Determining Integer-Valued Polynomials From Their Image

Vadim Ponomarenko

Department of Mathematics and Statistics
San Diego State University
(this year at Karl Franzens University Graz)

November 30, 2010
Third International Meeting on Integer-Valued Polynomials and Problems in Commutative Algebra
http://www-rohan.sdsu.edu/~vadim/marseille.pdf
Joint work with Scott Chapman, preprint available

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Given \( f, g \in \text{Int}(\mathbb{Z}) \) with \( f(\mathbb{Z}) = g(\mathbb{Z}) \), how are \( f, g \) related?

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An Equivalence Relation

We say $f \sim g$ if for some $n \in \mathbb{Z}$, $f(x) = g(x - n)$ or $f(x) = g(-x - n)$.

Note 1: $\sim$ is an equivalence relation.

Note 2: If $f \sim g$ then $f(\mathbb{Z}) = g(\mathbb{Z})$. Converse?

Note 3: If $f \in \text{Int}(\mathbb{Z})$ and $f \sim g$, then $g \in \text{Int}(\mathbb{Z})$. 
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Odd Degree

Even Degree

Horizontal Symmetry
**Odd Degree Theorem**

**Thm:** If $f, g$ have the same, odd, degree and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

**Proof:** Without loss of generality, $f : [0, \infty) \rightarrow f(\mathbb{Z})$ consecutively, and $g : [0, \infty) \rightarrow f(\mathbb{Z})$ consecutively. Choose $m, n \in \mathbb{N}$ with $f(m) = g(n)$. Then $f(m + x) = g(n + x)$ for all $x \in \mathbb{N}$, so $f(m + x) = g(n + x)$. 
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Image of left branch coincides with image of right branch
Type 1
Image of left branch alternates with image of right branch

Type 2  **Thm:** These are the only two types.
Another Picture

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Even Degree Theorems

**Thm:** If $f, g$ have the same, even, degree, are both of Type 1, and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

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**Thm:** If $f, g$ have the same, even, degree, are both of Type 2, and $f(\mathbb{Z}) = g(\mathbb{Z})$, then $f \sim g$.

**Proof:** Without loss of generality, $f \sim f_{\text{left}} \sim f_{\text{right}}$ with $f_{\text{left}} : [0, \infty) \to f(\mathbb{Z})$ and $f_{\text{right}} : [0, \infty) \to f(\mathbb{Z})$ alternating. Similarly $g_{\text{left}}, g_{\text{right}}$. Choose $m, n \in \mathbb{N}$ with $f_{\text{left}}(m) = g_{\text{left}}(n)$ [or $f_{\text{left}}(m) = g_{\text{right}}(n)$]. Then $f_{\text{left}}(m + 2x) = g_{\text{left}}(n + 2x)$ for all $x \in \mathbb{N}$, so $f_{\text{left}}(m + x) = g_{\text{left}}(n + x)$. 
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Four Types of Even-Degree Polynomials

A polynomial has at most one line of reflection $x = k$:

1. $k \in \mathbb{Z}$
2. $2k \in \mathbb{Z}$, $k \notin \mathbb{Z}$
3. $4k \in \mathbb{Z}$, $2k \notin \mathbb{Z}$
4. other, including no line of reflection
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Four Types of Even-Degree Polynomials

A polynomial has at most one line of reflection $x = k$:

(1a) $k \in \mathbb{Z}$

(1b) $2k \in \mathbb{Z}, k \notin \mathbb{Z}$

(2a) $4k \in \mathbb{Z}, 2k \notin \mathbb{Z}$

(2b) other, including no line of reflection

**Thm:** (1a),(1b) are of Type 1. (2a),(2b) are of Type 2.
Cross-Type Image Sharing

Thm: Suppose $f, g$ have the same, even, degree with $f(\mathbb{Z}) = g(\mathbb{Z})$. Suppose $f$ is of Type 1, $g$ is of Type 2. Then $f$ is of Type (1b), $g$ is of Type (2a), and $f(2x) \sim g(x)$

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