The Probability that Two Semigroup Elements Commute Can Be Anything

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http://www-rohan.sdsu.edu/~vadim/commute.pdf
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Introduction

A finite semigroup $S$ has an associative, closed, binary operation. Inverses, identity, commutativity are NOT assumed.

The commuting probability of $S$ is the probability that $x \ast y = y \ast x$ if $x, y$ are chosen uniformly at random. Which probabilities in $(0, 1]$ are possible?

Previously: the set of possible values are dense in $(0, 1]$. Now: the set of possible values are (all rationals in) $(0, 1]$. 
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First Semigroup Family

Set $x \star y = f(x)$, where $f : S \to S$ is:

\[ |S| = a + b + c + 2k \]
$x \star y = f(x)$

- $f$ is idempotent, hence this is associative
- Commuting probability is $\frac{a^2+b^2+c^2+4k}{(a+b+c+2k)^2}$.
- Using Lagrange’s four-square theorem, can achieve every rational in $(0, \frac{1}{3}]$. 
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Second Semigroup Family

Rank function $r : S \rightarrow \mathbb{N}$ is given below.

If $r(x) > r(y)$ then $x \star y = y \star x = x$.

If $r(x) = r(y)$ then $x \star y = x$.

$$|S| = a + b + c + 2k$$
\[x \star y = x \text{ or } y\]

- \(x \star y \star z\) is of those in \(\{x, y, z\}\) with highest rank, the one that is first in the expression. Associative.

- Commuting probability is
  
  \[1 - \frac{a^2 + b^2 + c^2 + 4k - (a + b + c + 2k)}{(a + b + c + 2k)^2}.
  
- Using Lagrange’s four-square theorem, can achieve every rational in \((2/3, 1]\).
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First Semigroup Alteration

Given semigroup $S$, we make two copies $S_1, S_2$.  

$$x_1 \star y_2 = y_2 \star x_1 = x_1; \quad x_i \star y_i = (x \star y)_i.$$

This is associative (several cases).

If $S$ had commuting probability $\frac{m}{n}$, then this alteration has commuting probability $(\frac{m}{n} + 1)/2$.

Applying it to the first construction gives all rationals in $(\frac{1}{2}, \frac{2}{3}]$. 

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Associative.

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Conclusions

\[(0, \frac{1}{3}] \cup (\frac{2}{3}, 1] \cup (\frac{1}{2}, \frac{2}{3}] \cup (\frac{1}{3}, \frac{1}{2}] = (0, 1].\]

Open: Find a single family.

Open: Use just two alterations with some trivial starter groups.

See http://www-rohan.sdsu.edu/~vadim for paper.
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