MATH 579 Exam 9 Solutions

1. Let a_n denote the number of ways to pay n cents using pennies, nickels, dimes and/or quarters. For example, $a_5 = 2$ (either five pennies or one nickel). Find the generating function for a_n.

   Let $A(x) = \sum_{n\geq 0} a_n x^n$. We have $A(x) = (1 + x + x^2 + \cdots)(1 + x^5 + x^{10} + \cdots)(1 + x^{10} + x^{20} + \cdots)(1 + x^{25} + x^{50} + \cdots) = \frac{1}{1-x}\frac{1}{1-x^5}\frac{1}{1-x^{10}}\frac{1}{1-x^{25}}$.

2. Let a_n be the number of subsets of $[n]$ in which the difference between any two elements is at least four. Find the generating function for a_n.

   Consider a valid subset. If it does not contain n, then it is in fact a valid subset from $[n-1]$; on the other hand if it does contain n then the rest is a valid subset from $[n-4]$; Hence $a_n = a_{n-1} + a_{n-4}$. Also we have $a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4$ (in all these base cases a valid subset can contain at most one element). Multiplying both sides of the recurrence by x^n and summing over all $n \geq 4$ we get $\sum_{n\geq 4} a_n x^n = x \sum_{n\geq 4} a_{n-1}x^{n-1} + x^4 \sum_{n\geq 4} a_{n-4}x^{n-4} = x\sum_{n\geq 3} a_n x^n + x^4 \sum_{n\geq 0} a_n x^n$. Hence $A(x) - 1 - 2x - 3x^2 - 4x^3 = x(A(x) - 1 - 2x - 3x^2) + x^4 A(x)$. Solving, we get $A(x) = \frac{1+x+x^2+x^3}{1-x-x^4}$.

3. Use generating functions to solve the recurrence $a_0 = 1, a_n = 3a_{n-1} + 2^{n-1}$.

   Multiplying both sides by x^n and summing over all $n \geq 1$ (note: not $n \geq 0$ since a_{-1} is undefined) we get $\sum_{n\geq 1} a_n x^n = 3x \sum_{n\geq 1} a_{n-1}x^{n-1} + x\sum_{n\geq 1} 2^{n-1}x^{n-1} = 3x\sum_{n\geq 0} a_n x^n + x\sum_{n\geq 0} 2^n x^n$. Hence $A(x) - 1 = 3xA(x) + \frac{x}{1-2x}$. Solving, we get $A(x) = \frac{1-x}{(1-2x)(1-3x)} = \frac{a}{1-2x} + \frac{b}{1-3x} = a\sum_{n\geq 0} 2^n x^n + b\sum_{n\geq 0} 3^n x^n$. Hence $a_n = a2^n + b3^n$. It remains to solve the partial fractions: $1 - x = a(1-3x) + b(1-2x)$. Substituting $x = 1/2$, we get $a = -1$; substituting $x = 1/3$, we get $b = 2$. Hence $a_n = -2^n + 2 \cdot 3^n$.

4. A permutation is called *indecomposable* if its one-line notation cannot be split into two parts such that every number in the first part is smaller than every number in the second part. Let a_n denote the number of indecomposable permutations of length n, with $a_0 = 0$. Find the generating function for a_n. You may use the generating function $B(x) = \sum_{n\geq 0} n!x^n$, the generating function of all permutations.

   The one-line notation of every permutation is one or more indecomposable permutations, one after the other. In other words, we get a permutation by breaking up $[n]$ into one or more nonempty blocks, then ordering each of those blocks indecomposably. Thus $B(x) = \frac{1}{1-A(x)}$. Solving, we get $A(x) = 1 - \frac{1}{B(x)}$.

5. Find the generating function for $a_n = n^3$.

   Solution 1: Start with $\frac{x}{(1-x)^2} = B(x) = \sum_{n\geq 0} nx^n$. $B'(x) = \sum_{n\geq 0} n^2 x^{n-1}$, and $xB'(x) = \sum_{n\geq 0} n^2 x^n$. Taking derivatives again, we have $(xB'(x))' = \sum_{n\geq 0} n^3 x^{n-1}$ and $x(xB'(x))' = \sum_{n\geq 0} n^3 x^n = A(x)$. Using the quotient rule and simplifying, we have $B'(x) = \frac{1+x}{(1-x)^3}$ and $xB'(x) = \frac{x+x^2}{(1-x)^3}$. $(xB'(x))' = \frac{x^2+4x+1}{(1-x)^4}$; hence $A(x) = \frac{x^3+4x^2+x}{(1-x)^4}$.

Solution 2: $n^3 = S(3,3)(n)_3 + S(3,2)(n)_2 + S(3,1)(n)_1 + S(3,0)(n)_0 = (n)_3 + 3(n)_2 + 1(n)_1 + 0 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1}$. Note: this only holds for $n > 0$, since $S(0,0) = 1$, so instead of $A(x) = \sum_{n\geq 0} a_n x^n$, we will consider $B(x) = \sum_{n\geq 1} a_n x^n$. As it happens, $a_0 = 0$, so $A(x) = B(x)$. Hence $A(x) = \sum_{n\geq 1} n^3 x^n = \sum_{n\geq 1} 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1} x^n = 6\sum_{n\geq 1}\binom{n}{3}x^n + 6\sum_{n\geq 1}\binom{n}{2}x^n + \sum_{n\geq 1}\binom{n}{1}x^n$. Now $\frac{1}{(1-x)^4} = \sum_{n\geq 0}\binom{n+3}{3}x^n$, so $\frac{x^3}{(1-x)^4} = \sum_{n\geq 0}\binom{n+3}{3}x^{n+3} = \sum_{n\geq 3}\binom{n}{3}x^n$. Similarly, $\frac{1}{(1-x)^3} = \sum_{n\geq 0}\binom{n+2}{2}x^n$, so $\frac{x^2}{(1-x)^3} = \sum_{n\geq 0}\binom{n+2}{2}x^{n+2} = \sum_{n\geq 2}\binom{n}{3}x^n$. Hence $A(x) = 6\left(0 + 0 + \frac{x^3}{(1-x)^4}\right) + 6\left(0 + \frac{x^2}{(1-x)^3}\right) + \frac{x}{(1-x)^2} = \frac{x^3+4x^2+x}{(1-x)^4}$.

6. Let a_n denote the number of ways to place n identical nonattacking rooks on an $n \times n$ board, with no rooks on the diagonal (such as on the previous exams). Prove that this equals $D(n)$, the number of derangements of $[n]$. Use this to calculate a_6.

    We establish a bijection between the two sets. Given a valid rook placement, suppose a rook is in row i and column j. We create a directed graph by forming an edge between vertex i and vertex j, for every rook in the placement. Note that each row and column of the valid placement has exactly one rook; hence every vertex in the directed graph has exactly one in-edge and exactly one out-edge. Hence this is the directed graph representation of a permutation. Furthermore, the diagonal is forbidden in the rook placement, hence no edge leads from a vertex back to itself, hence this permutation is in fact a derangement. On the other hand, given a derangement, we take each column of the two-line notation as coordinates for a rook. To be a valid rook placement, no two rooks can be in the same row or column (holds since this is a permutation), and no rook may be on the diagonal (holds since this is a derangement).

    Now, $a_1 = 0, a_2 = 1$, and we use the recursive formula $a_{n+1} = n(a_n + a_{n-1})$, that we proved for derangements. Hence $a_3 = 2(1+0) = 2, a_4 = 3(2+1) = 9, a_5 = 4(9+2) = 44, a_6 = 5(44+9) = 265$.

Exam results: High score=88, Median score=73, Low score=50 (before any extra credit)

Reminder: The final exam will be in our classroom, 10:30-12:30 on Tuesday, May 19. As usual, this will be open book, open notes, calculators required.