MATH601 Spring 2008
Handout 11: Multiplication and Transfinite Induction
Unit 5: Ordinals

Recall that addition of ordinals was defined recursively, as repeated $+1$. We define multiplication[1] recursively, as repeated addition. Let x, y be ordinals.

1. If $y = 0$, then $x \times y = 0$.
2. If y is a successor, then for some ordinal z, we have $y = z + 1$. We define $x \times y = (x \times z) + x$.
3. If y is a limit ordinal, then $y = \lim_{z<y} z$. We then define $x \times y = \lim_{z<y} (x \times z)$.

We now demonstrate *transfinite induction* by proving that $0 \times y = 0$, for every ordinal y. With regular induction, there are two steps (base case, inductive case). With transfinite induction, there are three steps (base case, successor case, limit case) that correspond to the three types of ordinals.

Suppose first that $y = 0$ (base case). By part (1) of the definition of $\times$, $0 \times y = 0$. Suppose next that y is a successor, that is $y = z + 1$ (successor case). We assume as inductive hypothesis that $0 \times z = 0$; then by part (2) of the definition of $\times$, $0 \times y = (0 \times z) + 0 = 0 + 0$. But by part (1) of the definition of $+$, $0 + 0 = 0$, so $0 \times y = 0$. Suppose now that y is a limit ordinal (limit case). We assume as inductive hypothesis that $0 \times z = 0$ for all $z < y$. By part (3) of the definition of $\times$, $0 \times y = 0 \times \lim_{z<y} z \underset{\uparrow}{=} \lim_{z<y} (0 \times z) = \lim_{z<y} 0 = 0$. Hence no matter which ordinal type y is, $0 \times y = 0$.

We will allow the limit symbol to move around in a rather sloppy way, as indicated with $\uparrow$, since making this precise is more trouble than it's worth. Use the intuition you built in Calculus 1 to justify such operations with limits; limits are well-behaved with ordinals, unlike addition or multiplication.

Exercises:

1. Calculate 3×5 and 5×3. Although calculated differently, multiplication of natural numbers coincides with your intuition.
2. Show that $2 \times \omega = \omega$, and that $\omega \times 2 = \omega + \omega$. These are not equal, or even isomorphic, because ω contains only successor ordinals, while $\omega + \omega = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots\}$ contains one limit ordinal (namely, ω).
3. For every ordinal x, prove that $0 + x = x$.
4. For all ordinals x, y, z with $x < y$, prove that $z + x < z + y$.
   Hint: Consider first the special case $x = 0$.
5. For all ordinals x, y, z with $x < y$, prove that $x + z \leq y + z$. Find ordinals x, y, z with $x < y$ but $x + z = y + z$.
6. For all ordinals x, y, z with $x \leq y$, prove that $x \times z \leq y \times z$.
7. Show that $(\omega + \omega) \times \omega = \omega \times \omega \neq \omega \times \omega \times 2 = (\omega \times \omega) + (\omega \times \omega)$.

   Exercise 7 demonstrates that distributivity on the right does not always hold: $(x + y) \times z$ does not always equal $(x \times z) + (y \times z)$. It can be proved (hint: transfinite induction on z) that distributivity on the left holds: $x \times (y + z) = (x \times y) + (x \times z)$.

[1] Multiplication is associative, but we again omit the proof.