On The Multi-Dimensional Frobenius Problem

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http://www-rohan.sdsu.edu/~vadim/frobenius2.pdf
Please encourage your students to apply to the San Diego State University Mathematics REU.

http://www.sci.sdsu.edu/math-reu/index.html

This talk presents $\approx 3\%$ of the results from 2005, 2006.
Fix a set $A$ of positive integers.
Frobenius numbers are defined via:

$$g(A) = \sup \left( \mathbb{Z} \setminus \mathbb{N}_0[A] \right) \quad f(A) = \sup \left( \mathbb{Z} \setminus \mathbb{N}[A] \right)$$

Classical results:

$$f(a_1, a_2) = a_1 a_2, \quad g(a_1, a_2) = a_1 a_2 - a_1 - a_2$$

$$g(A) = f(A) - \sum A.$$ 

Active area of research, hundreds of papers, special session at JM
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$$f(a_1, a_2) = a_1 a_2, \quad g(a_1, a_2) = a_1 a_2 - a_1 - a_2$$
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Vector Generalization

Take $A$ to be a set of $d$-vectors.
First problem: Find “right” definitions.

$$g(A) = \inf\{ x | \text{if } y > x \text{ then } y \in \mathbb{N}_0[A] \}$$
$$f(A) = \inf\{ x | \text{if } y > x \text{ then } y \in \mathbb{N}[A] \}$$

Scalars: New definitions coincide with old, are never in semigroup ($\mathbb{N}_0[A]$ or $\mathbb{N}[A]$), unique.
Vectors: Sometimes in semigroup, need not be unique.

Second problem: Choose a (partial) vector order.
Hidden third problem...
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Ordering $\mathbb{Z}^d$

$$A = \{(0, 4), (1, 1), (3, 0)\}$$

These form a cone with the origin. (first quadrant). This cone can be translated, such as to (8,7).

$a \leq b$ if $b$ is in the cone at $a$.

Unique $g$-vector at (8,7).
A second example

\[ A = \{(0, 4), (1, 1), (3, 0)\} \]

\[ A = \{(0, 4), (2, 1), (3, 0)\} \]
More typical cones

\[ A = \{(1, 4), (1, 1), (3, 1)\} \]

\[ A = \{(1, 4), (2, 2), (3, 1)\} \]
One with $|A| = 4$

\[ A = \{(0,5), (1,2), (2,1), (4,0)\} \]

Three $g$-vectors:
\( (7,13), (9,9), (11,5) \)

Missing from semigroup:
\( (9,13), (11,9) \), as indicated, and infinitely many points on \( x = 7, y = 5 \)
It gets worse

\[ A = \{(1, 6), (2, 2), (1, 3), (5, 1)\} \text{ has 11 lattice } g\text{-vectors.} \]

Problem since we insist \( g\)-vectors have \( \mathbb{Z} \) coordinates.
It gets better

2 rational $g$-vectors: $\left(\frac{190}{29}, \frac{647}{29}\right)$, $\left(\frac{248}{29}, \frac{415}{29}\right)$

Problem eliminated. 2 rational $g$-vectors.
Why 29?

\[ A = \{(5, 1), (1, 6), (2, 2), (1, 3)\} \]

Bounding vectors: \((5, 1), (1, 6)\)

\[
\left|\begin{array}{cc}
5 & 1 \\
1 & 6 \\
\end{array}\right| = 29
\]

In \(d\) dimensions, a cone is simple if \(d\) vectors determine the cone. For \(d = 1, 2\), all cones are simple. Henceforth, we assume our cone is simple.

We reorder \(A\) so that its first \(d\) vectors determine the cone.

**Thm[2005]:** Let \(a_1, a_2, \ldots, a_d\) determine the cone. All rational \(d\)-vectors can be written with \(|a_1 a_2 \cdots a_d|\) in the denominator.
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Bounding vectors: (5,1),(1,6)

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Existence of $g$-vectors

Novikov 92/94, Halter-Koch 93, made progress

Schur’s Thm: $f, g$ exist if and only if $\gcd(A) = 1$

Thm[2005]: Take vectors of $A$, $d$ at a time, form a square matrix, and take the absolute value of its determinant. Set $\gcd(A)$ to be the gcd of these $|A|_d$ values. Then $f, g$ exist if and only if $\gcd(A) = 1$.

Thm[2005]: $g(A) = f(A) - \sum A$ (easy)
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The simplest case

Suppose $|A| = d + 1$.

The scalar case: $f(a_1, a_2) = a_1 a_2$

Thm: $f(a_1, \ldots, a_d, a_{d+1}) = |a_1 a_2 \ldots a_d|a_{d+1}$

In particular, a unique $f$-vector, with integer coordinates.

[Simpson and Tijdeman 2003, rediscovered in 2005]
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Custom $g$-sets

**Thm[2006]:** Given $\{a_1, \ldots, a_d\}$, and $m > 0$. Then there is $A \supseteq \{a_1, \ldots, a_d\}$ with $|A| = d + m$ and

$$|g(A)| = \binom{|a_1 \cdots a_d| + m - 2}{m - 1}$$

**Thm[2006]:** For $m = 2$, $|g(A)| \leq |a_1 \cdots a_d|$.

**Thm[2005]:** Given an $x \in \mathbb{N}^d$, there is an $A$ with $|A| = d + 1$ and $g(A) = \{x\}$, if and only if at least one coordinate of $x$ is odd.
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An almost-common factor

In the following, assume all $f$ exist.

**Thm[Johnson 60]:** $f(a_1, na_2, \ldots, na_k) = nf(a_1, \ldots, a_k)$

**Thm[2005]:** $f(a_1, \ldots, a_d, na_{d+1}, \ldots, na_k) = nf(a_1, \ldots, a_k)$

Note: $n$ scalar, $a_i$'s vectors

**Thm[2006]:** $f(Na_1, \ldots, \frac{1}{|N|} Na_i, \ldots, Na_k) = Nf(a_1, \ldots, a_k)$

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An upper bound

Thm[Schur 35]:
\[ f(a_1, \ldots, a_k) \leq (a_1 - 1) \max\{a_2, \ldots, a_k\} + a_2 + \ldots + a_k \]

Thm[2005]:
\[ f(a_1, \ldots, a_k) \leq (|a_1 \cdots a_d| - 1) \text{LUB}(a_{d+1}, \ldots, a_k) + a_{d+1} + \ldots + a_k. \]

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