MATH601 Spring 2008
Handout 17: Impartial Games (Nim)
Unit 7: Games

We define "Impartial" games. They are very similar to Partisan games; the only difference is that now L's and R's moves are indistinguishable. The most well-known impartial game is Nim[1]. Nim is played with several piles of objects. Players alternate removing (and discarding) as many objects as they like from any one pile. A position will be written as $(3, 3, 5)$ indicating two piles of size 3 and one pile of size 5. Since the order of piles doesn't matter, we may as well write them in nondecreasing order.

We again compute the value of each position using $\prec L|R \succ$ notation. Because the games are impartial, we will always have $L = R$ so they will never be surreal numbers[2]. We call the value of a single pile of n objects [position (n)] a *nimber*, and denote it by $\star n$. Note that $\star n = \prec \star 0, \star 1, \ldots, \star(n-1) | \star 0, \star 1, \ldots, \star(n-1) \succ$, so they are defined recursively. Note also that $\star 0 = 0$, and this is a loss for the player about to move; on the other hand $\star n$ is a win for the player about to move, for any $n > 0$.

Remarkably, these nimbers are enough to value not only all other positions, but all impartial games[3]. A first observation is that $\star n + \star n = \star 0$, for every n. Whatever the first player does to one of the two piles of size n, the second player can copy with the other pile. Eventually both piles will be gone, and the first player will lose. The second key principle is that the nim-sum of several *different* powers of two will be their ordinary sum. The proof is tricky and we omit it. Examples: $\star 2 + \star 4 = \star 6$; $\star 1 + \star 8 + \star 16 = \star 25$. Other helpful properties: addition is commutative and associative, and $\star n + \star 0 = \star n$ for every n.

Hence, to add nimbers: (1) express every summand as a sum of different powers of two, (2) cancel repeats in pairs, then (3) recombine the powers of two into a new nimber. Examples: $\star 5 + \star 10 = (\star 1 + \star 4) + (\star 2 + \star 8) = \star 15$ since there were no repeats; $\star 5 + \star 12 = (\star 1 + \star 4) + (\star 4 + \star 8) = \star 1 + \star 8 = \star 9$; $\star 5 + \star 6 + \star 7 = (\star 1 + \star 4) + (\star 2 + \star 4) + (\star 1 + \star 2 + \star 4) = \star 4$, where the $\star 1$'s and $\star 2$'s cancelled, as well as two of the $\star 4$'s. $\star 3 + \star 5 + \star 6 = (\star 1 + \star 2) + (\star 1 + \star 4) + (\star 2 + \star 4) = \star 0$, since everything cancels.

Any position whose value is $\star 0$ is a loss for the player about to move; any other value is a win for the player about to move (given correct play). If it is your move and the value is nonzero, you must choose a move that leaves value $\star 0$ for your opponent, which will always be possible (this is proved in the exercises). To find such a move, nim-add the value of the game to each pile. If the result is smaller, then reducing the pile to this size is a winning move. For example $(1, 3, 4)$ has value $\star 1 + \star 3 + \star 4 = \star 6$. $\star 6 + \star 1 = \star 7$; $\star 6 + \star 3 = \star 5$; $\star 6 + \star 4 = \star 2$. Hence, the (only) winning move is to take 2 away from the biggest pile, yielding $(1, 2, 3)$.

Exercises:

1. Make an addition table for $\star 0$ through $\star 15$. (note: the table will be symmetric since $+$ is commutative)
2. Suppose that $\star a + \star b = \star c$. Prove that $\star a + \star b + \star c = \star 0$, $\star a = \star b + \star c$, and that $\star a + \star c = \star b$.
3. Prove that $\star a = \star b$ if and only if $\star a + \star b = \star 0$. (one direction is done already, the 'first observation')
4. Find all winning moves for the positions $(3, 4, 5)$, $(5, 6, 7)$, $(2, 3, 5, 7, 11)$, $(2, 3, 5, 7, 11, 13)$.
5. Suppose that a position has value $\star m$. On our move, we reduce one of the piles from size a to size b. Prove that the resulting position has value $\star m + \star a + \star b$.
6. Prove that every move from a position with value $\star 0$ will yield a position with nonzero value.
7. Prove that the moves suggested in the last paragraph above will yield a position with value $\star 0$, and no other moves will. Hence this is a complete winning strategy.
8. Prove that at least one of the moves suggested will always be possible.
   HINT: Write the game value as a sum of powers of 2, and consider the largest summand.

---

[1] This game is equivalent to Tri-Hackenbush, where all edges are unlabeled, and each bush is just a collection of stalks of various lengths.

[2] unless $L = R = \emptyset$ – the empty position is of value 0 as before

[3] By the Sprague-Grundy theorem [ca. 1935] all impartial games can be valued with nimbers.