MATH 623 Chapter 0 Exercises

1. Suppose that S is a vector space over field $\mathbb{F}$, and S_1, S_2 are both subspaces of S. Prove that $S_1 + S_2$ is a subspace of S.
2. Suppose that S is a vector space over field $\mathbb{F}$, and S_1, S_2 are both subspaces of S. Prove that $S_1 \cap S_2$ is a subspace of S.
3. Consider the vector space $\mathbb{R}^2$. Find two subspaces S_1, S_2 such that $S_1 \cup S_2$ is *not* a subspace.
4. Prove that every nonempty sublist of an independent list of vectors is again independent.
5. Prove that every superlist of a dependent list of vectors is again dependent.
6. For matrix $A \in M_{m,n}(\mathbb{F})$, prove that the rowspace and nullspace are both subspaces of $\mathbb{F}^n$.
7. Find an infinite-dimensional vector space V, with two proper nontrivial subpaces V_1, V_2 such that V_1 is finite-dimensional and V_2 is infinite-dimensional.
8. Set $P_2(t)$ to be the set of all polynomials of degree at most 2, in variable t, with real coefficients. Prove that $P_2(t)$ is isomorphic to $\mathbb{R}^3$.
9. Let $P_2(t)$ be as in (8). Prove that $T : P_2(t) \to P_2(t)$ given by $T(f(t)) = t\frac{df(t)}{dt}$ is a linear transformation.
10. Let T be as in (9). Find its rank and nullity.
11. For matrices A, B where AB is defined, prove that $(AB)^T = B^T A^T$ and $(AB)^* = B^* A^*$.
12. For complex-valued matrix $A = [a_{ij}]$, prove that $A^* = A^T$ if and only if $a_{ij} \in \mathbb{R}$ for all i, j.
13. For complex-valued matrix $A = [a_{ij}]$, prove that $A + A^T$ is symmetric, $A + \overline{A}$ is real, and $A + A^*$ is Hermitian.
14. Calculate the determinant and permanent of $\left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right)$.
15. Calculate the inverse of $\left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right)$.
16. Prove that the inverse of an elementary matrix is elementary.
17. For $A, B \in M_n(\mathbb{F})$, prove that AB is invertible if and only if both A, B are invertible.
18. Suppose that square matrix A has RREF of I. Prove that A may be written as the product of elementary matrices.
19. If $A \in M_{m,n}(\mathbb{F})$, prove that $\text{rank}A \leq \min(m, n)$.
20. If $A \in M_{m,n}(\mathbb{F})$, and $B \in M_{n,n}(\mathbb{F})$, prove that $\text{rank}A \geq \text{rank } AB$.
21. If $A \in M_{m,n}(\mathbb{F})$, and $B \in M_{n,n}(\mathbb{F})$ is nonsingular, prove that $\text{rank}A = \text{rank } AB$.
22. If $A \in M_{m,n}(\mathbb{C})$, prove that $\text{rank}A = \text{rank } A^*A$.
23. Prove that if square matrix A has a left inverse, then it also has a right inverse, and they are the same.
24. Let $V = \mathbb{C}^2$. Define $\langle x, y\rangle = y^* \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right) x$. Prove that this defines an inner product on V.
25. With $V, \langle\cdot,\cdot\rangle$ as in (24), calculate the angle between $x = (-1, 2)$ and $y = (1, 1)$.
26. With $V, \langle\cdot,\cdot\rangle$ as in (24), use Gram-Schmidt starting with $\{e_1, e_2\}$ to find an orthonormal basis for V.
27. Suppose S is a subspace of $\mathbb{C}^n$. Prove that $(S^\perp)^\perp = S$.
28. Suppose S_1, S_2 are subspaces of $\mathbb{C}^n$. Prove that $(S_1 + S_2)^\perp = S_1^\perp \cap S_2^\perp$.
29. Calculate $C_2(A)$ for $A = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right)$.
30. Calculate $C_2(A^2)$ for $A = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right)$, and verify that $C_2(A^2) = C_2(A)^2$.
31. Calculate the adjugate of $A = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right)$.