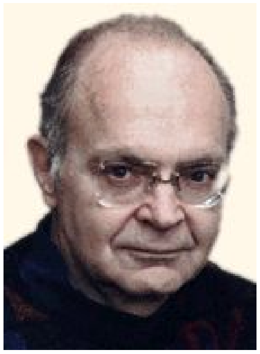
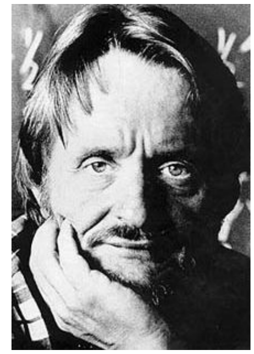
MATH601 Spring 2008 Handout 13: Introduction
Unit 6: Surreal Numbers



Donald Knuth 1938-
Stanford
Renowned computer scientist, TEX
Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness

John Horton Conway 1937-
Princeton
Renowned mathematician, Game of Life, Doomsday



Surreal numbers, $\mathbb{S}$, first described in a work of fiction, were discovered by John Conway about 35 years ago. They nicely extend most of the ideas we've studied thus far in this course: they are built from nothing, contain the reals/hyperreals/ordinals, and are an ordered field (e.g., for any $x, y \in \mathbb{S}$, either $x \leq y$ or $y \leq x$ or both).

Let $L, R \subseteq \mathbb{S}$, that satisfy the important condition: $\forall a \in L, \forall b \in R, a < b$. We define the number $\prec L|R \succ \in \mathbb{S}$. We use notation $x = \prec L(x)|R(x)\succ$; note that $L(x), R(x)$ are sets. We also write x^L to denote the "typical" element (i.e. every element) of $L(x)$; e.g. $x^L \geq 3$ means every element of $L(x)$ is ≥ 3. x^R is used similarly for the typical element of $R(x)$. x is a surreal number if $x^L < x^R$. For notational convenience we will drop curly braces: $x = \prec\{1,2\}|\{3\}\succ = \prec 1,2|3\succ$ has $L(x) = \{1,2\}, R(x) = \{3\}$.

There are several differences from the method we used to build $\mathbb{R}$. L, R are themselves subsets of $\mathbb{S}$ – when we built $\mathbb{R}$, they were subsets of a different field $\mathbb{Q}$. Also, L, R can be small. In fact, if $a < b$, $\prec a, b|c\succ = \prec b|c\succ$ (this is proved in the exercises), so we can often discard parts of L, R and make them quite small indeed.

All elements of $\mathbb{S}$ have a birthday, which is an ordinal number. They are given names, in the following table. They are built only out of surreals that have been born previously (i.e. have a smaller birthday). Note: This table just gives names, these names alone do not justify the arithmetic you would expect (i.e. $1 + 1 = 2$), which we will do later. However, the intuition you have for these numbers is generally correct.

Birthday 0: $0 = \prec|\succ$
Birthday 1: $-1 = \prec|0\succ$, $1 = \prec 0|\succ$
Birthday 2: $-2 = \prec|-1\succ$, $-1/2 = \prec -1|0\succ$, $1/2 = \prec 0|1\succ$, $2 = \prec 1|\succ$
Birthday 3: $-3 = \prec|-2\succ$, $-3/2 = \prec -2|-1\succ$, $3/4 = \prec 1/2|1\succ$, and 5 more
Birthday ω: $\omega = \prec 1,2,3,\ldots|\succ$, $1/\omega = \prec 0|1/2, 1/4, 1/8, \ldots\succ$, the rest of $\mathbb{R}$ (irrationals, $1/3$, etc.)
Birthday $\omega+1$: $\omega + 1 = \prec\omega|\succ$, $1/2\omega = \prec 0|1/\omega\succ$, $\omega - 1 = \prec 1,2,3,\ldots|\omega\succ$
Birthday $\omega+\omega$: $\omega/2 = \prec 1,2,3,\ldots|\omega, \omega-1, \omega-2, \ldots\succ$, $1/\omega^2 = \prec 0|1/\omega, 1/2\omega, 1/4\omega, \ldots\succ$

The largest surreal with birthday x is exactly the ordinal x. Those elements with finite birthdays are *dyadic rationals*; fractions whose denominator is a power of two. They can be expressed in binary arithmetic with a terminating expansion, e.g. 1.11 in binary means $1\,3/4$.

The above constructions give names for the "simplest" pairs of sets L, R. If L, R are more complicated, we can still determine the name $\prec L|R\succ$ with the following.

Seniority Principle: $x = \prec L|R\succ$ is the earliest-born surreal that satisfies $x^L < x < x^R$.

For example, $\prec 1/2|7\succ = 1$, $\prec|-3/2\succ = -2$, $\prec 5|\omega\succ = 6$.

Given $x, y \in \mathbb{S}$ we define $x \geq y$ and $x > y$ as follows:

	(i)		(ii)
$x \geq y$:	$\forall a \in R(x), a > y$	AND	$\forall b \in L(y), x > b$
$x \geq y$:	$x^R > y$	AND	$x > y^L$
$x > y$:	$\exists a \in R(y), x \geq a$	OR	$\exists b \in L(x), b \geq y$

$x \leq y$ means $y \geq x$, and $x < y$ means $y > x$. $x = y$ means $x \geq y$ and $y \geq x$.
$x \geq y$ has two equivalent formulations; $x > y$ doesn't. $x > y$ iff $y \not\geq x$; $x \geq y$ iff $y \not> x$.
Note: if $R(x)$ is empty, then (i) of $x \geq y$ is considered vacuously true (similarly for $L(y)$ and (ii)).

Check $1 \geq 0$: (i) $R(1)$ is empty, so $1^R > 0$ is vacuously true. (ii) $L(0)$ is empty, so $1 > 0^L$ vacuously.

Check $0 \geq 1$: (i) $R(0)$ is empty, so $0^R > 1$ vacuously. (ii) $L(1) = \{0\}$, which contains an element ≥ 0. Hence $0 \not\geq 1$. Together with the previous we have shown that $1 > 0$.

Check $1 \geq 1$: (i) $R(1)$ is empty, so $1^R > 1$ vacuously. (ii) $L(1) = \{0\}$, each element of which is < 1, since we proved above that $0 < 1$. Hence $1 \leq 1$, and therefore $1 = 1$. Nice to know!

Check $1 \geq 1/2$: (i) $R(1)$ is empty, so $1^R > 1/2$ vacuously. (ii) $L(1/2) = \{0\}$, each element of which is < 1, since we proved $0 < 1$. Hence $1 \geq 1/2$.

Check $1/2 \geq 1$: (i) $R(1/2) = \{1\}$. This contains an element, namely 1, with $1 \geq 1$. Hence $1/2 \not\geq 1$. With the previous we have shown that $1 > 1/2$. It is not necessary to check (ii), since BOTH (i) and (ii) are required for the inequality.

Check $3/4 \geq 1/2$: (i) $R(3/4) = \{1\}$, so we need to verify that $1 > 1/2$. We did this already. (ii) $L(1/2) = \{0\}$, so we need to verify that $3/4 > 0$.

> Check $3/4 > 0$: (i) $R(0)$ is empty, so this won't work. (ii) $L(3/4) = \{1/2\}$. We now need to determine if $1/2 \geq 0$. If so, we will have shown $3/4 > 0$, and hence $3/4 \geq 1/2$.
>
> > Check $1/2 \geq 0$: (i) $R(1/2) = \{1\}$. We have already shown $0 \not\geq 1$. (ii) $L(0)$ is empty, so $1/2 > 0^L$ vacuously. Hence $1/2 \geq 0$.

Exercises:

1. Find the remaining elements of $\mathbb{S}$ with birthday 3.
2. For each of the following names, find their birthdays and L, R sets: $35,\ 3/16,\ 2+1/\omega,\ \sqrt{2}+1/\omega,\ 1/3,\ \omega+1/2$
3. Apply the seniority principle to find the names for: $\prec 1/\omega | 1/2 \succ$, $\prec 1/\omega | \omega \succ$, $\prec -\omega | 1/\omega \succ$, $\prec 1/3 | 1/2 \succ$, $\prec 1/7 | 1/5 \succ$, $\prec e | \pi \succ$, $\prec e | \succ$, $\prec | e \succ$
4. Check if $1 \geq 2,\ 2 \geq 1,\ 3/4 \geq 1,\ 3/4 \geq 1/4,\ 1/2 \geq -1/2,\ 5 \geq \omega,\ \omega \geq 5,\ 1/\omega \geq 0,\ \omega - 1 \geq \omega/2$.
5. Check if $1 > 2,\ 2 > 1,\ 3/4 > 1,\ 3/4 > 1/4,\ 1/2 > -1/2,\ 5 > \omega,\ \omega > 5,\ 1/\omega > 0,\ \omega - 1 > \omega/2$.
   Do not use the results of the previous exercise.
6. Using induction, prove that $n \geq 0$ and $1/2^n \geq 0$, for every natural n.
7. For every $x \in \mathbb{S}$, prove that $x \geq x$. You will need the seniority principle.
8. Suppose that a, b, c are surreals with $a < b < c$. Prove that $\prec a, b | c \succ = \prec b | c \succ$.
   HINT:Prove $\prec a, b | c \succ \leq \prec b | c \succ$ and $\prec b | c \succ \leq \prec a, b | c \succ$.
9. For k finite, put all surreals with birthday $< k$ on a number line. Exactly one surreal with birthday k will fit into each gap, and one more on either side. Using this fact, determine with proof how many surreals have birthday k.