MATH 579 Exam 3 Solutions

1. Our class has 19 students, 14 males and 5 females. How many ways are there to form a study group of five students that contains at least one male and at least one female?

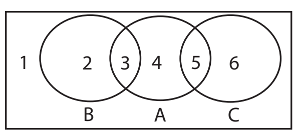
   There are $\binom{19}{5} = 11628$ ways to form a study group irrespective of gender. Of these, $\binom{14}{5} = 2002$ are all-male, and $\binom{5}{5} = 1$ are all-female. Hence there are $11628 - 2002 - 1 = 9625$ mixed-gender study groups.

2. How many odd, four digit numbers, have distinct digits?

   There are 5 choices for the units digit, since the number must be odd. Regardless of which choice was made, there will be 9 choices for the tens digit (cannot be the units digit), 8 choices for the hundreds digit (cannot be either of the previously chosen digits), and 7 choices for the thousands digit. This yields $5\times 9\times 8\times 7 = 2520$ numbers. However this includes numbers like 0123, which are not four digits. Hence we must subtract $5 \times 8 \times 7 = 280$ (the tens digit cannot be the units digit or zero, the hundreds digit may not be the previously chosen digits or zero). The result is $2520 - 280 = 2240$.

   Alternate solution: There are 5 choices for the units digit. Then, we move to the other side; there are 8 choices for the thousands digit (not the units digit or zero). There are 8 choices for the hundreds digit (not either of the previously chosen digits). There are 7 choices for the tens digit. This yields $5 \times 8 \times 8 \times 7 = 2240$.

3. We want to select three subsets A, B, C from $[n]$, so that $A \neq \emptyset$ and $B \cap C = \emptyset$. How many ways are there to do this?

   The Venn diagram at left reflects $B\cap C = \emptyset$, and has six disjoint regions. We label each element from $[n]$ by the region in which it appears; these labellings are bijective with what we are trying to count. There are 6^n ways to do this without concern for A; however if we don't label any element $3, 4$, or 5, then A is empty. There are 3^n ways to have this forbidden labelling (choosing from $1, 2, 6$); hence there are $6^n - 3^n$ valid labellings.

4. I want to play racquetball on five occasions in January; however I need 3 full days of rest between game days. How many schedules are there?

   Let a_1, a_2, a_3, a_4, a_5 denote the days I play. $a_1 \geq 1$, $a_2 \geq a_1 + 4$, $a_3 \geq a_2 + 4$, $a_4 \geq a_3 + 4$, $a_5 \geq a_4 + 4$, and $a_5 \leq 31$ are the conditions specified by the problem. We can write this compactly as $1 \leq a_1 \leq a_2 - 4 \leq a_3 - 8 \leq a_4 - 12 \leq a_5 - 16 \leq 15$. Set $b_1 = a_1, b_2 = a_2 - 4, b_3 = a_3 - 8, b_4 = a_4 - 12, b_5 = a_5 - 16$, and the problem's conditions are equivalent to $1 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq b_5 \leq 15$. Hence there is a bijection between valid schedules and multisets of size 5 drawn from $[15]$. There are $\left(\binom{15}{5}\right) = \binom{19}{5} = 11628$ possible schedules.

   Alternate solution: We write the same conditions compactly as $1 \leq a_1 < a_2 - 3 < a_3 - 6 < a_4 - 9 < a_5 - 12 \leq 19$. Setting $b_1 = a_1, b_2 = a_2 - 3, b_3 = a_3 - 6, b_4 = a_4 - 9, b_5 = a_5 - 12$, we have $1 \leq b_1 < b_2 < b_3 < b_4 < b_5 \leq 19$. Now there is a bijection between valid schedules and sets of size 5 drawn from [19], giving $\binom{19}{5} = 11628$ again.

5. To play the California Fantasy 5 lottery game, you pick five numbers from [39], as does the state. The prizes vary depending on receipts, but the best day historically paid \$402,112 for matching all five, \$645 for matching four, \$20 for matching three, and \$1 for matching two. On that day, what was the expected value of one ticket? You may assume that the prizes are not divided among multiple winners.

   To find the expected value of one ticket, we imagine playing each possible ticket once, calculate the total winnings, and divide by the number of tickets. There is one possible ticket that wins the grand prize, winning $\$402,112$. To match four from five, we need to choose four winning numbers (where we agree), and one losing number (where we disagree). There are $\binom{5}{4}\binom{34}{1} = 170$ tickets that win second prize, winning $170 \times \$645 = \$109,650$. To match three from five, we need to choose three winning numbers and two losing numbers. There are $\binom{5}{3}\binom{34}{2} = 5610$ tickets that win third prize, winning $5610 \times \$20 = \$112,200$. To match two from five, we need two winning numbers and three losing numbers; there are $\binom{5}{2}\binom{34}{3} = 59840$ such tickets, winning $\$59,840$. Our total winnings are therefore $\$683,802$. To win this we would need to buy $\binom{39}{5} = 575,757$ tickets, for an expected value of $\frac{\$683{,}802}{575{,}757} \approx \$1.1877$. Since tickets cost \$1, this is a net gain of almost 19 cents per play.

   Caution: most days have significantly lower payoffs, and players are not told the payoffs in advance. This problem should not be construed as investment advice.

6. How many surjective (onto) functions are there that have as domain $\{A, B, C, D, E, F\}$ and have as codomain $\{X, Y, Z\}$?

   If we ignore the surjective restriction, there are $3^6 = 729$ functions with the specified domain and codomain. We will subtract from these 729 the functions that are not surjective, leaving behind the desired value. A function that is not surjective can be thought of as having a smaller codomain. Some of the undesired functions have codomain $\{X, Y\}$; specifically, there are $2^6 = 64$ of these. Some have codomain $\{X, Z\}$; there are 64 of these. Similarly, there are 64 undesired functions with codomain $\{Y, Z\}$. Before we conclude that there are $3 \times 64 = 192$ undesired functions, there is another factor. There are three special functions: one with codomain $\{X\}$, one with codomain $\{Y\}$, and one with codomain $\{Z\}$. These have been overcounted among the undesired functions; each has been counted exactly twice (e.g. the function with codomain $\{X\}$ has been counted once with those having codomain $\{X, Y\}$ and again with those having codomain $\{X, Z\}$), but needs to be counted exactly once. Hence there are not 192 but 189 undesired functions, and hence $729 - 189 = 540$ surjective ones.

   This is a first step in the direction of what is called the inclusion-exclusion principle, covered in detail in Chapter 7. This tool consists of calculating certain values on smaller and smaller sets, then alternately adding and subtracting these values. In this case the result is $+729 - 192 + 3$.

Exam results: High score=96, Median score=72, Low score=56 (before any extra credit)