MATH 579 Exam 1 Solutions

Part I:

Your nemesis places five points on the surface of a sphere. You may slice the sphere in half with a great circle, and keep either half. Prove that there is some way to do this to get at least four of the five points in your half. (points on the great circle are considered to be in your half).

> Choose any two points of the five. These define a great circle; slice the sphere in half along this circle. Generously allow your nemesis to move any other points off the great circle; this can only help him since points on the great circle count for you in both hemispheres. The two hemispheres are pigeonholes, and the remaining three points are pigeons. By the PHP, at least two of the three points must be in the same hemisphere; together with the two that were already on the great circle, this hemisphere contains at least four points.

Part II:

1. An airport has 1500 takeoffs per day. Prove that there are two planes taking off within a minute of each other.

   A day contains $60 \times 24 = 1440$ minutes. By the PHP, since $1500 > 1440$, some minute must contain at least two takeoffs.

2. Your nemesis chooses 1001 distinct integers from $[1, 2000]$. Prove that some pair of these must have no nontrivial (greater than 1) common factor.

   We partition $[1, 2000]$ as $[1, 2], [3, 4], [5, 6], \ldots, [1999, 2000]$. There are 1000 parts, hence by PHP there is some part where your nemesis picked both numbers. But then these are n and $n + 1$ for some n; these never have a nontrivial common factor.

3. Your nemesis chooses 55 distinct integers in $[1, 100]$. Prove that some pair of these must differ by 12.

   We partition $[1, 100]$ as $[1, 13], [2, 14], [3, 15], [4, 16], [5, 17], [6, 18], [7, 19], [8, 20], [9, 21], [10, 22], [11, 23], [12, 24], [25, 37], [26, 38], [27, 39], [28, 40], [29, 41], [30, 42], [31, 43], [32, 44], [33, 45], [34, 46], [35, 47], [36, 48], [49, 61], [50, 62], [51, 63], [52, 64], [53, 65], [54, 66], [55, 67], [56, 68], [57, 69], [58, 70], [59, 71], [60, 72], [73, 85], [74, 86], [75, 87], [76, 88], [77, 89], [78, 90], [79, 91], [80, 92], [81, 93], [82, 94], [83, 95], [84, 96], [97], [98], [99], [100]$. This is 52 parts; by PHP since $55 > 52$ there is some part where the nemesis picked both numbers. These two numbers differ by 12.

4. Prove that the sequence $2010, 20102010, 201020102010, \ldots$ contains an element that is divisible by 2011.

   Let a_n denote the n^{th} term in the sequence; i.e. $a_1 = 2010, a_2 = 20102010$, etc. Consider the remainders upon division by 2011. There are only 2011 possible remainders, so by PHP among $a_1, \ldots, a_{2012}$ two terms must have the same remainder, say a_i and a_j (with $i > j$ without loss of generality). Then $a_i - a_j$ must have no remainder, hence 2011 divides $a_i - a_j = (2010\ldots2010)0000\ldots0000 = a_{i-j}(1000)^j$. But 2011 does not have any common factors with 1000^j, hence 2011 divides a_{i-j}.
   Note: It is not enough to observe that $2011 \nmid 1000$ to conclude that $2011|a_{i-j}$. For example, $4|6 \cdot 10 = 60$, and $4 \nmid 10$, but we cannot conclude that $4|6$.

5. (5-12 points) Your nemesis colors each point of 3-dimensional space one of red, blue, or green. Prove that there is some rectangular box (all six faces are rectangles, like a shoebox), such that all eight corners of this box are the same color.

   We define a grid of size $i \times j \times k$ (with numbers to be specified later), and prove that this will contain the desired rectangular box. Consider an $i \times j$ rectangle. There are $3^{i \times j}$ ways to color the grid points; hence if $k > 3^{i \times j}$, then by PHP we are guaranteed two $i \times j$ rectangles (each a slice) with identically colored grid points. Consider now a line segment of length i within an $i \times j$ rectangle. There are 3^i ways to color the grid points of the line segment; hence if $j > 3^i$, then by PHP we are guaranteed two length i line segments (in every $i \times j$ rectangle) with identically colored grid points. Finally, if we choose $i > 3$, then by PHP some two grid points on the line segment must have the same color. The smallest values that work for this strategy are $i = 4, j = 3^4 + 1 = 82, k = 3^{328} + 1$. Hence, two of the $3^{328} + 1$ rectangles must be identical. Within these two identical rectangles, two of the 82 lines must be identical. This line must have two points the same color, but there are four such lines (two on each rectangle), hence there are eight points of the same color determining a rectangular box. This construction requires $4 \times 82 \times (3^{328} + 1) \approx 10^{159}$ points, a lot.

Exam grades: High score=104, Median score=85, Low score=62