MATH601 Spring 2008 Handout 7
Unit 4: Cardinals

A cardinal is a number that measures the size of a set. For set S, we use $|S|$ to denote the cardinal that measures the size of S. Important comment: Our number sense breaks down with infinite cardinals. Beware! For example, given two sets S, T that have sizes $|S|, |T|$ respectively, we are not guaranteed that either of $|S|, |T|$ is larger than the other.

We say that $|S| \leq |T|$ if there is an injective (one-to-one) function from S into T. We say $|S| = |T|$ if there is a bijective (one-to-one and onto) function from S into T. In this latter case we say that S, T are *equicardinal*.

Cantor-Schröder-Bernstein Theorem [ca. 1900]
Let S, T be any two sets. If $|S| \leq |T|$ and $|T| \leq |S|$, then $|T| = |S|$.

Set $S = \mathbb{N}_0$, the set of nonnegative integers; $T = \mathbb{Z}$, the set of integers. We prove that S, T are equicardinal.

CSB method: Set $f : S \to T$ via $f(x) = x$; this is injective, so $|S| \leq |T|$. Set $g(x) = \begin{cases} x^2 & x \geq 0 \\ x^2 + 1 & x < 0 \end{cases}$, for the reverse $g : T \to S$. To prove g is injective, suppose that $g(x) = g(x')$. If x, x' are both positive, then $x^2 = (x')^2$, hence we take square roots and use that both are positive to find $x = x'$. If x, x' are both negative, then $x^2 + 1 = (x')^2 + 1$, and again we find $x = x'$. Finally, if x, x' are of different signs, then there are two perfect squares (namely $f(x), f(x')$) whose difference is one. This is only possible for $0, 1$, but then $x = x' = 0$. Hence g is injective, and $|T| \leq |S|$. By the CSB theorem, $|S| = |T|$. Note that neither of f, g are bijective.

Direct method: Set $f : T \to S$ via $f(x) = \begin{cases} 2x & x \geq 0 \\ -1 - 2x & x < 0 \end{cases}$. To prove that f is injective, suppose that $f(x) = f(x')$. If x, x' are both positive, then $2x = 2x'$, and hence $x = x'$. If x, x' are both negative, then $-1 - 2x = -1 - 2x'$, and again $x = x'$. However, x, x' cannot be of opposite signs since then one of $f(x), f(x')$ would be odd and the other would be even. To prove that f is surjective, let $y \in S$. If y is even, then set $x = y/2 \in T$; we have $f(x) = y$. If y is odd, then set $x = \frac{-y-1}{2}$. Note that $x < 0$ so $f(x) = -1 - 2(\frac{-y-1}{2}) = y$. Hence f is a bijection and $|S| = |T|$.

Exercises:

1. Prove that $\{1, 2, 3\}$ is equicardinal with $\{2, 3, 4\}$.
2. Prove that $2\mathbb{Z}$ (the set of even integers) is equicardinal with $\mathbb{Z}$.
3. Set $S = \mathbb{Z} - 2\mathbb{Z}$, the set of odd integers. Prove that S is equicardinal with $\mathbb{Z}$.
4. Prove that $\mathbb{N}$ (the set of positive integers) is equicardinal with $\mathbb{Z}$.
5. Suppose that R is equicardinal with S, and S is equicardinal with T. Prove that R is equicardinal with T.
6. Prove that the real interval $(0, 1)$ is equicardinal with the real interval (a, b) for any $a < b$.
7. Prove that the real interval $(0, 1)$ is equicardinal with $\mathbb{R}$, the set of all reals.
   Hint: $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$.
8. Prove that $\mathbb{Q}^+$ (the set of positive fractions) is equicardinal with $\mathbb{N}$.
   Hint: arrange $\mathbb{Q}^+$ in a diagram, then take a path through this diagram.