**MATH601 Spring 2008 Handout 2: Multiplication**
Unit 1: Natural Numbers

Recall your techniques for doubling and halving, from the exercises of the previous handout. A good method for halving, if you didn't find one, is as follows: Go from left to right, hand by hand. For Z,B,D, write down Z,A,B respectively. For A,C,E, again write down Z,A,B, but 'carry' a C to the right, and add this to the result of halving the next hand. Since that result must be Z,A,B, when adding C it becomes C,D,E. If the number ends in A,C,E, then the number is odd and there is a remainder when we halve. A good method for doubling, if you didn't find one is as follows: Work from right to left. Z,A,B become Z,B,D respectively. C,D,E become Z,B,D respectively as well, with a carry to the left of A. This A gets added to the result of the next doubling.

This is how we multiply two numbers. Write them side-by-side on a line. On the next line, halve the one on the left (ignoring the remainder, if any), and double the one on the right. Repeat this process again and again, until the number on the left is just A. Mark each row where the left number is odd (ending in A,C,E). Add the numbers on the right, just for the circled rows. This is the result. Here are some examples:

| | | |
|---|---|---|
| D | AAA | |
| B | BBB | |
| A | DDD | ← |
| | DDD | |

| | | |
|---|---|---|
| E | ABC | ← |
| B | BEZ | |
| A | EDZ | ← |
| ABC+EDZ= | AAZC | |

| | | |
|---|---|---|
| BC | BC | ← |
| AA | EZ | ← |
| C | ADZ | ← |
| A | CBZ | ← |
| BC+EZ+ADZ+CBZ= | AZAC | |

Conclusions: D×AAA=DDD, E×ABC=AAZC, BC×BC=AZAC

This method is called 'Russian Peasant Multiplication', and is frequently used by computers. The reason is that halving and doubling are very fast in binary, as they involve moving all the bits to the right or left, respectively. It is also similar to multiplication done in ancient Egypt. For more details see:
`cut-the-knot.org/Curriculum/Algebra/PeasantMultiplication.shtml`
`mathforum.org/dr.math/faq/faq.peasant.html`
`www-groups.dcs.st-andrews.ac.uk/~history/HistTopics/Egyptian_papyri.html`

Exercises to complete for next class:

A. Calculate ED×AA, AA×ED, BAD×BEE. Express the completed problems as statements with spelled-out numbers.

B. We know that multiplication is commutative: $x \times y = y \times x$. We therefore have a choice when we multiply using this method. Which way should we choose, to try to minimize our effort?

C. Prove that this method gives the correct answer, for A×$n$, B×$n$, C×$n$, D×$n$, E×$n$, AZ×$n$ (for every number $n$). Do these six special cases directly; do not prove that this method works in general.

D. What is the maximum number of lines required, if the first number is one hand? two hands? three hands? $n$ hands? (hint: think about powers of two)

E. Choose five largish (four to six hands) numbers, 'at random'. For each of these numbers, write down the last (rightmost) hand. Also, write down the last two hands, and the last three hands. Calculate the sum of the hands; if the result has more than one hand, calculate the sum of those hands, and so on until a single hand is left. Write this down. (e.g. DAB → D+A+B=AA → A+A=B). Calculate the alternating sum and difference of the hands from the right; if the result has more than one hand, calculate the alternating sum and difference of the hands, and so on until a single hand is left. Write this down. (e.g. DAC → C-A+D=AZ → Z-A = -A).
Now, for each of the thirty numbers you have, factor them completely into primes (you might want to temporarily cheat and use decimal, and a calculator). See table on back of handout. We will use this experimental data next class period.

| | Largish Number | Last Digit | Last Two | Last Three | Sum | +/- |
|---|---|---|---|---|---|---|
| Number: | DEZABC | C | BC | ABC | E | C |
| Factors: | C×E×BA×EBA | C | C×E | C×BE | E | C |
| Number: | | | | | | |
| Factors: | | | | | | |
| Number: | | | | | | |
| Factors: | | | | | | |
| Number: | | | | | | |
| Factors: | | | | | | |
| Number: | | | | | | |
| Factors: | | | | | | |
| Number: | | | | | | |
| Factors: | | | | | | |

Notes:
A. One sample is completed; you need to do five more like that.
B. DEZABC is 37635=3×5×13×193 (in decimal).
C. D+E+Z+A+B+C=BC, B+C=E.
D. C-B+A-Z+E-D=C
E. One-hand numbers are very easy to factor; the only one that factors is D=B×B.