i + j \equiv k \bmod m \qquad \text{and} \qquad c \in \mathbb{Z}_{\geq 1},$$

so by similar reasoning to part (a), $\Delta_0(S) \subseteq \{1, 2\}$. Moreover, since $m \geq 3$, applying the trade with $i = 1$ and $j = 2$ to the factorization $z = e_0 + \cdots + e_{m-1}$ yields a factorization z' with $\ell_0(z') = m - 2$. Moreover, no other factorization z'' can have $\ell_0(z'') = m - 1$, as then the trade $z \sim z''$ would be between distinct factorizations for a minimal generator of S.

For part (c), in the minimal presentation for S presented in [**17**, Theorem 2.16], each trade is between factorizations with 0-length at most 2, so $\Delta_0(S) \subseteq \{1, 2\}$. Moreover, writing $a - 1 = qk + r$ with $0 \leq r < k$, the minimal presentation in [**17**] also implies

$$x = a + (ah + (r+1)d) + q(ah + kd) = a(d + h(q+1))$$

are the only two factorizations of x, so $\Delta_0(x) = \{2\}$. □

We next characterize $\Delta_0(S)$ when S is 3-generated. Recall that an expression

$$S = t'S' + t''S'' \qquad \text{with} \qquad S' = \langle b_1, \ldots, b_r \rangle \qquad \text{and} \qquad S'' = \langle c_1, \ldots, c_{k-r} \rangle$$

is called a *gluing* if $t' \in S'' \setminus \{c_1, \ldots, c_{k-r}\}$, $t'' \in S' \setminus \{b_1, \ldots, b_r\}$, and $\gcd(t', t'') = 1$; see [**21**, Chapter 9] for more on gluings. Note that such an expression for S need not be unique. In particular, if $S = \langle a_1, a_2, a_3 \rangle$, then there can be up to 3 such expressions for S as a gluing, each of the form $S = \langle a_i \rangle + t'S'$ for some $i \in \{1, 2, 3\}$.

THEOREM 4.3. *Suppose $S = \langle a_1, a_2, a_3 \rangle$. If S has at most 1 expression as a gluing, then $\Delta_0(S) = \{1, 2\}$. Otherwise, $\Delta_0(S) = \{1\}$.*

PROOF. If S has at least 2 distinct expressions $S = \langle a_i \rangle + t'S' = \langle a_j \rangle + t''S''$ as a gluing, then we can write

$$S = \langle t'b_1, t't''b_2, t''b_3 \rangle.$$

Since $t'b_1 \in S' = \langle t'b_2, b_3 \rangle$, there exist $z_2, z_3 \in \mathbb{Z}_{\geq 0}$ with $t'b_1 = z_2 t'b_2 + z_3 b_3$, and since $\gcd(t', b_3) = 1$, we must have $t' \mid z_2$. As such, $b_1 = z_1 b_2 + z_2 b_3$ and thus $b_1 \in \langle b_2, b_3 \rangle$. By similar reasoning, we know $b_3 \in \langle b_1, b_2 \rangle$. Assuming $b_1 \leq b_3$ without loss of generality, this is only possible if $b_1 = b_3$ or $b_2 \mid b_1$. In particular, $t'b_1$ has a factorization in S' with singleton support. As such, by [**21**, Theorem 9.2], S has a minimal presentation within which every factorization has singleton support, so $\Delta_0(S) = \{1\}$ by Theorem 4.2(a).

Conversely, suppose $S = \langle a_1 \rangle + tS'$ with $S' = \langle b_1, b_2 \rangle$ is the only expression of S as a gluing. Then writing $a_1 = z_1 b_1 + z_2 b_2$, we cannot have $z_2 = 0$, as otherwise

$$S = \langle z_1 b_1, t'b_1, t'b_2 \rangle = \langle t'b_2 \rangle + b_1 \langle z_1, t' \rangle$$

is a second expression of S as a gluing. Analogously, $z_1 > 0$. As such,

$$x = (t+1)a_1 = a_1 + z_2 t'b_1 + z_3 t'b_2$$

has $\mathcal{L}_0(x) = \{1, 3\}$, so $\Delta_0(x) = \{2\}$.

This leaves the case where S cannot be expressed as a gluing. By [**21**, Section 10.3], S has a unique minimal presentation comprised of trades

$$c_1 e_1 \sim r_{12} e_2 + r_{13} e_3, \qquad c_2 e_2 \sim r_{21} e_1 + r_{23} e_3, \qquad \text{and} \qquad c_3 e_3 \sim r_{31} e_1 + r_{32} e_2$$

where each $r_{ij} > 0$ and each $c_k = r_{ik} + r_{jk}$ for $\{i, j, k\} = \{1, 2, 3\}$. We consider cases.

- If $r_{ij} = r_{ik} = 1$ for some i, then $x = a_1 + a_2 + a_3$ has at least one factorization without full support, and any such factorization must have singleton support, so $\Delta_0(x) = \{2\}$.
- If $r_{ji} \geq 2$ and $r_{ki} \geq 2$ for some i, then

$$x = (c_i + 1)a_i = a_i + r_{ij} a_j + r_{ik} a_k$$

  are the only factorizations of x, so $\Delta_0(x) = \{2\}$.
- If $r_{ji} = r_{ki} = 1$ for some i, then either $r_{jk} \geq 2$ and $r_{kj} \geq 2$, in which case

$$x = (c_j + 1)a_j = a_j + r_{ji} a_i + r_{jk} a_k$$

  are the only factorizations of x and $\Delta_0(x) = \{2\}$, or $r_{jk} = 1$ or $r_{kj} = 1$, meaning we are in the first case above.
- In all other cases, up to reordering i, j, and k, we have $r_{ij} = r_{jk} = r_{ki} = 1$ while $r_{ji}, r_{kj}, r_{ik} \geq 2$. In this case,

$$x = (c_j + 1)a_j = a_j + r_{ji} a_i + a_k$$

  are the only factorizations of x, so $\Delta_0(x) = \{2\}$.

In all cases above, we conclude $\Delta_0(S) = \{1, 2\}$. □

Thus far, all semigroups S presented have $\max \Delta_0(S) \leq 2$. We close by presenting two families of numerical semigroups exhibiting more interesting behavior: one demonstrating $\Delta_0(S)$ can be an arbitrarily large interval (Theorem 4.4), and another demonstrating $\Delta_0(S) \setminus [1, \max \Delta_0(S)]$ can be arbitrarily large (Theorem 4.5).

THEOREM 4.4. *For each $k \geq 2$, there exists a numerical semigroup S such that $\Delta_0(S) = \{1, 2, \ldots, k-1\}$.*

Proof. Fix distinct primes p_1, p_2 with $p_1, p_2 > k$. Let $S_2 = \langle p_1, p_2 \rangle$, so $\Delta_0(S_2) = \{1\}$. Proceeding inductively, assume $S_{i-1} = \langle a_1, \ldots, a_{i-1} \rangle$ has Betti elements $b_1, \ldots, b_{i-2}$ with $\mathsf{Z}(b_1) = \{p_2 e_1,\, p_1 e_2\}$ and for each $j \ge 2$,

$$\mathsf{Z}(b_j) = \{p_{j+1} e_{j+1},\, (k+1-j)e_1 + e_2 + \cdots + e_j\}$$

for some prime p_{j+1}. Since each $j \le k$, we have $\Delta_0(b_j + a_{j+1}) = \{j\}$ for each j. Letting

$$a_i = (k+1-i)a_1 + a_2 + \cdots + a_{i-1},$$

we see (i) the above factoriation of a_i is not preceded (under the component-wise partial order) by any factorizations of $b_1, \ldots, b_{i-2}$ (meaning a_i is uniquely factorable in S_{i-1}), and (ii) the above factorization of a_i does not precede a factorization of any b_j. As such, choosing a prime $p_i > a_i$, the semigroup

$$S_i = p_i S_{i-1} + \langle a_i \rangle$$

is a gluing, so we have $\mathrm{Betti}(S_i) = \{p_i b_1, \ldots, p_i b_{i-2}, p_i a_i\}$ and

$$\mathsf{Z}(p_i a_i) = \{p_i e_i,\, (k+1-i)e_1 + e_2 + \cdots + e_{i-1}\}.$$

This ensures $\Delta_0((p_i+1)a_i) = \{i-1\}$ and $\Delta_0(S_i) = \{1, 2, \ldots, i-1\}$. Thus, at the conclusion of this process, the semigroup S_k has $\Delta_0(S_k)$ as claimed. □

Theorem 4.5. *For $k \ge 16$, there is a numerical semigroup $S = \langle a_1, \ldots, a_{k+1} \rangle$ with $\Delta_0(S) \cap [\frac{7}{8}k, k] = \{k-1, k\}$.*

Proof. Let $S_2 = \langle 2, 3 \rangle$. Next, for each $i = 3, \ldots, k$, let

$$S_i = 2S_{i-1} + \langle 2a_{i-2} + a_{i-1} \rangle \qquad \text{where} \qquad S_{i-1} = \langle a_1, \ldots, a_{i-1} \rangle.$$

Lastly, let

$$S = S_{k+1} = 2S_k + \langle a_1 + \cdots + a_k \rangle \qquad \text{where} \qquad S_k = \langle a_1, \ldots, a_k \rangle.$$

As each S_i is easily shown to be a gluing, the trades

$$2e_2 \sim 3e_1, \qquad 2e_{k+1} \sim e_1 + \cdots + e_k, \qquad \text{and} \qquad 2e_i \sim 2e_{i-2} + e_{i-1} \text{ for } 3 \le i \le k$$

form a minimal presentation ρ of S.

In what follows, write $S = \langle a_1, \ldots, a_{k+1} \rangle$. We see by inspection that

$$\mathsf{Z}(3a_{k+1}) = \{3e_{k+1},\, e_1 + \cdots + e_{k+1}\},$$

since no other trades in ρ can be performed, so in particular $\Delta_0(3a_{k+1}) = \{k\}$ and $\Delta_0(2a_{k+1}) = \{k-1\}$. We claim every other $x \in S$ with $\Delta_0(x)$ nonempty has $\max \Delta_0(x) \le \frac{7}{8}k$. Indeed, any two factorizations of x can be connected by a sequence of trades in ρ, and of such trades, the only one that can yield a change in 0-length of more than 2 is the trade $2e_{k+1} \sim e_1 + \cdots + e_k$. As such, consider factorizations $z, z' \in \mathsf{Z}(x)$ of the form

$$z = u + e_1 + \cdots + e_k \qquad \text{and} \qquad z' = u + 2e_{k+1}$$

for some $u \in \mathbb{Z}^k_{\ge 0}$. By way of contradiction, suppose $\ell_0(u) \le \frac{1}{8}k$, so that

$$\ell_0(z) - \ell_0(z') \ge k - \tfrac{1}{8}k = \tfrac{7}{8}k.$$

First, suppose $u_i \ge 1$ for some $i \le \frac{1}{2}k$, and fix j maximal with $i + 2j \le k$. Performing the trade

$$2e_i + e_{i+1} + e_{i+3} + \cdots + e_{i+2j-1} \sim 2e_{i+2j}$$

to z yields a factorization z'' in which $j \geq \frac{1}{4}k$ entries are strictly smaller than in z. However, since $\ell_0(u) \leq \frac{1}{8}k$, at least $\frac{1}{8}k$ entries of z'' must be zero. As such,

$$\ell_0(z) - \ell_0(z'') \leq \tfrac{1}{4}k \qquad \text{and} \qquad \ell_0(z'') - \ell_0(z') \leq \tfrac{7}{8}k.$$

Next, suppose $u_i \geq 1$ for $\frac{1}{2}k < i \leq k$. Performing the trade

$$\begin{aligned} 2e_i + e_{i-1} + e_{i-3} + e_{i-5} + \cdots &\sim 2e_{i-1} + 2e_{i-3} + 2e_{i-5} + \cdots \\ &\sim e_{i-2} + 2e_{i-4} + 3e_{i-6} + \cdots \end{aligned}$$

to z yields a factorization z'' in which at least $\frac{1}{4}k$ entries are strictly smaller than in z. As in the previous case, we obtain

$$\ell_0(z) - \ell_0(z'') \leq \tfrac{1}{4}k \qquad \text{and} \qquad \ell_0(z'') - \ell_0(z') \leq \tfrac{7}{8}k.$$

Lastly, since $x \neq 2a_{k+1}, 3a_{k+1}$, the only remaining case is when $u = ca_{k+1}$ with $c \geq 2$. In this case, one may performe the trade

$$\begin{aligned} 4e_{k+1} &\sim 2e_1 + 2e_2 + \cdots + 2e_k \\ &\sim 3e_{k-1} + 4e_{k-3} + 5e_{k-5} + \cdots \end{aligned}$$

to obtain a factorization with at least $\frac{1}{2}k$ zero entries, which completes the proof. □

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