MATH 579 Final Exam; 5/15/12
Please read the exam instructions.

Please indicate what work goes with which problem and put your name or initials on every sheet. Cross out work you do not wish graded; incorrect work can lower your grade, even compared with no work at all. Show all necessary work in your solutions; if you are unsure, show it. Simplify all numerical answers to be integers, if possible. You are welcome to use your book, notes, and calculators; if you use an earlier result be sure to cite it. You have 120 minutes. This exam is out of 60 points maximum.

Part I: Do all three problems.

1. (5-10 points) How many ways are there to place five black and two white rooks on a chessboard so that none of them attack any other?

2. (5-10 points) How many positive integers in $[1000]$ are divisible by at least one of four, six, and ten?

3. (5-10 points) We color each square of a $1 \times n$ chessboard with the colors red, blue, green, and yellow. We insist on an even number of red squares, an even number of blue squares, an odd number of green squares, and an odd number of yellow squares. How many ways can we do this?

Part II: Choose three of the following six problems.

4. (5-8 points) Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where the p_i are distinct primes, and the $a_i \in \mathbb{N}$. How many positive divisors does n have?

5. (5-10 points) For all integers $n \geq 2$, prove that $\binom{2n}{n} > 2^n$.

6. (5-10 points) Let $a_0 = 0, a_1 = 1, a_n = a_{n-1} + 12a_{n-2} (n \geq 2)$. Find a closed form for a_n.

7. (5-10 points) For all integers n, k satisfying $n \geq k \geq 1$, prove that $S(n, k) \geq k^{n-k}$.

8. (5-10 points) Let $F(n)$ be the number of all partitions of $[n]$ with no singleton blocks. Prove that $B(n) = F(n) + F(n + 1)$.

9. (5-12 points) Use generating functions to prove Vandermonde's convolution: For r, s, n arbitrary integers,

$$\sum_{k=0}^{n} \binom{r}{k} \binom{s}{n - k} = \binom{r + s}{n}$$

Extra credit: Predict your score on this exam, out of 60. If close (within 2 points), you'll earn a bonus point. If exactly right, you'll earn two bonus points.