MATH601 Spring 2008
Handout 10: Addition
Unit 5: Ordinals

The ordinal numbers represent not the size of a set, but the position within the ordered sequence $0, 1, 2, \ldots$. They are well-ordered[1] and every ordinal has a unique successor, obtained by adding 1 and denoted by $+\!\!1$, which is larger than the original. The von Neumann definition of ordinals is as the set containing all smaller ordinals (and nothing else). The $+\!\!1$ operation is equivalent to 'the set containing all ordinals up to and including this one', i.e. $x +\!\!1 = x \cup \{x\}$. The naturals $\mathbb{N}_0$ are defined in a very precise way: 5 is defined as 'one more than' 4, etc. Under this definition, x is an ordinal if every element of x is also a subset of x. For example, 3 has three elements: $0, 1, 2$. $0 = \{\}$, the empty set, which is a subset of every set (and hence of 3). $1 = \{0\} \subseteq \{0, 1, 2\} = 3$. Finally, $2 = \{0, 1\} \subseteq \{0, 1, 2\} = 3$. For ordinals x, y, we define $x < y$ if and only if $x \in y$ if and only if $x \subseteq y$. Be careful, as it is easy to get confused in all this.
$0 = \{\},\ 1 = \{0\} = 0 +\!\!1,\ 2 = \{0, 1\} = 1 +\!\!1,\ 3 = \{0, 1, 2\} = 2 +\!\!1,\ \omega = \{0, 1, 2, \ldots\},\ \omega +\!\!1 = \{0, 1, \ldots, \omega\}$.

Every ordinal n is necessarily one of three following types. Statements about ordinals will typically handle the three cases separately.

1. $n = 0$, the unique smallest ordinal, a special case. 0 is not a successor or a limit.
2. $n = m +\!\!1$ for some other ordinal m. We say that n is a *successor* ordinal, and m is its *predecessor*.
3. n has no predecessor; for every $m < n$ we also have $m +\!\!1 < n$. We say that n is a *limit ordinal*.

Every element of $\mathbb{N}$ (note that $0 \notin \mathbb{N}$) is a successor; the smallest ordinal after all of $\mathbb{N}$ is called ω, omega. We can now define addition of ordinals[2], as follows. Let x, y be ordinals.

1. If $y = 0$, then $x + y = x$.
2. If y is a successor, then for some ordinal z, we have $y = z +\!\!1$. We define $x + y$ as the successor of $x + z$; i.e. $x + y = (x + z) +\!\!1$.
3. If y is a limit ordinal, then $y = \lim\limits_{z<y} z$. We then define $x + y = \lim\limits_{z<y} (x + z)$.

Note: $+\!\!1$ means successor, which is a set operation (it is not addition). However, these two coincide:

Lemma For any ordinal x, $x + 1 = x +\!\!1$.

Proof. $1 = 0 +\!\!1$. Hence $x + 1$ by property 2 equals $(x + 0) +\!\!1$, which by property 1 equals $x +\!\!1$. ☐

Exercises:

1. Prove that $3 < 5$ in three ways: $3 \subseteq 5, 3 \in 5$, and we can get from 3 to 5 using the successor operation.
2. Prove that $3 < \omega$ in two ways: $3 \subseteq \omega, 3 \in \omega$. Can we get from 3 to ω using the successor operation?
3. Express $3, 4$ in von Neumann notation using only symbols $\{, \}$. e.g., $1 = \{\{\}\}, 2 = \{\{\}, \{\{\}\}\}$.
4. Prove that $\omega = \{0, 1, 2, \ldots\}$ is a limit ordinal.
5. Calculate $4 + 2$ and $2 + 4$. Note that although the answer is the same the method is quite different.
6. Show that $1 + \omega = \omega \neq \omega + 1$.
7. For any nonzero ordinals x, y, prove: (1) If y is a successor then $x + y$ is a successor; and (2) if y is a limit then $x + y$ is a limit. Hence if one of x, y is a successor and the other is a limit, $x + y \neq y + x$.
8. Calculate $\big(3 + (\omega + 4)\big) + \big((\omega + 5) + 6\big)$.

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[1] This does not *necessarily* mean we assume the axiom of choice (there may be non-ordinal numbers that are not in this well-ordering), however we may as well assume AC since it makes things conceptually simpler.

[2] This addition is associative, but the proof is difficult and we omit it.