Membership and Elasticity in Certain Affine Monoids

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July 20, 2018

Abstract

For affine monoids of dimension 2 with embedding dimension 2 and 3, we study the problem of determining when a vector is an element of the monoid, and the problem of determining the elasticity of a monoid element.

1 Introduction

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N}_0$ denote the set of nonnegative integers, and $\mathbb{Q}^\star$ denote the set of nonnegative rational numbers adjoined with $+\infty$. An *affine monoid*, S, is a finitely generated submonoid of $\mathbb{N}_0^r$, with operation $+$, for some positive integer r. They are of substantial interest (see, e.g., [4, 8, 11]). In the remainder, we restrict to the case $r = 2$. Any affine monoid is cancellative ($\mathbf{a}+\mathbf{b} = \mathbf{a}+\mathbf{c}$ implies $\mathbf{b} = \mathbf{c}$), reduced (its only unit is 0, the identity element), and torsion free ($k\mathbf{a} = k\mathbf{b}$ for $k \in \mathbb{N}$ implies $\mathbf{a} = \mathbf{b}$). Let S be an affine monoid minimally generated by $\mathcal{A} := \{\mathbf{a}_1, \dots, \mathbf{a}_p\} \subset \mathbb{N}_0^r$, that is to say $S = \langle \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_p\rangle := \mathbb{N}_0\mathbf{a}_1 + \cdots + \mathbb{N}_0\mathbf{a}_p$ and no proper subset of $\mathcal{A}$ generates S. We say the *embedding dimension* of S is p. For a general introduction to monoids and their invariants, see [5].

The monoid map

$$\pi_{\mathcal{A}} : \mathbb{N}_0^p \longrightarrow S; \mathbf{u} = (u_1, \dots, u_p) \longmapsto \sum_{i=1}^{p} u_i\mathbf{a}_i$$

is sometimes known as the *factorization homomorphism* associated to $\mathcal{A}$, and if $\pi_{\mathcal{A}}(\mathbf{u}) = s$, $\mathbf{u}$ is called a *factorization* of s. For every $s \in S$, the set $\mathsf{Z}(s) := \pi_{\mathcal{A}}^{-1}(s)$ is called the *set of factorizations* of s. Given $s \in S$, for $\mathbf{u} = (u_1, \dots, u_p) \in \mathsf{Z}(s)$, define the *length* of the factorization $\mathbf{u}$, to be $|\mathbf{u}| = u_1 + \cdots + u_p$, and define the *set of lengths* of s as $\mathsf{L}(s) = \{|\mathbf{u}| : \mathbf{u} \in \mathsf{Z}(\mathbf{a})\}$. Define the *elasticity* of $s \in S$

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as $\rho(s) = \frac{\max(\mathsf{L}(s))}{\min(\mathsf{L}(s))}$, and the *elasticity* of S to be $\rho(S) = \sup\{\rho(s) : s \in S \setminus \{0\}\}$. The elasticity is a very important monoid invariant (see, e.g., [2, 3, 6, 7]).

The monoid elasticity $\rho(S)$ for affine monoids is known (see, e.g., [9]). In this note, our main tool will be the function $\phi : \mathbb{Z}^2 \to \mathbb{Q}^\star$ given by $\phi : \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \mapsto \frac{a}{b}$, with $\frac{a}{0}$ conventionally taken to be $+\infty$. Our main focus will be $S \subseteq \mathbb{N}_0^2$, with embedding dimension 2 and 3.

We will compute the elasticity of individual monoid elements. We also provide membership tests for arbitrary elements of $\mathbb{N}_0^2$. We will show that for a given $s \in \mathbb{N}_0^2$, membership in S and $\rho(s)$ are largely determined by $\phi(s)$.

2 Preliminaries

We begin with the observation that $\mathbb{Q}^\star$ is ordered, and the semigroup operation (commonly known as the mediant) preserves this order. This property is well-known; its proof is included for completeness.

Lemma 1. *Let $a, b, c, d \in \mathbb{N}_0$ with $\phi(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]) < \phi(\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right])$. Then*

$$\phi(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]) < \phi(\left[\begin{smallmatrix} a+c \\ b+d \end{smallmatrix}\right]) < \phi(\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]).$$

Proof: We prove only the nontrivial case $bd \neq 0$. Then $ad < bc$ by hypothesis. If we add ab to both sides and divide by $b(b+d)$, we conclude $\frac{a}{b} < \frac{a+c}{b+d}$ which gives the first inequality. If we instead add cd to both sides and divide by $d(b+d)$, we get the second inequality. QED

Corollary 2. *Let $u, v \in \mathbb{N}_0^2$ with $\phi(u) < \phi(v)$. Let $s \in \langle u, v\rangle$. Then $\phi(u) \leq \phi(s) \leq \phi(v)$.*

Proof: Strict inequality is lost if $s = u + u$ or similar. QED

Let $GL(2)$ denote the set of 2×2 unimodular matrices (i.e. with determinant ± 1), with entries from $\mathbb{Z}$. Let $\left[\, u\ v \,\right]$ denote the 2×2 matrix whose first column is u, and whose second column is v. Let $\left[\, \mathcal{A} \,\right]$ denote a similar matrix whose columns are the monoid generators.

Corollary 3. *Let $u, v \in \mathbb{N}_0^2$ with $\phi(u) < \phi(v)$. Let $s \in \langle u, v\rangle$. Let $A \in GL(2)$. Suppose that $Au, Av \in \mathbb{N}_0^2$. Then $As \in \langle Au, Av\rangle$, and either $\phi(Au) \leq \phi(As) \leq \phi(Av)$ or $\phi(Av) \leq \phi(As) \leq \phi(Au)$.*

Proof: Since $s \in \langle u, v\rangle$, there is some vector w with $\left[\, u\ v \,\right]\left[\, w \,\right] = \left[\, s \,\right]$. Then $A\left[\, u\ v \,\right]\left[\, w \,\right] = A\left[\, s \,\right]$, hence $\left[\, Au\ Av \,\right]\left[\, w \,\right] = \left[\, As \,\right]$. Hence $As \in \langle Au, Av\rangle$. We apply Corollary 2 in one of two ways, depending on whether $\phi(Au) \leq \phi(Av)$ or $\phi(Au) \geq \phi(Av)$. QED

Given some $u = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathbb{N}_0^2$, we say that it is ϕ-minimal if $\gcd(a, b) = 1$; otherwise we could take a smaller $u' = \left[\begin{smallmatrix} a/\gcd(a,b) \\ b/\gcd(a,b) \end{smallmatrix}\right]$ with $\phi(u) = \phi(u')$. Henceforth we assume that all of our monoid generators are ϕ-minimal.

Lemma 4. *Let $u = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right], v = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right] \in \mathbb{N}_0^2$. Suppose that both are ϕ-minimal and $\phi(u) = \phi(v)$. Then $u = v$.*

Proof: If $bd = 0$, then $b = d = 0$ and $a = c = 1$; hence, $u = v$. Otherwise $ad = bc$. Since $\gcd(a, b) = 1$, $a|c$. Since $\gcd(c, d) = 1$, $c|a$. Since $a, c \in \mathbb{N}_0$, $a = c$. Similarly, $b = d$. QED

Since all monoid generators are distinct, by Lemma 4, they must also have distinct ϕ-values. Henceforth, we may assume, without loss of generality, that our monoid generators are given in strictly increasing ϕ order.

We now recall Hermite Normal Form, an analog of row echelon form for matrices over non-fields like $\mathbb{Z}$. For every rectangular matrix M with integer entries, there is an associated square unimodular matrix U such that UM is (a) upper triangular; and (b) the pivot in each nonzero row is strictly to the right of the previous row; and (c) all entries of M are nonnegative integers. For an introduction to these and other properties of HNF, see [1].

Now, for $M = [\, u \; v \,]$, applying HNF we have the first column of UM as $\left[\begin{smallmatrix} g \\ 0 \end{smallmatrix}\right]$, where g is the gcd of the entries of u. Since u is ϕ-minimal, $g = 1$. Hence, we have $UM = \left[\begin{smallmatrix} 1 & b \\ 0 & a \end{smallmatrix}\right]$, with $a, b \in \mathbb{N}_0$. We now consider a row-swapped HNF, defined as $U' = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] U$, so $U'M = \left[\begin{smallmatrix} 0 & a \\ 1 & b \end{smallmatrix}\right]$. Note that $U'u, U'v \in \mathbb{N}_0^2$, so by Corollary 3, if $s \in \langle u, v\rangle$ then $\phi(U'u) \leq \phi(U's) \leq \phi(U'v)$. Further, note that $0 = \phi(U'u)$ and $\phi(U'v) > 0$. Henceforth we will assume without loss of generality that our first generator is $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$.

We now recall Smith Normal Form, a non-field analog of the linear algebra theorem giving invertible U, V with $UMV = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$, a block matrix. For any rectangular matrix M with integer entries, there are associated square unimodular matrices U, V such that $UMV = \left[\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix}\right]$, where $D = diag(d_1, d_1 d_2, \ldots, d_1 d_2 \cdots d_k)$. Of particular interest to us are the d_i, the so-called determinantal divisors of M, which satisfy that d_i is the gcd of all the $i \times i$ minors of M. For example, $d_1(M)$ is the gcd of all the entries of M.

The determinantal divisors of M are not disturbed upon multiplication (on either side) by any unimodular matrix. Further, they are not disturbed by appending a column that is a $\mathbb{Z}$-linear combination of the other columns. For an introduction to these and other properties of SNF, see [10] or [1].

Given a single generator u, because we have assumed it is ϕ-minimal, the determinantal divisor $d_1([\, u \,]) = 1$. Consequently, for any invertible U', we must have $d_1([\, U'u \,]) = 1$. In particular, applying our row-swapped HNF preserves ϕ-minimality.

We provide our first membership test for our affine monoid, of arbitrary embedding dimension.

Lemma 5. *Let $S = \langle \mathcal{A} \rangle$, and let $v \in \mathbb{N}_0^2$. Set $M = [\, \mathcal{A} \,]$ and $M' = [\, \mathcal{A} \; v \,]$. If $d_2(M) \neq d_2(M')$, then $v \notin S$.*

Proof: If $v \in S$, then removing the last column of M' (which gives M) will not change the determinantal divisors. QED

3 Embedding Dimension 2

In this section, we fix the case of $S = \langle u, v \rangle$, with $u = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], v = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, and $\gcd(a, b) = 1$. Note that $d_2([\, u\ v\,]) = a$. Consider some $s = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{N}_0^2$. We have proved that if $s \in S$, then $0 \le \phi(s) \le \frac{a}{b}$, and that $d_2([\, u\ v\ s\,]) = d_2([\, u\ v\,]) = a$. It turns out that these two necessary conditions for membership are sufficient.

Theorem 6. *With notation as above, $s \in S$ if and only if both of the following hold:*

1. $0 \le \frac{x}{y} \le \frac{a}{b}$*; and*
2. $a|x$.

Further, if $s \in S$, then $\rho(s) = 1$.

PROOF: Suppose first that $s \in \langle u, v \rangle$. By Corollary 2, $\phi(u) \le \phi(s) \le \phi(v)$. Note that $d_2(\left[\begin{smallmatrix} 0 & a \\ 1 & b \end{smallmatrix}\right]) = a$, as the 2×2 minor is $-a$. Note also that one of the 2×2 minors of $[\, \mathcal{A}\ s\,]$ has determinant $-x$, so we must have $a|x$.

Suppose now that the two conditions hold, i.e. there is some $k \in \mathbb{N}_0$ with $x = ka$. If $k = 0$, then $s = y \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. No other factorization is possible, as even one copy of v will disturb the 0.

Otherwise, since $\frac{x}{y} \le \frac{ka}{kb} = \frac{x}{kb}$, we must have $y \ge kb$. Hence we may write $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = k \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] + (y - kb) \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, which proves $s \in \langle u, v \rangle$. No other factorization is possible, by a back-substitution-type argument: u does not affect the first coordinate, so we must have k copies of v and hence $y - kb$ copies of u. QED

This provides an alternate proof of the well-known fact that in embedding dimension 2, $\rho(S) = 1$.

4 Embedding Dimension 3

We turn now to the case of embedding dimension 3. Henceforth, we fix the case of $S = \langle u, v, w \rangle$, with $u = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], v = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right], w = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$, $\phi(u) < \phi(v) < \phi(w)$, and $\gcd(a, b) = 1 = \gcd(c, d)$. Set $M = \left[\begin{smallmatrix} 0 & a & c \\ 1 & b & d \end{smallmatrix}\right]$. We will also fix $s = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{N}_0^2$.

We first offer a simple way to compute the determinantal divisor d_2 below.

Lemma 7. *With notation as above, $d_2(M) = \gcd(a, c)$.*

PROOF: Since $\gcd(a, c)$ divides each entry of the first row of each 2×2 submatrix, it divides each minor. Hence $\gcd(a, c)|d_2(M)$. Considering the submatrices $\left[\begin{smallmatrix} 0 & a \\ 1 & b \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & c \\ 1 & d \end{smallmatrix}\right]$, we find that $d_2(M)$ divides each of a, c. Hence $d_2(M)|\gcd(a, c)$. QED

Similarly to the embedding dimension 2 case, if $s \in S$, we must have $0 \le \phi(s) \le \frac{c}{d}$, and $d_2([\, M\ s\,]) = d_2([\, M\,]) = \gcd(a, c)$. Further, we must have $x \in \langle a, c \rangle$, since only v, w have nonzero first coordinates to contribute to x. Unfortunately, in general these necessary conditions are not sufficient, as the following example demonstrates.

Example 8. *Consider* $u = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], v = \left[\begin{smallmatrix}11\\10\end{smallmatrix}\right], w = \left[\begin{smallmatrix}10\\3\end{smallmatrix}\right], s = \left[\begin{smallmatrix}199\\119\end{smallmatrix}\right]$. *Note that* $\phi(s) < 2 < \phi(w)$, *and that* $d_2([\,M\ s\,]) = d_2([\,M\,]) = 1$. 199 *can be factored (uniquely) in* $\langle 11, 10\rangle$ *as* $199 = 9 \cdot 11 + 10 \cdot 10$. *However,* $9v + 10w = \left[\begin{smallmatrix}199\\120\end{smallmatrix}\right]$. *Including* u*'s will not help, so* $s \notin S$.

If $x \in \langle a, c\rangle$, then we can impose a restriction on its representation, as follows.

Proposition 9. *Let* $a, c \in \mathbb{N}$ *with* $\gcd(a, c) = 1$. *If* $x \in \langle a, c\rangle$, *then there are* $\alpha, \beta \in \mathbb{N}_0$ *with* $x = \alpha a + \beta c$ *and* $0 \le \alpha < c$.

Proof: Since $x \in \langle a, c\rangle$, there are some $\alpha', \beta' \in \mathbb{N}_0$ with $x = \alpha' a + \beta' b$. But also $x = (\alpha' - tc)a + (\beta' + ta)c$ for all integer t. Choose $t \ge 0$ maximal with $\alpha' - tc \ge 0$, set $\alpha = \alpha' - tc, \beta = \beta' + ta$, and observe that $0 \le \alpha < c$. QED

We will frequently use the canonical factorization of x in $\langle a, c\rangle$ from Proposition 9, which we call $\alpha(x), \beta(x)$.

Despite the setback of Example 8, with an additional restriction, we can solve the membership problem. Henceforth, we add the following standing hypothesis.

$$bc - ad = 1 \tag{$\star$}$$

Note that ($\star$) implies that $1 = \gcd(a, b) = \gcd(a, c) = \gcd(b, d) = \gcd(c, d) = 1$. Hence, condition ($\star$) alone implies ϕ-minimality on v, w, and also $d_2(M) = 1$.

Theorem 10. *With notation as above,* $s \in S$ *if and only if both*

1. $0 \le \frac{x}{y} \le \frac{c}{d}$; *and*
2. $x \in \langle a, c\rangle$.

Proof: If $s \in S$, both conditions are easily seen to hold.

Suppose now that the two conditions hold. Take α, β as in Proposition 9. We now prove that $y \ge \alpha b + \beta d$. Supposing otherwise, we have $y \le \alpha b + \beta d - 1$. Since $\alpha < c$, $-\alpha > -c$, and hence $(ad - bc)\alpha > -c$. Adding βcd to both sides, with a bit of algebra we get $\alpha ad + \beta cd > \alpha bc + \beta cd - c$, or $\frac{\alpha a + \beta c}{\alpha b + \beta d - 1} > \frac{c}{d}$. But then $\frac{x}{y} > \frac{c}{d}$, which contradicts hypothesis. Hence $y \ge \alpha b + \beta d$. Then we write $s = (y - \alpha b - \beta d)\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] + \alpha\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right] + \beta\left[\begin{smallmatrix}c\\d\end{smallmatrix}\right]$, and hence $s \in S$. QED

We turn now to the elasticity problem. The different factorizations of s in S all come from different factorizations of x in $\langle a, c\rangle$, by the following.

Lemma 11. *With notation as above, given* $\alpha', \beta' \in \mathbb{N}_0$ *with* $x = \alpha' a + \beta' c$, *there is exactly one* $\delta = \delta(\alpha', \beta') \in \mathbb{Z}$ *with* $s = \delta u + \alpha' v + \beta' w$.

Proof: If $s = \delta u + \alpha' v + \beta' w$, then $y = \delta + \alpha' b + \beta' d$. We solve for δ uniquely. If $\delta \ge 0$, then $s = \delta u + \alpha' v + \beta' w$ is a factorization of s in S. QED

Henceforth, we define function $\delta(\alpha, \beta)$, applying Lemma 11 to the factorization from Proposition 9.

We call a factorization of s *extreme* if it is either of minimal or maximal length. The extreme factorizations are given in the following theorem; there are two cases based on whether $\frac{x}{y}$ is in $(0, \frac{a}{b}]$ or $[\frac{a}{b}, \frac{c}{d})$. Recall that $\lfloor z \rfloor$ denotes the greatest integer that is less than or equal to z.

Theorem 12. *With notation as above, the extreme factorizations of s are*

$$s = (\delta - t)u + (\alpha + ct)v + (\beta - at)w$$

for $t = 0$ and for

$$t = \begin{cases} \lfloor \frac{\beta}{a} \rfloor & \frac{x}{y} \leq \frac{a}{b} \\ \delta & \frac{x}{y} \geq \frac{a}{b} \end{cases}.$$

These extreme factorizations have lengths $\delta + \alpha + \beta$ and

$$\begin{cases} \delta + \alpha + \beta + \lfloor \frac{\beta}{a} \rfloor (c - a - 1) & \frac{x}{y} \leq \frac{a}{b} \\ \delta + \alpha + \beta + \delta(c - a - 1) & \frac{x}{y} \geq \frac{a}{b} \end{cases},$$

respectively.

PROOF: Note that, since $\gcd(a, c) = 1$, all factorizations of x in $\langle a, c\rangle$ are given by $x = (\alpha + ct)a + (\beta - at)c$, for various integer t. Note that $\alpha + ct \geq 0$ precisely when $t \geq 0$, by our choice of α.

By Lemma 11, for each choice of t there is a unique $\delta_t = \delta(\alpha + ct, \beta - at)$ with $s = \delta_t u + (\alpha + ct)v + (\beta - at)w$. Hence $y = \delta_t + (\alpha + ct)b + (\beta - at)d = \delta_t + \alpha b + \beta d + t$, so $\delta_t = y - \alpha b - \beta d - t$. The factorization length (of s in S) is $(\alpha + ct) + (\beta - at) + (y - \alpha b - \beta d - t) = (\alpha + \beta + y - \alpha b - \beta d) + t(c - a - 1)$. In particular, the length varies linearly with t; one extreme is when $t = 0$, and the other is when t is maximal.

There are two upper bounds on t, both of which must hold. One is that $\beta - at \geq 0$ (else the coefficient of w would not be in $\mathbb{N}_0$), while the other is that $0 \leq \delta_t = y - \alpha b - \beta d - t = \delta - t$. Now we compare the two bounds of $\frac{\beta}{a}$ and δ. We have $\frac{\beta}{a} \leq \delta$ exactly when $\alpha ab + \beta cb \leq \alpha ab + \beta ad + \delta a$, which holds exactly when $xb \leq ya$ or $\frac{x}{y} \leq \frac{a}{b}$. In this case, we use the $\frac{\beta}{\alpha}$ bound and get the other for free; in the other case it is the reverse.

Substituting $t = 0$ and $t = \lfloor \frac{\beta}{a} \rfloor$ (or $t = \delta$), we find the lengths as above. QED

Note that the sign of $c - a - 1$ determines which of the two extreme factorizations is minimal and which is maximal. In particular, we have the following.

Corollary 13. *With notation as above, if $c = a + 1$, then $\rho(S) = 1$.*

PROOF: By Theorem 12, each $s \in S$ has $|\mathsf{L}(\mathbf{s})| = 1$. QED

Corollary 14. *With notation as above, we fix $a, b, c, d, x, \alpha, \beta$ and suppose that $\beta(x) < a$. Then, for every $y \geq \frac{bx}{a}$, $\rho([\begin{smallmatrix} x \\ y \end{smallmatrix}]) = 1$.*

PROOF: Our hypotheses force $\frac{x}{y} \leq \frac{a}{b}$ and $\lfloor \frac{\beta}{a} \rfloor = 0$. Although δ will vary based on y, all factorizations of $[\begin{smallmatrix} x \\ y \end{smallmatrix}]$ have the same length. QED

5 Multiples of $s \in S$

We now fix $s \in S$, and consider factorizations of $ks = \left[\begin{smallmatrix} kx \\ ky \end{smallmatrix}\right] \in S$ for various $k \in \mathbb{N}$. For any individual k, we can of course compute $\rho(ks)$ using Theorem 12, but we seek $\rho(ks)$, or estimates thereto, for all the various choices of k. We offer three such results, two specific and one general. For convenience, we recall the sign function given by

$$\mathrm{sign}(z) = \begin{cases} 1 & z > 0 \\ 0 & z = 0 \\ -1 & z < 0 \end{cases}.$$

Our special results determine $\rho(ks)$ exactly, independently of k, but are for periodic values of k only. There are two, based on whether or not $\frac{x}{y} \leq \frac{a}{b}$.

Theorem 15. *With notation as above, set $\tau = sign(c - a - 1)$. Suppose thet $ac|k$ and $\frac{x}{y} \leq \frac{a}{b}$. Then*

$$\rho(ks) = \left(\frac{c}{a}\frac{ya - x(b-1)}{yc - x(d-1)}\right)^{\tau}.$$

Proof: Let $k' \in \mathbb{N}$ with $k = k'ac$. We have $\alpha(kx) = 0$ and $\beta = \beta(kx) = k'ax$. We calculate $\delta = \delta(0, \beta) = ky - \beta d = ak'(cy - dx)$. One of the extreme factorization lengths will be $\delta + \beta = ak'(cy - dx) + ak'x = ak'(cy - (d-1)x)$. The other will be $\delta + \beta + \lfloor \frac{\beta}{a} \rfloor (c - a - 1) = ak'(cy - (d-1)x) + k'x(c - a - 1)$. QED

We now give our second special result, for the case of k a multiple of c and $\frac{x}{y} \geq \frac{a}{b}$. Note that again the elasticity is independent of k.

Theorem 16. *With notation as above, set $\tau = sign(c - a - 1)$. Suppose thet $c|k$ and $\frac{x}{y} \geq \frac{a}{b}$. Then*

$$\rho(ks) = \left(c\frac{y(c-a) - x(d-b)}{yc - x(d-1)}\right)^{\tau}.$$

Proof: Let $k' \in \mathbb{N}$ with $k = k'c$. We have $\alpha(kx) = 0$ and $\beta = \beta(kx) = k'x$. We calculate $\delta = \delta(0, \beta) = ky - \beta d = k'(cy - dx)$. One of the extreme factorization lengths will be $\delta + \beta = k'(cy - dx) + k'x = k'(cy - (d-1)x)$. The other will be $\delta + \beta + \delta(c - a - 1) = k'(cy - (d-1)x) + k'(cy - dx)(c - a - 1)$. QED

The following is a general result for all k. In particular, it implies that $\rho(ks)$ is largely predicted by $\phi(s)$, with this prediction becoming more accurate as $k \to \infty$. Note also that the limiting values agree, as expected, with the values in Theorems 15, 16.

Theorem 17. *With notation as above, set $\tau = sign(c - a - 1)$. Then*

$$\lim_{k \to \infty} \rho(ks) = \begin{cases} \left(\frac{c}{a}\frac{ya - x(b-1)}{yc - x(d-1)}\right)^{\tau} & \frac{x}{y} \leq \frac{a}{b} \\ \left(c\frac{y(c-a) - x(d-b)}{yc - x(d-1)}\right)^{\tau} & \frac{x}{y} \geq \frac{a}{b} \end{cases}.$$

Proof: We set $\alpha = \alpha(kx), \beta = \beta(kx)$, with $kx = \alpha a + \beta c$, and $0 \leq \alpha < c$. Note that $\beta = \frac{kx - \alpha a}{c}$. We calculate $\delta = ky - \alpha b - \beta d = ky - \alpha b - (kx - \alpha a)\frac{d}{c} = k(y - x\frac{d}{c}) - \alpha(b - \frac{ad}{c}) = k(y - x\frac{d}{c}) - \frac{\alpha}{c}$.

Rather than taking $\rho(ks)$ as the ratio of $\max \mathsf{L}(ks)$ to $\min \mathsf{L}(ks)$, we will instead take ρ as the ratio of $\frac{1}{k}\max \mathsf{L}(ks)$ to $\frac{1}{k}\min \mathsf{L}(ks)$. One of these will be $\frac{1}{k}(\delta + \alpha + \beta) = \frac{1}{k}\left(k(y - x\frac{d}{c}) - \frac{\alpha}{c} + \alpha + \frac{kx - \alpha a}{c}\right) = y - x\frac{d-1}{c} + \frac{\alpha}{k}\frac{c-a-1}{c}$. In the limit, the last term vanishes, leaving $y - x\frac{d-1}{c}$.

We consider the case of $\frac{x}{y} \leq \frac{a}{b}$. The other term we will have in our ratio limit will be $\frac{1}{k}\Big(\delta + \alpha + \beta + \lfloor\frac{\beta}{a}\rfloor(c - a - 1)\Big) = y - x\frac{d-1}{c} + \frac{\alpha}{k}\frac{c-a-1}{c} + \frac{1}{k}\lfloor\frac{\beta}{a}\rfloor(c - a - 1)$ Now, $\frac{\beta}{a} = k\frac{x}{ac} - \frac{\alpha}{c}$. In the limit we will get $y - x\frac{d-1}{c} + \frac{x}{ac}(c - a - 1)$. We simplify to $y - x\frac{b-1}{a}$. This gives the first formula.

Finally, we turn to the case of $\frac{x}{y} \geq \frac{a}{b}$. The other term we will have in our ratio limit will be $\frac{1}{k}(\delta + \alpha + \beta + \delta(c - a - 1)) = y - x\frac{d-1}{c} + \frac{\alpha}{k}\frac{c-a-1}{c} + \frac{c-a-1}{k}\left(k(y - x\frac{d}{c}) - \frac{\alpha}{c}\right)$. In the limit we will get $y - x\frac{d-1}{c} + (c-a-1)(y - x\frac{d}{c}) = (c-a)y - (d-b)x$. This gives the second formula. QED

We close by noting that the functions appearing in Theorems 15, 16, and 17 are quite simple, being linear fractional transformations in the variable $\frac{x}{y} = \phi(s)$.

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