MATH 579 Exam 8 Solutions

1. Solve the recurrence given by $a_0 = a_1 = 2, a_n = -2a_{n-1} - a_{n-2}$ $(n \geq 2)$.

   The characteristic equation is $r^2 + 2r + 1 = 0$, which has double root $r = -1$. Hence the general equation is $a_n = \alpha(-1)^n + \beta n(-1)^n$. $2 = a_0 = \alpha(-1)^0 + \beta 0(-1)^0 = \alpha$, and $2 = a_1 = \alpha(-1)^1 + \beta 1(-1)^1 = -\alpha - \beta$, so $\beta = -4$. Hence the solution is $a_n = 2(-1)^n - 4n(-1)^n$.

2. Solve the recurrence given by $a_0 = 3, a_n = 3a_{n-1} - 4$ $(n \geq 1)$.

   The characteristic equation of the homogeneous relation is $r - 3 = 0$, which has root $r = 3$. Hence the general homogeneous solution is $a_n = \alpha 3^n$. To solve the nonhomogeneous relation, we guess a constant polynomial $a_n = A$. $A = 3A - 4$, so $A = 2$. Combining, the general nonhomogeneous solution is $a_n = \alpha 3^n + 2$. $3 = a_0 = \alpha 3^0 + 2 = \alpha + 2$, so $\alpha = 1$. Hence, the solution is $a_n = 3^n + 2$.

3. How many ways are there to climb a flight of n stairs, where each of your steps may move you one or two stairs higher?

   Let a_n denote the desired quantity. We have $a_0 = 1, a_1 = 1$ by inspection. Your last step might have been two stairs (in which case you just climbed $n-2$ stairs in any legal way), or it might have been one stair (in which case you just climbed $n-1$ stairs in any legal way). Hence $a_n = a_{n-1} + a_{n-2}$, the familiar Fibonacci relation. It has characteristic equation $r^2 - r - 1 = 0$, with roots $r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$. The general solution is $a_n = \alpha r_1^n + \beta r_2^n$. $1 = a_0 = \alpha + \beta$, $1 = a_1 = \alpha r_1 + \beta r_2$. This has solution $\alpha = \frac{r_1}{\sqrt{5}}, \beta = -\frac{r_2}{\sqrt{5}}$, so the solution is $a_n = (\frac{1}{\sqrt{5}})(r_1^{n+1} - r_2^{n+1})$.

4. Codewords (strings) from the alphabet $\{0, 1, 2\}$ are called *legitimate* if they have an even number of 0's. How many legitimate codewords are there of length k?

   Let a_k denote the desired quantity. There are $3^k - a_k$ illegitimate codewords of length k. Starting with a legitimate codeword of length k, remove the last letter. If we removed a 1 or 2, what remains is legitimate; if we removed a 0, what remains is illegitimate. Hence $a_k = 2a_{k-1} + (3^{k-1} - a_{k-1}) = a_{k-1} + 3^{k-1}$ is the relation. We have $a_0 = 1$. The homogeneous relation has general solution $a_k = \alpha(1)^k = \alpha$. We guess $a_k = A3^k$ for a nonhomogeneous solution. $A3^k = A3^{k-1} + 3^{k-1}$, so $A = 0.5$. Our general nonhomogeneous solution is $a_k = \alpha + 3^k/2$. $1 = a_0 = \alpha + 0.5$, so $\alpha = 0.5$ and our solution is $a_k = \frac{1+3^k}{2}$.

5. You open a holiday savings account in early January with \$500 you won in a scratch game. It pays the princely sum of 1% interest, compounded monthly. You have \$20 automatically deposited at the end of each of your twice-monthly pay periods; your deposits begin to earn interest in the month after they are made. On December 19, you're ready to shop. How much will you have saved up?

   Let a_k denote the amount at the end of month k, where a_1 is how much at the end of January. $a_0 = 500$, and $a_{k+1} = (1 + \frac{0.01}{12})a_k + 40 = \frac{1201}{1200}a_k + 40$. The homogenous relation has general solution $a_k = \alpha(\frac{1201}{1200})^k$. We guess constant polynomial A for the nonhomogeneous relation, getting $A = \frac{1201}{1200}A + 40$. This has solution $A = -48000$. Hence the general nonhomogeneous solution is $a_k = \alpha(\frac{1201}{1200})^k - 48000$. We have $500 = a_0 = \alpha - 48000$, so $\alpha = 48500$. Hence the solution is $a_k = 48500(\frac{1201}{1200})^k - 48000$. That's a lot of work, when all we really want is $a_{11} = \$946.44$, the balance at the end of November. On Dec. 15 you put another 20 in, for a grand total of \$966.44. Of that, you've put in \$960 yourself, and earned \$6.44 in interest. Compound interest is very powerful, but not piddly amounts like 1% for a year.