Factorizations Common to Some Subsemigroups of \((\mathbb{N}, +)\) and \((\mathbb{N}, \times)\)

Vadim Ponomarenko

Department of Mathematics and Statistics
San Diego State University

Iberian Meeting on Numerical Semigroups
Granada  February 4, 2010

http://www-rohan.sdsu.edu/~vadim/granada.pdf
Shameless advertising

My trip is partially sponsored by the
San Diego State University Mathematics REU.

http://www.sci.sdsu.edu/math-reu/index.html

Please consider a visit, and send your (U.S. citizen or permanent resident) undergraduates.
Main Result, very broadly

**Theorem:** Given $M$, a subsemigroup of $(\mathbb{N}, \times)$, and $m \in \mathbb{N}$.

Then there is a “partial homomorphism” $f : M \rightarrow (\mathbb{N}, +)$.

$f$ is a homomorphism between $M \cap [1, m]$ and its image.

i.e., $f(xy) = f(x) + f(y)$ provided $\{xy, x, y\} \subseteq M \cap [1, m]$.

i.e., $f(xy) = f(x) + f(y)$ provided $\{x, y\} \subseteq M$, and $xy \leq m$.
Theorem: Given $M$, a subsemigroup of $(\mathbb{N}, \times)$, and $m \in \mathbb{N}$.

Then there is a “partial homomorphism” $f : M \rightarrow (\mathbb{N}, +)$.

$f$ is a homomorphism between $M \cap [1, m]$ and its image. i.e., $f(xy) = f(x) + f(y)$ provided $\{xy, x, y\} \subseteq M \cap [1, m]$

i.e., $f(xy) = f(x) + f(y)$ provided $\{x, y\} \subseteq M$, and $xy \leq m$
Main Result, very broadly

**Theorem:** Given $M$, a subsemigroup of $(\mathbb{N}, \times)$, and $m \in \mathbb{N}$.

Then there is a “partial homomorphism” $f : M \rightarrow (\mathbb{N}, +)$.

$f$ is a homomorphism between $M \cap [1, m]$ and its image.
i.e., $f(xy) = f(x) + f(y)$ provided $\{xy, x, y\} \subseteq M \cap [1, m]$
i.e., $f(xy) = f(x) + f(y)$ provided $\{x, y\} \subseteq M$, and $xy \leq m$
Main Result, in a picture

\[ M \cap [1, m] \]
atomic representation
\{a_1, a_2, \ldots, a_j\}
Main Result, in a picture

\[ M \cap [1, m] \]
prime representation
\[ \{p_1, p_2, \ldots, p_k\} \]

\[ A \]

\[ M \cap [1, m] \]
atomic representation
\[ \{a_1, a_2, \ldots, a_j\} \]
Main Result, in a picture

\[ M \cap [1, m] \]
prime representation
\{p_1, p_2, \ldots, p_k\}

\[ M \cap [1, m] \]
atOMIC representation
\{a_1, a_2, \ldots, a_j\}

\begin{align*}
\text{natural} & \quad \text{free group on} \\
\{p_1, p_2, \ldots, p_k\} & \quad \mathbb{N}_0^k
\end{align*}
Main Result, in a picture

\[
\begin{align*}
M \cap [1, m] & \quad \text{prime representation} \\
\{p_1, p_2, \ldots, p_k\} & \\

\downarrow \quad \text{natural} & \\

\text{free group on} & \\
\{p_1, p_2, \ldots, p_k\} & \quad \nu \\

\Leftrightarrow & \\

N^k_0 & \\

\downarrow & \\

Q & \\

(\mathbb{N}_0, +) & \\
\end{align*}
\]
Main Result, in a picture

\[ M \cap [1, m] \]
prime representation
\{p_1, p_2, \ldots, p_k\}

\[ \text{natural} \]
\[ \text{free group on} \]
\{p_1, p_2, \ldots, p_k\}

\[ \nu \]
\[ \mathbb{N}_0^k \]

\[ A \]
\[ f \]
\[ (\mathbb{N}_0, +) \]
**Trivial but Important Lemma**

**Lemma:** Consider $x_0, x_1, \ldots, x_{k-1} \in \mathbb{Z}$ and $q \in \mathbb{N}$. Suppose that $x_0 q^0 + x_1 q^1 + x_2 q^2 + \cdots + x_{k-1} q^{k-1} = 0$. Then $x_0 = x_1 = \cdots = x_{k-1} = 0$, or $|x_i| \geq q$ for some $i$.

**Proof:** Assume $k$ is minimal to give a counterexample. All terms but the last: $|x_0 q^0 + x_1 q^1 + \cdots + x_{k-2} q^{k-2}| \leq (q - 1)(q^0 + q^1 + \cdots + q^{k-2}) = (q - 1) \frac{q^{k-1} - 1}{q - 1} < q^{k-1}$. Hence $x_{k-1} = 0$, and $k$ was not minimal.

**Consequence:** Choose sufficiently large $q$, then take $Q = [1, q, q^2, \ldots, q^{k-1}]$. 
Trivial but Important Lemma

**Lemma:** Consider $x_0, x_1, \ldots, x_{k-1} \in \mathbb{Z}$ and $q \in \mathbb{N}$. Suppose that $x_0 q^0 + x_1 q^1 + x_2 q^2 + \cdots + x_{k-1} q^{k-1} = 0$. Then $x_0 = x_1 = \cdots = x_{k-1} = 0$, or $|x_i| \geq q$ for some $i$.

**Proof:** Assume $k$ is minimal to give a counterexample. All terms but the last: $|x_0 q^0 + x_1 q^1 + \cdots + x_{k-2} q^{k-2}| \leq (q - 1)(q^0 + q^1 + \cdots + q^{k-2}) = (q - 1)\frac{q^{k-1} - 1}{q - 1} < q^{k-1}$. Hence $x_{k-1} = 0$, and $k$ was not minimal.

**Consequence:** Choose sufficiently large $q$, then take $Q = [1, q, q^2, \ldots, q^{k-1}]$. 
Trivial but Important Lemma

**Lemma:** Consider \( x_0, x_1, \ldots, x_{k-1} \in \mathbb{Z} \) and \( q \in \mathbb{N} \). Suppose that \( x_0 q^0 + x_1 q^1 + x_2 q^2 + \cdots + x_{k-1} q^{k-1} = 0 \). Then \( x_0 = x_1 = \cdots = x_{k-1} = 0 \), or \( |x_i| \geq q \) for some \( i \).

**Proof:** Assume \( k \) is minimal to give a counterexample. All terms but the last: \( |x_0 q^0 + x_1 q^1 + \cdots + x_{k-2} q^{k-2}| \leq (q - 1)(q^0 + q^1 + \cdots + q^{k-2}) = (q - 1) \frac{q^{k-1} - 1}{q - 1} < q^{k-1} \). Hence \( x_{k-1} = 0 \), and \( k \) was not minimal.

**Consequence:** Choose sufficiently large \( q \), then take \( Q = [1, q, q^2, \ldots, q^{k-1}] \).
**Trivial but Important Lemma**

**Lemma:** Consider \( x_0, x_1, \ldots, x_{k-1} \in \mathbb{Z} \) and \( q \in \mathbb{N} \). Suppose that \( x_0 q^0 + x_1 q^1 + x_2 q^2 + \cdots + x_{k-1} q^{k-1} = 0 \). Then \( x_0 = x_1 = \cdots = x_{k-1} = 0 \), or \( |x_i| \geq q \) for some \( i \).

**Proof:** Assume \( k \) is minimal to give a counterexample. All terms but the last: \( |x_0 q^0 + x_1 q^1 + \cdots + x_{k-2} q^{k-2}| \leq (q - 1)(q^0 + q^1 + \cdots + q^{k-2}) = (q - 1) \frac{q^{k-1} - 1}{q - 1} < q^{k-1} \). Hence \( x_{k-1} = 0 \), and \( k \) was not minimal.

**Consequence:** Choose sufficiently large \( q \), then take \( Q = [1, q, q^2, \ldots, q^{k-1}] \).
Main Result, one more time

\[ Q = [1, q, q^2, \ldots, q^{k-1}]; \quad Qx = 0 \text{ implies } x = 0 \text{ or } \|x\|_\infty \geq q. \]
A Detailed Example

Let $M = \langle 144, 216, 108, 162 \rangle \subseteq (\mathbb{N}, \times)$

$j = 4; 144 = 2^43^2, 216 = 2^33^3, 108 = 2^23^3, 162 = 2^13^4$

$k = 2; p_1 = 2, p_2 = 3$

$A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 3 & 3 & 4 \end{pmatrix}$

Q1: Find all factorizations of $(216)^{10}$.

Q2: Find all factorizations of $(216)^{20}$. 
A Detailed Example

Let $M = \langle 144, 216, 108, 162 \rangle \subseteq (\mathbb{N}, \times)$

$j = 4; 144 = 2^43^2, 216 = 2^33^3, 108 = 2^23^3, 162 = 2^13^4$

$k = 2; p_1 = 2, p_2 = 3$

$A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 3 & 3 & 4 \end{pmatrix}$

Q1: Find all factorizations of $(216)^{10}$.

Q2: Find all factorizations of $(216)^{20}$. 
Let $M = \langle 144, 216, 108, 162 \rangle \subseteq (\mathbb{N}, \times)$

$j = 4; \ 144 = 2^43^2, \ 216 = 2^33^3, \ 108 = 2^23^3, \ 162 = 2^13^4$

$k = 2; \ p_1 = 2, \ p_2 = 3$

\[ A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 3 & 3 & 4 \end{pmatrix} \]

Q1: Find all factorizations of $(216)^{10}$.
Q2: Find all factorizations of $(216)^{20}$. 
\[ M = <144, 216, 108, 162>, \quad m = (216)^{10} \]

\[
(144)^{10} \leq (216)^{10}, \quad (144)^{11} > (216)^{10}, \quad (144)^{10} = 2^{40}3^{20}
\]

\[
(216)^{10} \leq (216)^{10}, \quad (216)^{11} > (216)^{10}, \quad (216)^{10} = 2^{30}3^{30}
\]

\[
(108)^{11} \leq (216)^{10}, \quad (108)^{12} > (216)^{10}, \quad (108)^{11} = 2^{22}3^{33}
\]

\[
(162)^{10} \leq (216)^{10}, \quad (162)^{11} > (216)^{10}, \quad (162)^{10} = 2^{10}3^{40}
\]

We take \( q = \max\{40, 20, 30, 30, 22, 33, 10, 40\} + 1 = 41 \).

Simpler strategy:

\[
2^{77} \leq (216)^{10}, \quad 2^{78} > (216)^{10}, \quad 3^{48} \leq (216)^{10}, \quad 3^{49} > (216)^{10}
\]

We could take the larger \( q = \max\{77, 48\} + 1 = 78 \).
\[ M = \langle 144, 216, 108, 162 \rangle, \quad m = (216)^{10} \]

\[(144)^{10} \leq (216)^{10}, \quad (144)^{11} > (216)^{10}, \quad (144)^{10} = 2^{40} 3^{20} \]
\[(216)^{10} \leq (216)^{10}, \quad (216)^{11} > (216)^{10}, \quad (216)^{10} = 2^{30} 3^{30} \]
\[(108)^{11} \leq (216)^{10}, \quad (108)^{12} > (216)^{10}, \quad (108)^{11} = 2^{22} 3^{33} \]
\[(162)^{10} \leq (216)^{10}, \quad (162)^{11} > (216)^{10}, \quad (162)^{10} = 2^{10} 3^{40} \]

We take \( q = \max\{40, 20, 30, 30, 22, 33, 10, 40\} + 1 = 41. \)

Simpler strategy:
\[ 2^{77} \leq (216)^{10}, \quad 2^{78} > (216)^{10}, \quad 3^{48} \leq (216)^{10}, \quad 3^{49} > (216)^{10} \]

We could take the larger \( q = \max\{77, 48\} + 1 = 78. \)
$M = \langle 144, 216, 108, 162 \rangle$, $m = (216)^{10}$

$(144)^{10} \leq (216)^{10}$, $(144)^{11} > (216)^{10}$, $(144)^{10} = 2^{40}3^{20}$
$(216)^{10} \leq (216)^{10}$, $(216)^{11} > (216)^{10}$, $(216)^{10} = 2^{30}3^{30}$
$(108)^{11} \leq (216)^{10}$, $(108)^{12} > (216)^{10}$, $(108)^{11} = 2^{22}3^{33}$
$(162)^{10} \leq (216)^{10}$, $(162)^{11} > (216)^{10}$, $(162)^{10} = 2^{10}3^{40}$

We take $q = \max\{40, 20, 30, 30, 22, 33, 10, 40\} + 1 = 41$.

Simpler strategy:
$2^{77} \leq (216)^{10}$ $2^{78} > (216)^{10}$, $3^{48} \leq (216)^{10}$ $3^{49} > (216)^{10}$

We could take the larger $q = \max\{77, 48\} + 1 = 78$. 
\[ M = \langle 144, 216, 108, 162 \rangle, \; m = (216)^{10} \]

\[
(144)^{10} \leq (216)^{10}, \; (144)^{11} > (216)^{10}, \; (144)^{10} = 2^{40}3^{20} \\
(216)^{10} \leq (216)^{10}, \; (216)^{11} > (216)^{10}, \; (216)^{10} = 2^{30}3^{30} \\
(108)^{11} \leq (216)^{10}, \; (108)^{12} > (216)^{10}, \; (108)^{11} = 2^{22}3^{33} \\
(162)^{10} \leq (216)^{10}, \; (162)^{11} > (216)^{10}, \; (162)^{10} = 2^{10}3^{40}
\]

We take \( q = \max\{40, 20, 30, 30, 22, 33, 10, 40\} + 1 = 41. \)

**Simpler strategy:**

\[
2^{77} \leq (216)^{10} \; 2^{78} > (216)^{10}, \; 3^{48} \leq (216)^{10} \; 3^{49} > (216)^{10}
\]

We could take the larger \( q = \max\{77, 48\} + 1 = 78. \)
$M = \langle 144, 216, 108, 162 \rangle$, $m = (216)^{10}$

$q = 41$, $k = 2$, so $Q = \begin{pmatrix} 1 & 41 \end{pmatrix}$.

$A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 3 & 3 & 4 \end{pmatrix}$

$QA = \begin{pmatrix} 86 & 126 & 125 & 165 \end{pmatrix}$

Hence we consider $N = \langle 86, 126, 125, 165 \rangle \subseteq (\mathbb{N}, +)$.

We factor $1260 = [0, 10, 0, 0]$ in $N$. GAP+numericalsgps factors: $[0, 10, 0, 0], [3, 2, 6, 0], [4, 1, 5, 1], [5, 0, 4, 2]$ $(216)^{10}, (144)^3 (216)^2 (108)^6, (144)^4 (216) (108)^5 (162), (144)^5 (108)^4 (162)^2$
\[ M = \langle 144, 216, 108, 162 \rangle, \ m = (216)^{10} \]

\[ q = 41, \ k = 2, \text{ so } Q = \begin{pmatrix} 1 & 41 \end{pmatrix}. \]

\[ A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 3 & 3 & 4 \end{pmatrix} \]

\[ QA = \begin{pmatrix} 86 & 126 & 125 & 165 \end{pmatrix} \]

Hence we consider \[ N = \langle 86, 126, 125, 165 \rangle \subseteq (\mathbb{N}, +) \].

We factor \[ 1260 = [0, 10, 0, 0] \] in \[ N \]. GAP+numericalsgps factors:

\[ [0, 10, 0, 0], [3, 2, 6, 0], [4, 1, 5, 1], [5, 0, 4, 2], (216)^{10}, (144)^{3}(216)^{2}(108)^{6}, (144)^{4}(216)(108)^{5}(162), (144)^{5}(108)^{4}(162)^{2} \]
\[ M = \langle 144, 216, 108, 162 \rangle, \ m = (216)^{10} \]

\[ q = 41, \ k = 2, \text{ so } Q = \begin{pmatrix} 1 & 41 \end{pmatrix}. \]

\[ A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 3 & 3 & 4 \end{pmatrix} \]

\[ QA = \begin{pmatrix} 86 & 126 & 125 & 165 \end{pmatrix} \]

Hence we consider \( N = \langle 86, 126, 125, 165 \rangle \subseteq (\mathbb{N}, +) \).

We factor \( 1260 = [0, 10, 0, 0] \) in \( N \). GAP+numericalsgps factors: \( [0, 10, 0, 0], [3, 2, 6, 0], [4, 1, 5, 1], [5, 0, 4, 2] \)

\( (216)^{10}, (144)^3(216)^2(108)^6, (144)^4(216)(108)^5(162), (144)^5(108)^4(162)^2 \)
\[ M = \langle 144, 216, 108, 162 \rangle, \quad m = (216)^{20} \]

We turn now to \( m = (216)^{20} \).

**NOTE:** Our previous semigroup \( < 86, 126, 125, 165 > \) based on \( q = 41 \) does not work.

GAP factors \([0, 20, 0, 0]\) as (among others) \([0, 0, 3, 13]\) but \((216)^{20} \neq (108)^3(162)^{13}\)

Set \( p(n) = n \left\lfloor \frac{\log(m)}{\log(n)} \right\rfloor \). \( p(144) = 2^{84}3^{42} \), \( p(216) = 2^{60}3^{60} \), \( p(108) = 2^{44}3^{66} \), \( p(162) = 2^{21}3^{84} \)

We take \( q = \max\{84, 42, 60, 60, 44, 66, 21, 84\} + 1 = 85 \).
\[ M = \langle 144, 216, 108, 162 \rangle, \quad m = (216)^{20} \]

We turn now to \( m = (216)^{20} \).

NOTE: Our previous semigroup \( \langle 86, 126, 125, 165 \rangle \) based on \( q = 41 \) does not work.

GAP factors \([0, 20, 0, 0]\) as (among others) \([0, 0, 3, 13]\) but \((216)^{20} \neq (108)^3 (162)^{13}\)

Set \( p(n) = n \left\lfloor \frac{\log(m)}{\log(n)} \right\rfloor \).

\[
p(144) = 2^{84} 3^{42}, \quad p(216) = 2^{60} 3^{60}, \quad p(108) = 2^{44} 3^{66}, \quad p(162) = 2^{21} 3^{84}
\]

We take \( q = \max\{84, 42, 60, 60, 44, 66, 21, 84\} + 1 = 85.\)
\[ M = \langle 144, 216, 108, 162 \rangle, \quad m = (216)^{20} \]

We turn now to \( m = (216)^{20} \).

NOTE: Our previous semigroup \( \langle 86, 126, 125, 165 \rangle \) based on \( q = 41 \) does not work.

GAP factors \([0, 20, 0, 0]\) as (among others) \([0, 0, 3, 13]\) but

\((216)^{20} \neq (108)^3(162)^{13}\)

Set \( p(n) = n \left\lfloor \frac{\log(m)}{\log(n)} \right\rfloor \).

\( p(144) = 2^{84}3^{42}, \quad p(216) = 2^{60}3^{60}, \quad p(108) = 2^{44}3^{66}, \quad p(162) = 2^{21}3^{84} \)

We take \( q = \max\{84, 42, 60, 60, 44, 66, 21, 84\} + 1 = 85 \).
\[ M = \langle 144, 216, 108, 162 \rangle, \quad m = (216)^{20} \]

\[ q = 85, \text{ so } Q = \begin{pmatrix} 1 & 85 \end{pmatrix}, \quad A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 3 & 3 & 4 \end{pmatrix} \]

\[ QA = \begin{pmatrix} 174 & 258 & 257 & 341 \end{pmatrix} \]

Hence we consider \( N = \langle 174, 258, 257, 341 \rangle \subseteq (\mathbb{N}, +) \).

We factor \( 5160 = [0, 20, 0, 0] \) in \( N \).

GAP+numericalsgps instantly gives the 13 factorizations:

\[ [0, 20, 0, 0], [3, 12, 6, 0], [4, 11, 5, 1], [5, 10, 4, 2], [6, 9, 3, 3], [6, 4, 12, 0], [7, 8, 2, 4], [7, 3, 11, 1], [8, 7, 1, 5], [8, 2, 10, 2], [9, 6, 0, 6], [9, 1, 9, 3], [10, 0, 8, 4] \]
\[ M = \langle 144, 216, 108, 162 \rangle, \quad m = (216)^{20} \]

\[ q = 85, \quad Q = \begin{pmatrix} 1 & 85 \end{pmatrix}, \quad A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 3 & 3 & 4 \end{pmatrix} \]

\[ QA = \begin{pmatrix} 174 & 258 & 257 & 341 \end{pmatrix} \]

Hence we consider \( N = \langle 174, 258, 257, 341 \rangle \subseteq (\mathbb{N}, +) \).

We factor 5160 = [0, 20, 0, 0] in \( N \).

GAP+numericalsgps instantly gives the 13 factorizations:

\[ [0, 20, 0, 0], [3, 12, 6, 0], [4, 11, 5, 1], [5, 10, 4, 2], [6, 9, 3, 3], [6, 4, 12, 0], [7, 8, 2, 4], [7, 3, 11, 1], [8, 7, 1, 5], [8, 2, 10, 2], [9, 6, 0, 6], [9, 1, 9, 3], [10, 0, 8, 4] \]
\[
M = \langle 144, 216, 108, 162 \rangle, \quad m = (216)^{20}, \quad q = 85, \\
N = \langle 174, 258, 257, 341 \rangle
\]

Note: \( q = 85 \) equally works for all of \([1, (216)^{20}]\).

For example, back to Q1, to factor \((216)^{10} = [0, 10, 0, 0]\).
GAP factors \( 2580 = [0, 10, 0, 0] \) in the same four ways.

2580: catenary degree=9, elasticity=\( \frac{11}{10} \), delta set={1}.

elasticity of \( N \) is \( \frac{341}{174} \geq \) elasticity of \( M \cap [1, m] \).
catenary degree of \( N \) is 29 \( \geq \) catenary deg. of \( M \cap [1, m] \).
\[ M = \langle 144, 216, 108, 162 \rangle, \quad m = (216)^{20}, \quad q = 85, \]
\[ N = \langle 174, 258, 257, 341 \rangle \]

Note: \( q = 85 \) equally works for all of \([1, (216)^{20}]\).

For example, back to \( Q1 \), to factor \((216)^{10} = [0, 10, 0, 0]\).
GAP factors \( 2580 = [0, 10, 0, 0] \) in the same four ways.

2580: catenary degree=9, elasticity=\( \frac{11}{10} \), delta set=\{1\}.

elasticity of \( N \) is \( \frac{341}{174} \geq \) elasticity of \( M \cap [1, m] \).

catenary degree of \( N \) is 29 \( \geq \) catenary deg. of \( M \cap [1, m] \).
$M = \langle 144, 216, 108, 162 \rangle, \ m = (216)^{20}, \ q = 85, \ N = \langle 174, 258, 257, 341 \rangle$

Note: $q = 85$ equally works for all of $[1, (216)^{20}]$.

For example, back to Q1, to factor $(216)^{10} = [0, 10, 0, 0]$. GAP factors $2580 = [0, 10, 0, 0]$ in the same four ways.

2580: catenary degree=9, elasticity=$\frac{11}{10}$, delta set=${1}$.

elasticity of $N$ is $\frac{341}{174} \geq$ elasticity of $M \cap [1, m]$.
catenary degree of $N$ is $29 \geq$ catenary deg. of $M \cap [1, m]$. 
Genesis

Chapman, Herr, Rooney
“A Factorization Formula for Class Number Two”, 1999

\[ M \cong < 4, 9, 25, 6, 10, 15 >, \ m = 900 = 4 \cdot 15^2. \]

\[
A = \begin{pmatrix}
2 & 0 & 0 & 1 & 1 & 0 \\
0 & 2 & 0 & 1 & 0 & 1 \\
0 & 0 & 2 & 0 & 1 & 1
\end{pmatrix}
\]

Calculate \( q = 9 \) as before, take \( Q = [13, 118, 1063] \)
(differs from previous \( Q \) for technical reasons)
\( N = < 26, 236, 2126, 131, 1076, 1181 > \)

\( 2388 = [1, 0, 0, 0, 0, 2] \) has five factorizations.
Genesis

Chapman, Herr, Rooney
“A Factorization Formula for Class Number Two”, 1999

\[ M \cong < 4, 9, 25, 6, 10, 15 >, \ m = 900 = 4 \cdot 15^2. \]

\[
A = \begin{pmatrix}
2 & 0 & 0 & 1 & 1 & 0 \\
0 & 2 & 0 & 1 & 0 & 1 \\
0 & 0 & 2 & 0 & 1 & 1
\end{pmatrix}
\]

Calculate \( q = 9 \) as before, take \( Q = [13, 118, 1063] \) 
(differs from previous \( Q \) for technical reasons)

\[ N = < 26, 236, 2126, 131, 1076, 1181 > \]

\[ 2388 = [1, 0, 0, 0, 0, 2] \text{ has five factorizations.} \]
Genesis

Chapman, Herr, Rooney
“A Factorization Formula for Class Number Two”, 1999

\[ M \cong < 4, 9, 25, 6, 10, 15 >, \ m = 900 = 4 \cdot 15^2. \]

\[
A = \begin{pmatrix}
2 & 0 & 0 & 1 & 1 & 0 \\
0 & 2 & 0 & 1 & 0 & 1 \\
0 & 0 & 2 & 0 & 1 & 1
\end{pmatrix}
\]

Calculate \( q = 9 \) as before, take \( Q = [13, 118, 1063] \)
(differs from previous \( Q \) for technical reasons)
\[ N = < 26, 236, 2126, 131, 1076, 1181 > \]

\[ 2388 = [1, 0, 0, 0, 0, 2] \] has five factorizations.
Genesis

Chapman, Herr, Rooney
“A Factorization Formula for Class Number Two”, 1999

\[ M \cong < 4, 9, 25, 6, 10, 15 >, \quad m = 900 = 4 \cdot 15^2. \]

\[
A = \begin{pmatrix}
2 & 0 & 0 & 1 & 1 & 0 \\
0 & 2 & 0 & 1 & 0 & 1 \\
0 & 0 & 2 & 0 & 1 & 1
\end{pmatrix}
\]

Calculate \( q = 9 \) as before, take \( Q = [13, 118, 1063] \)
(differs from previous \( Q \) for technical reasons)
\[ N = < 26, 236, 2126, 131, 1076, 1181 > \]

2388 = [1, 0, 0, 0, 0, 2] has five factorizations.
Factorizations Common to Some Subsemigroups of \((\mathbb{N}, +)\) and \((\mathbb{N}, \times)\)

Vadim Ponomarenko

Department of Mathematics and Statistics
San Diego State University

Iberian Meeting on Numerical Semigroups
Granada   February 4, 2010

http://www-rohan.sdsu.edu/~vadim/granada.pdf