MATH601 Spring 2008
Exam 5 Solutions

1. Simplify $(\omega^\omega \times 2 + \omega^{\omega^\omega} \times 3 + \omega^2 \times 4 + \omega^2 \times 2 + 7) \times (\omega^\omega \times 5 + \omega^2 + \omega \times 8 + 5)$ and place the result into Cantor Normal Form.

   We first note that $\omega^\omega \times 2 + \omega^{\omega^\omega} \times 3 = \omega^{\omega^\omega} \times 3$, and that $\omega^2 \times 4 + \omega^2 \times 2 = \omega^2 \times 6$, so the problem reduces to $(\omega^{\omega^\omega} \times 3 + \omega^2 \times 6 + 7) \times (\omega^\omega \times 5 + \omega^2 + \omega \times 8 + 5)$.

   Because multiplication is distributive on the left, this equals $(\omega^{\omega^\omega} \times 3 + \omega^2 \times 6 + 7) \times (\omega^\omega \times 5) + (\omega^{\omega^\omega} \times 3 + \omega^2 \times 6 + 7) \times (\omega^2) + (\omega^{\omega^\omega} \times 3 + \omega^2 \times 6 + 7) \times (\omega \times 8) + (\omega^{\omega^\omega} \times 3 + \omega^2 \times 6 + 7) \times (5)$.

   $(\omega^{\omega^\omega} \times 3 + \omega^2 \times 6 + 7) \times (\omega^\omega \times 5) = \omega^{\omega^\omega} \times \omega^\omega \times 5 = \omega^{\omega^\omega + \omega} \times 5$
   $(\omega^{\omega^\omega} \times 3 + \omega^2 \times 6 + 7) \times (\omega^2) = \omega^{\omega^\omega} \times \omega^2 = \omega^{\omega^\omega + 2}$
   $(\omega^{\omega^\omega} \times 3 + \omega^2 \times 6 + 7) \times (\omega \times 8) = \omega^{\omega^\omega} \times \omega \times 8 = \omega^{\omega^\omega + 1} \times 8$.
   $(\omega^{\omega^\omega} \times 3 + \omega^2 \times 6 + 7) \times (5) = \omega^{\omega^\omega} \times 15 + \omega^2 \times 6 + 7$.

   Putting it all together we get $\omega^{\omega^\omega + \omega} \times 5 + \omega^{\omega^\omega + 2} + \omega^{\omega^\omega + 1} \times 8 + \omega^{\omega^\omega} \times 15 + \omega^2 \times 6 + 7$, which is already in CNF.

2. Prove that $1 \times x = x = x \times 1$, for every ordinal x.

   We prove $1 \times x = x$ by transfinite induction. If $x = 0$, then $1 \times 0 = 0$ by the definition of $\times$. If $x = z +\!\!\!1$ (x is a successor), then $1 \times x = (1 \times z) + 1$ by the definition of $\times$. By the inductive hypothesis, $1 \times z = z$. By a lemma proved in handout 10, $z + 1 = z +\!\!\!1$. Putting these together, $1 \times x = z + 1 = z +\!\!\!1 = x$. Finally, if $x = \lim\limits_{z<x} z$ (x is a limit ordinal), then $1 \times x = \lim\limits_{z<x} 1 \times z$. By the inductive hypothesis, $1 \times z = z$ for all $z < x$; hence $1 \times x = \lim\limits_{z<x} z = x$.

   The other part is simpler: $x \times 1 = (x \times 0) + x = 0 + x$, using the definition of $\times$ twice. $0 + x = x$ by one of the homework exercises.

3. Exam grades: 98, 92, 91, 88, 84, 83, 77, 66, 60, 57