1. Find a formula involving the connectives $\lor$, $\land$, and $\neg$ that has this truth table:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>??</th>
<th>$P \land Q$</th>
<th>$\neg(P \land Q)$</th>
<th>$P \lor Q$</th>
<th>$(\neg(P \land Q)) \land (P \lor Q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
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<td>F</td>
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<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

Answer: $(\neg(P \land Q)) \land (P \lor Q)$. Other answers are possible.

2. What can we put in the blank to make the identity correct?

$(A \triangle B) \cap C = (C \setminus A) \triangle _________$

Answer: $C \setminus B$. The simplest justification is by Venn diagram. $A \triangle B$ is the regions 1, 2, 5, 6; intersecting with $C$ gives the regions 5, 6. $C \setminus A$ is the regions 4, 6, while $C \setminus B$ is the regions 4, 5. Taking the symmetric difference gives the regions 5, 6.

3. Find a formula involving only the connectives $\neg$ and $\rightarrow$ that is equivalent to $P \leftrightarrow Q$.

Answer: $\neg((P \rightarrow Q) \rightarrow \neg(Q \rightarrow P))$. This is equivalent to $\neg((P \rightarrow Q) \lor \neg(Q \rightarrow P))$ by the conditional law, which is equivalent to $\neg(P \rightarrow Q) \land \neg(Q \rightarrow P)$ by the second DeMorgan’s law, which is equivalent to $(P \rightarrow Q) \land (Q \rightarrow P)$ by the double negation law (twice), which is equivalent to $P \leftrightarrow Q$ by the definition of biconditional.

4. Determine whether or not the following statements are equivalent: $(\exists x \in A P(x)) \land (\exists x \in B P(x))$ and $\exists x \in (A \cap B) P(x)$.

Answer: no. Here is a counterexample: let $A = \{2, 4\}$, $B = \{3\}$, and let $P(x)$ stand for the sentence “$x$ is prime”. $A$, $B$ each contains a prime, but $A \cap B$ is empty, so does not contain a prime.

5. Prove that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

$\subseteq$: Let $x \in \mathcal{P}(A \cap B)$. Then $x \subseteq A \cap B$. For all $y \in x$, $y \in A \cap B$. In particular, $\forall y \in x$, $y \in A$ (thus $x \subseteq A$) and $\forall y \in x$, $y \in B$ (thus $x \subseteq B$). Because $x \subseteq A$, $x \in \mathcal{P}(A)$; because $x \subseteq B$, $x \in \mathcal{P}(B)$. Combining these we get $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

$\supseteq$: Let $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then $x \in \mathcal{P}(A)$ and $x \in \mathcal{P}(B)$, so $x \subseteq A$ and $x \subseteq B$. For all $y \in x$, $y \in A$ and $y \in B$, so $y \in A \cap B$. Hence $x \subseteq A \cap B$ and thus $x \in \mathcal{P}(A \cap B)$. 


6. Suppose that \( A \setminus B \subseteq C \cap D \) and \( x \in A \). Prove that if \( x \notin D \) then \( x \in B \).

Suppose that \( A \setminus B \subseteq C \cap D \), \( x \in A \), and \( x \notin B \). Combining \( x \in A \) and \( x \notin B \) we get \( x \in A \setminus B \). Because \( A \setminus B \subseteq C \cap D \) we get \( x \in C \cap D \), and in particular \( x \in D \). We have proved that \( x \notin B \) implies \( x \in D \); the contrapositive of this is the desired goal.

7. Suppose that \( x, y \in \mathbb{R} \). Prove that if \( x \neq 0 \), then if \( y = \frac{3x^2 + 2y}{x^2 + 2} \) then \( y = 3 \).

Suppose that \( x \neq 0 \) and \( y = \frac{3x^2 + 2y}{x^2 + 2} \). Multiplying by the nonzero \( x^2 + 2 \) we get \( yx^2 + 2y = 3x^2 + 2y \). Subtracting \( 2y \) we get \( yx^2 = 3x^2 \). Dividing by the nonzero \( x^2 \) we get \( y = 3 \), as desired.

8. Prove that if \( A \) and \( B \setminus C \) are disjoint, then \( A \cap B \subseteq C \).

Suppose that \( A \) and \( B \setminus C \) are disjoint. Let \( x \in A \cap B \); hence \( x \in A \) and \( x \in B \). We argue by contradiction. Suppose that \( x \notin C \). Combining with \( x \in B \) we get \( x \in B \setminus C \).

But also \( x \in A \); yet \( A \) and \( B \setminus C \) are disjoint. This contradiction proves that \( x \in C \).

Since \( x \) was arbitrary in \( A \cap B \), we have shown that \( A \cap B \subseteq C \).

9. Prove that for every integer \( n \), \( n^3 \) is even iff \( n \) is even.

Let \( n \) be an integer. We proceed by cases, depending on if \( n \) is even or odd. If \( n \) is even, then for some integer \( m \), \( n = 2m \). Then \( n^3 = (2m)^3 = 8m^3 = 2(4m^3) \), twice an integer, which is even. This proves that if \( n \) is even then \( n^3 \) is even.

If \( n \) is odd, then for some integer \( k \), \( n = 2k + 1 \). Then \( n^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1 \), which is odd. This proves that if \( n \) is not even, then \( n^3 \) is not even.

10. Prove that for any sets \( A \) and \( B \), if \( \mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B) \) then either \( A \subseteq B \) or \( B \subseteq A \).

(method 1) Suppose that \( \mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B) \). We argue by contradiction. Suppose that \( \neg(A \subseteq B \cup B \subseteq A) \) holds. Hence \( \neg(A \subseteq B) \land \neg(B \subseteq A) \) holds, by DeMorgan’s second law. Hence \( \neg(\forall x \in A \ x \in B) \land \neg(\forall y \in B \ y \in A) \). Hence \( (\exists x \in A \ x \notin B) \land (\exists y \in B \ y \notin A) \). Now, consider the set \( \{x, y\} \); let’s name it \( z \).

Since \( x, y \in A \cup B \), we have \( z \subseteq A \cup B \) so \( z \in \mathcal{P}(A \cup B) \). But \( z \notin A \) since \( y \notin A \); hence \( z \notin \mathcal{P}(A) \). Also, \( z \notin B \) since \( x \notin B \); hence \( z \notin \mathcal{P}(B) \). But this contradicts \( z \in \mathcal{P}(A) \cup \mathcal{P}(B) \), which completes the proof.

(method 2) Suppose that \( \mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B) \). We argue by cases. Either \( A \subseteq B \) or \( B \subseteq A \). In the first case, we are done. In the second case, \( \neg(A \subseteq B) \) holds, so \( \neg(\forall x \in A \ x \in B) \). This implies that \( \exists x \in A \ x \notin B \). Now, let \( y \in B \). Consider the set \( \{x, y\} \); let’s name it \( z \). Since \( x, y \in A \cup B \), we have \( z \subseteq A \cup B \) so \( z \in \mathcal{P}(A \cup B) \). But \( z \notin B \) since \( x \notin B \); hence \( z \notin \mathcal{P}(B) \). Because \( z \in \mathcal{P}(A) \cup \mathcal{P}(B) \), in fact \( z \in \mathcal{P}(A) \). Hence \( y \in A \). Since \( y \in B \) was arbitrary, we have proved that \( B \subseteq A \).