MATH 579 Exam 5 Solutions

Part I: Prove that $p(n) \leq \frac{p(n-1)+p(n+1)}{2}$, for $n \in \mathbb{N}$.

A bit of algebra shows that the statement is equivalent to $q(n+1) \geq q(n)$, for the function $q(n) = p(n) - p(n-1)$. Let's call a "nice" partition one where each part is at least 2. Thm 5.20 in the text states that $q(n)$ counts nice partitions. We now establish a bijection between nice partitions of n and certain nice partitions of $n+1$, namely the ones that have their largest part strictly bigger than the second-largest part. Given a nice partition of n, we add 1 to the largest part. This gives a nice partition of $n+1$, which is a bijection between the two sets in question. Hence $q(n+1) \geq q(n)$.
NOTE: It isn't enough to add 1 to an arbitrary part of a nice partition of n; that is not 1-1.

Part II:

1. Find a formula for $S(n,2)$, for $n \geq 2$.

   The number of surjective functions from $[n]$ to $[2]$ is $2!S(n,2)$. There are 2^n functions altogether; however two are not surjective: the one that sends everything to 1, and the one that sends everything to 2. Solving $2^n - 2 = 2S(n,2)$ we get $S(n,2) = 2^{n-1} - 1$.

2. Find a formula for $S(n,3)$, for $n \geq 3$.

   The number of surjective functions from $[n]$ to $[3]$ is $3!S(n,3)$. There are 3^n functions altogether; however three send everything to just one place, and $\binom{3}{2}(2^n-2)$ send everything to two places (applying the previous problem). Hence $3^n - 3(2^n - 2) - 3 = 6S(n,3)$; solving, we get $S(n,3) = 0.5(3^{n-1} - 2^n + 1)$.

3. Find the number of compositions of 25 into 5 odd parts.

   By subtracting one from each part, we get a bijection between compositions of 25 into 5 odd parts, and weak compositions of 20 (=25-5) into 5 even parts. By dividing each part in half, we get a bijection between weak compositions of 20 into 5 even parts, and weak compositions of 10 into 5 parts. For this we have a formula, namely $\binom{14}{10} = 1001$.

4. Prove that $p_k(n) \le (n-k+1)^{k-1}$, for $1 \le k \le n$.

   We give a process that will yield various partitions with k parts, among them all partitions of n into k parts. For each part, we select from $[1, n-k+1]$, and we do this $k-1$ times. For the last part, there is at most one possible choice to make the sum n; if possible, we take it, otherwise it doesn't matter what we take. This process has $(n-k+1)^{k-1}$ outcomes. We now show that every possible partition of n into k parts occurs, by showing that each such partition must have each part at most $n-k+1$. If not, then some part must be greater than this, but the other $k-1$ parts have sum at least $k-1$, so together the sum would be greater than n.

5. Prove that $B(n) \ge \binom{n}{2}$, for $n \ge 0$.

   SOLUTION 1: Thm 5.12 states: $B(n+1) = \sum_i \binom{n}{i} B(i)$. We first prove the lemma that $B(n) \ge n$. We proceed by strong induction on n; for $n = 0$ the claim is $0 \ge 0$, which is true. Now $B(n+1) = \sum_i \binom{n}{i} B(i) \ge \sum_{i \in [0,n]} 1 = n+1$.
   Now we use the lemma to prove our result. $B(n+1) = \sum_i \binom{n}{i} B(i) \ge \sum_{i \in [0,n]} i = \frac{n(n+1)}{2} = \binom{n+1}{2}$, as desired.

   SOLUTION 2: Induction on n; we need extra base cases because we need $n \ge 3$ in our induction: $B(0) = 1 \ge 0 = \binom{0}{2}$, $B(1) = 1 \ge 0 = \binom{1}{2}$, $B(2) = 2 \ge 1 = \binom{2}{2}$, and $B(3) = 3 \ge 3 = \binom{3}{2}$. By Thm. 5.12, $B(n+1) = \sum_i \binom{n}{i} B(i) \ge = \sum_i \binom{n}{i}\binom{i}{2} \ge \binom{n}{2}\binom{2}{2} + \binom{n}{n}\binom{n}{2} = \frac{n(n-1)}{2} + \frac{n(n-1)}{2} = \frac{n(2n-2)}{2} \ge \frac{n(n+1)}{2} = \binom{n+1}{2}$, where we used the inductive hypothesis at the beginning and $n \ge 3$ at the end (to prove $2n-2 \ge n+1$).

   SOLUTION 3: Bell numbers are defined as $B(n) = \sum_k S(n,k) \ge S(n, n-1) = \binom{n}{2}$ (for $n \ge 2$). For $n = 0, 1$, we use $B(0) = 1 \ge 0 = \binom{0}{2}$, $B(1) = 1 \ge 0 = \binom{1}{2}$.

Exam grades: High score=104, Median score=66 (ouch!), Low score=52