Graph Realization for Atomic Simplexes

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http://vadim.sdsu.edu/2018-JMM-talk.pdf

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Summer 2018 projects:
HIV modeling, machine learning regression

`http://www.sci.sdsu.edu/math-reu/index.html`

This work was done jointly with Jackson Autry, Paige Graves, Jessie Loucks, Christopher O'Neill, and Samuel Yih.

Numerical Semigroups: definition

A numerical semigroup is a set S, satisfying:

- $\{0\} \subseteq S \subseteq \mathbb{N}_0$
- S is closed under $+$
- $\mathbb{N}_0 \setminus S$ is finite

Given $\{t_1, t_2, \ldots, t_k\} \subseteq S$, we define subsemigroup $\langle t_1, t_2, \ldots, t_k \rangle = \{\sum_{i=1}^{k} \alpha_i t_i : \alpha_i \in \mathbb{N}_0\} \subseteq S$

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Numerical Semigroups: irreducibles

A nonzero element $t \in S$ is irreducible if:

there are no nonzero $t_1, t_2 \in S$ with $t = t_1 + t_2$

There is a unique set of irreducibles $\{n_1, n_2, \ldots, n_d\}$ with

$S = \langle n_1, n_2, \ldots, n_d \rangle$.

We call d the embedding dimension of S.

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Numerical Semigroups: Frobenius Numbers

The Frobenius number of S, denoted $F(S)$, is the largest element of $\mathbb{N}_0 \setminus S$.

Finding $F(S)$ is a major open problem, for over 120 years.

- $d = 2$: $F(S) = n_1 n_2 - n_1 - n_2$
- $d = 3$: messy formula
- otherwise: special cases only

We call $x \in \mathbb{N}_0 \setminus S$, a pseudo-Frobenius number if $x + s \in S$ for all $s \in S \setminus \{0\}$. We denote the set of all such $PF(S)$, and note that $F(S) \in PF(S)$.

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Atomic Complexes: Definition

S numerical semigroup, with irreducibles $\{n_1, n_2, \ldots, n_d\}$.
Let Σ be the simplex with vertex set $\{1, 2, \ldots, d\}$.

For a face F of Σ, we set $n_F = \sum_{i \in F} n_i$.
Given $m \in S$, we define the atomic complex of m to be the simplicial complex

$$\Delta_m = \{F \in \Sigma : n_F \text{ divides } m\} = \{F \in \Sigma : m - n_F \in S\}.$$

Also known as: squarefree divisor complex

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Atomic Complexes: Examples

$$\langle 5, 7\rangle = \{0, 5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, \overset{F(S)}{23}, 24+\}$$

$\Delta_0 = \{\emptyset\}, \Delta_{15} = \{5\}, \Delta_7 = \{7\}, \Delta_{35} = \{5, 7\}, \Delta_{17} = \{57\}$
(only facets indicated)

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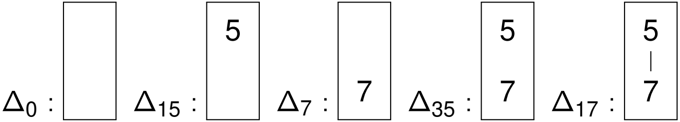
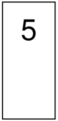
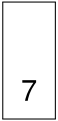


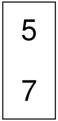
Atomic Complexes: Euler characteristic

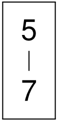
We define Euler characteristic as $\chi(\Delta) = \sum_{F \in \Delta} (-1)^{|F|}$.

Back to $\langle 5, 7 \rangle$: $\Delta_0 = \{\emptyset\}$, $\chi(\Delta_0) = 1$

$\Delta_{15} = \{5\}$, $\chi(\Delta_{15}) = 0$ 

$\Delta_7 = \{7\}$, $\chi(\Delta_7) = 0$ 

$\Delta_{35} = \{5, 7\}$, $\chi(\Delta_{35}) = -1$ 

$\Delta_{17} = \{57\}$, $\chi(\Delta_{17}) = 0$ 

The most interesting m have $\chi(\Delta_m) \neq 0$. "shaded"
Related to: Betti elements (disconnected).

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$\Delta_{35} = \{5, 7\}$, $\chi(\Delta_{35}) = -1$ $\boxed{\begin{array}{c} 5 \\ \\ 7 \end{array}}$

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Atomic Complexes: nonzero Euler characteristic

$\chi(\Delta_0) = 1$

If Δ_m is a nontrivial simplex, then $\chi(\Delta_m) = 0$

Take $x \in PF(S)$. $x \notin S$, but $x + s \in S$ for all $s \in S \setminus \{0\}$.
Consider $m = x + n_1 + n_2 + \cdots + n_d$.
Then Δ_m is Σ, without its maximal face.
Hence $\chi(\Delta_m) = -(-1)^d = (-1)^{d+1}$.

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What are the questions?

Which atomic complexes can arise? HARD

Which Euler characteristics can arise?

Which atomic complexes can arise, that are graphs?

What shaded elements arise, from important numerical semigroups?

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Gluing and Disjoint Gluing

Let $S = \langle n_1, n_2, \dots, n_d \rangle$ and $S' = \langle n'_1, n'_2, \dots, n'_{d'} \rangle$.

Let $k \in S \setminus \{n_1, n_2, \dots, n_d\}$ and $k' \in S' \setminus \{n'_1, n'_2, \dots, n'_{d'}\}$, with $\gcd(k, k') = 1$.

Set $k'S + kS' = \langle k'n_1, k'n_2, \dots, k'n_d, kn'_1, kn'_2, \dots, kn'_{d'} \rangle$.

Thm[Rosales, Garcia-Sanchez] Dimension $d + d'$.

Thm[Bruns, Herzog][REU] $\underbrace{\Delta_{kk'}}_{\text{in } k'S+kS'} = \underbrace{\Delta_k}_{\text{in } S} \dot\cup \underbrace{\Delta_{k'}}_{\text{in } S'}$

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Disjoint Gluing Example

Let $S = \langle 5, 7 \rangle$ and $S' = \langle 3, 5 \rangle$.

Let $k = 17 \in S$. $\Delta_k = \{57\}$. $\boxed{5 - 7}$

Let $k' = 15 \in S'$. $\Delta_{k'} = \{3, 5\}$. $\boxed{3 \quad 5}$

$k'S + kS' = \langle 15 \cdot 5, 15 \cdot 7, 17 \cdot 3, 17 \cdot 5 \rangle = \langle 75, 105, 51, 85 \rangle$.

$$\boxed{\begin{array}{cc} 75 - 105 \\ \\ 51 \quad 85 \end{array}}$$

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Take $F \in \Sigma$, $F' \in \Sigma'$.

$k - n_F \in S$ if and only if $k'(k - n_F) \in k'S$

If F, F' nonempty, $kk' - k'n_F - kn_{F'} \in k'S + kS'$.

$kk' = k't + kt'$, where $t \in S \setminus \{0\}$, $t' \in S' \setminus \{0\}$

$0 \equiv k't \pmod{k}$, so $k|t$, so $t \geq k$. Contradiction.

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Disjoint Gluing Consequences

Recall $\underbrace{\Delta_{kk'}}_{\text{in } k'S+kS'} = \underbrace{\Delta_k}_{\text{in } S} \dot\cup \underbrace{\Delta_{k'}}_{\text{in } S'}$

We calculate $\chi(\Delta_{kk'}) = \chi(\Delta_k) + \chi(\Delta_{k'}) - 1$.

Hence, if $\chi(\Delta_k) = 0$, and we repeatedly glue S to "itself", we get all negative Euler characteristics. Easy: $\langle 1 \rangle$

If $\chi(\Delta_k) \geq 2$, and we try same game, we would get all positive Euler characteristics. Hard: $\geq K_4$, gcd, wait

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Gluing Pendants

Set $S = \langle n_1, n_2, \ldots, n_d \rangle$. Let $m \in S$, consider Δ_{m+n_1}.

Let p be prime with $p \nmid m$. Set $T = pS + m\langle 1 \rangle$.

Then $\underbrace{\Delta_{p(m+n_1)}}_{\text{in } T} = \underbrace{\Delta_{(m+n_1)}}_{\text{in } S} \cup \underbrace{\{p \cdot n_1, m \cdot 1\}}_{\text{in } \langle 1 \rangle}$ pendant edge

Step 1: $(m + n_1 - n_F) \in S$, iff $p(m + n_1 - n_F) \in pS$.

Step 2: $p(m + n_1) - pn_1 - m = (p - 1)m \in T$.

Step 3: Suppose $p(m + n_1) = pn_F + mt$ with $t > 0$.

$p|t$, so $p(m + n_1) = pn_F + mp + mpt'$, for some $t' \in \mathbb{N}_0$.

$n_1 = n_F + mt'$. But $m \in S$, so $mt' \in S$. But n_1 irreducible!

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Supersymmetric Numerical Semigroups

Take $t_1, t_2, \ldots, t_d \in \mathbb{N} \setminus \{1\}$, pairwise coprime.
Set $T = t_1 t_2 \cdots t_d$.
Consider $S = \langle \frac{T}{t_1}, \frac{T}{t_2}, \ldots, \frac{T}{T_d} \rangle$. Called "supersymmetric".

Choose $k \in [1, d]$, and consider Δ_{kT}.
Because $kT = t_1(\frac{T}{t_1}) + t_2(\frac{T}{t_2}) + \cdots + t_k(\frac{T}{t_k})$,
face $\{12 \cdots k\}$ is in Δ_{kT}. Also any face of this dim.
Suppose $kT = a_1(\frac{T}{t_1}) + a_2(\frac{T}{t_2}) + \cdots + a_d(\frac{T}{t_d})$.
t_i divides each term, so $t_i | a_i$.
If $k+1$ of the $a_i > 0$, then RHS $\geq (k+1)T$, contradiction.
Hence Δ_{kT} is uniform simplicial complex.

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Supersymmetric with $k = 2$

Take $t_1, t_2, \ldots, t_d \in \mathbb{N} \setminus \{1\}$, pairwise coprime.
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Taking $k = 2$, we see that Δ_{2T} is uniform with facet size 2 (dimension 1).

Observation 1: This is complete graph K_d.

Observation 2: $\chi(\Delta_{2T}) = 1 - d + \binom{d}{2} = \frac{(d-1)(d-2)}{2}$

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Graph Realization Problem

1: Using disjoint gluing, can focus WLOG on connected components.

2: Using pendant gluing, can focus WLOG on graphs without pendant edges. (all trees/forests realizable)

3: Using supersymmetric numerical semigroups, can realize K_n.

4: Can also realize $K_{m,n}$, C_n, $K_{2,n}$ with extra edge.

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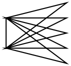
4: Can also realize $K_{m,n}$, C_n, $K_{2,n}$ with extra edge. 

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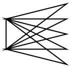
4: Can also realize $K_{m,n}$, C_n, $K_{2,n}$ with extra edge. 

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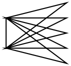
4: Can also realize $K_{m,n}$, C_n, $K_{2,n}$ with extra edge. 

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Four nonzero Euler characteristics:

Δ_0	$\Delta_{k'n_1n_2}$	$\Delta_{k'k}$	$\Delta_{k'n_1n_2+k'k}$
$\chi = 1$	$\chi = -1$	$\chi = -1$	$\chi = 1$

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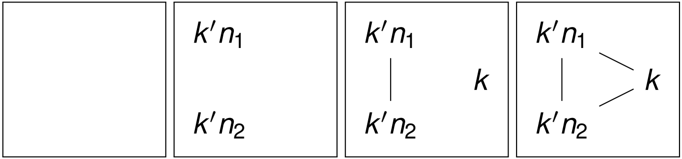

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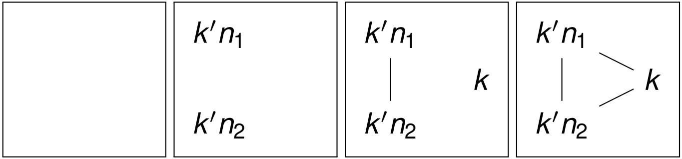

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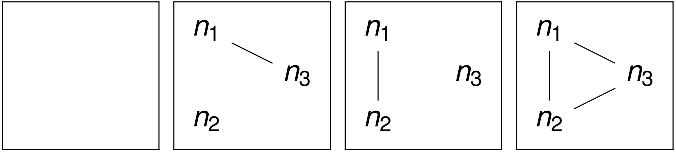

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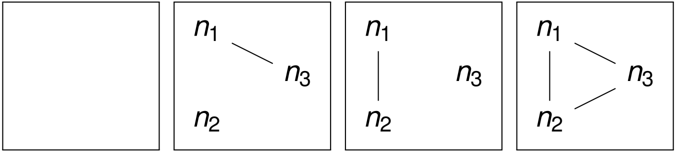

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For Further Reading

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  J. Pure Appl. Algebra 122 (3) 1997, pp. 185-208.
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