Gelfand's Question in Different Bases

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http://www-rohan.sdsu.edu/~vadim/gelfand-talk.pdf

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This work was done jointly with Jason Thoma, Master's student.

Background

Let $\langle\langle n \rangle\rangle$ denote the leading digit of positive integer n.
e.g. $\langle\langle 12 \rangle\rangle = 1$, $\langle\langle 345 \rangle\rangle = 3$, $\langle\langle 7 \rangle\rangle = 7$

Question 1: (Gelfand? 1965?)
Is there any $n \in \mathbb{N}$ with $\langle\langle 2^n \rangle\rangle = 9$?

Question 2: Set $D = \{1, 2, \ldots, 9\}$, the nonzero digits.
Given $d, t \in D$, is there any $n \in \mathbb{N}$ with $\langle\langle d^n \rangle\rangle = t$?

Note: $d = 1$ is trivial, as $d^n = 1$.

More Background

Question 3:

Given a vector t, i.e. $t \in D^8$, is there any $n \in \mathbb{N}$ with t achieved, i.e. $(\langle\langle 2^n \rangle\rangle, \langle\langle 3^n \rangle\rangle, \ldots, \langle\langle 9^n \rangle\rangle) = t$?

Special cases:

$t = (2, 3, \ldots, 9)$, insisting that $n > 1$

$t = (a, a, \ldots, a)$, for some $a \in D$

$t_1 t_2 \cdots t_8$, viewed as an 8-digit number, is prime

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Eising, Radcliffe, Top paper

American Mathematical Monthly 122 (3) 2015

Eising, Radcliffe, Top

"A Simple Answer to Gelfand's Question"

Q1: $\langle\langle 2^n \rangle\rangle = 9$? Yes

Q2: $d, t \in D$, $\langle\langle d^n \rangle\rangle = t$? Yes

Q3: $t \in D^8$, t achieved?

17596 vectors t are achieved (out of $9^8 = 43046721$)

23456789 is not achieved, nor is any *aaaaaaaa*

1127 primes are achieved

Principal Technique

Kronecker's Theorem (1884):

Let $x_1, x_2, \ldots, x_k \in \mathbb{R}$. Set y to be the natural projection of $(x_1, x_2, \ldots, x_k)$ into the additive group $\mathbb{R}^k/\mathbb{Z}^k$. TFAE:

(1) $\{1, x_1, \ldots, x_k\}$ is $\mathbb{Q}$-linearly independent

(2) $\langle y \rangle$ is dense in $\mathbb{R}^k/\mathbb{Z}^k$.

Kronecker's Theorem, special case

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Take $k = 1$. Then $x \notin \mathbb{Q}$, if and only if $\langle x \rangle$ is dense in $\mathbb{R}/\mathbb{Z}$.

Kronecker's Theorem, in ERT

Special Case: $x \in \mathbb{R} \setminus \mathbb{Q}$, if and only if $\langle x \rangle$ is dense in $\mathbb{R}/\mathbb{Z}$.

Set $\pi : \mathbb{R} \to \mathbb{R} \cap [0, 1)$ be the natural projection (mod 1).
ERT: $\langle\langle x \rangle\rangle = \lfloor 10^{\pi(\log_{10} x)} \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function.

Now set $x = 2^n$. $\langle\langle x \rangle\rangle = \lfloor 10^{\pi(n \log_{10} 2)} \rfloor$.

Since $\log_{10} 2 \notin \mathbb{Q}$, $\langle \log_{10} 2 \rangle$ is dense in $\mathbb{R}/\mathbb{Z}$. Hence for some n, the exponent must be in $[\log_{10} 9, 1)$.

(Question #1) Note: $n = 53$ is smallest such n.

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What about other bases?

Let $B \in \mathbb{N}$ be our base, and $D = \{1, 2, \ldots, B-1\}$.

Kronecker: $x \in \mathbb{R} \setminus \mathbb{Q}$, if and only if $\langle x \rangle$ is dense in $\mathbb{R}/\mathbb{Z}$.

ERT technique: $\langle\langle x \rangle\rangle = \lfloor B^{\pi(\log_B x)} \rfloor$

If B is not a perfect power, then $\log_B d \notin \mathbb{Q}$, and the same argument works; i.e. all t are achieved.

If B is a perfect power, then for certain d, $\log_B d \in \mathbb{Q}$, and $\langle \log_B d \rangle$ is not dense. What about $\langle\langle d^n \rangle\rangle$?

PT Thm: In that case certain t are not achieved.

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General Kronecker's Theorem

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(2) $\langle y \rangle$ is dense in $\mathbb{R}^k/\mathbb{Z}^k$.

ERT: $\log_{10} 2 + \log_{10} 5 = 1$: $\langle(\log_{10} 2, \log_{10} 5)\rangle$ NOT dense in $\mathbb{R}^2/\mathbb{Z}^2$. In particular, $(\langle\langle 2^n \rangle\rangle, \langle\langle 5^n \rangle\rangle) \neq (2, 5)$ for $n \neq 1$.

Results

- If B is a perfect power, not all t achieved for certain d.
- If B is not a perfect power, all t achieved for every digit d, singly.
- If $B = uv$ for $u, v > 1$ and $\gcd(u, v) = 1$, then $(\langle\langle u^n \rangle\rangle, \langle\langle v^n \rangle\rangle) \neq (u, v)$ for $n \neq 1$.
  Also, $(a, a, \ldots, a)$ is not achieved.
- If B is a prime, work in progress.

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For Further Reading

Jaap Eising, David Radcliffe, Jaap Top
A Simple Answer to Gelfand's Question
American Mathematical Monthly 122 (3) 2015, pp. 234-245.