## MATH 579 Spring 2007 Supplement: Recurrences

A recurrence is a sequence of numbers, defined by some positional relationship. This positional relationship is called a recurrence relation. That is, the $n^{\text{th}}$ number is a function of the previous numbers. Some examples of recurrence relations are $a_n = 2a_{n-1}$, $b_n = b_{n-2} + 2$, $c_n = c_{n-1} + c_{n-2}$ (Fibonacci numbers if $c_1 = c_2 = 1$), $d_{n+1} = n(d_{n-1} + d_n)$ (derangements if $d_1 = 0, d_2 = 1$). To fully specify the sequence, 'enough' initial conditions are necessary. For example, $\{a_n\}$ requires one initial condition (e.g. $a_1 = 3$). $\{b_n\}$ requires two; $b_1 = 3$ is enough to specify all the odd terms in the sequence, but to specify the even terms we need $b_2 = 4$.

To solve a recurrence means to find a closed-form expression for the sequence, that does not depend on previous terms. We have already learned two ways to solve recurrences. The first is guessing; a recurrence is completely specified by its initial conditions and recurrence. If you guess the answer, you can show that your guess satisfies the recurrence and satisfies the initial conditions – this is enough to prove your answer. The second is by generating functions.

**Example 1a**: $a_1 = 1, a_n = 2a_{n-1}$ $(n \geq 2)$

Guess $a_n = 2^{n-1}$. Check that $2^{1-1} = 1$, so the initial condition is satisfied. Also, $2^{n-1} = 2 \times 2^{(n-1)-1}$, so the recurrence relation is satisfied.

**Example 1b**: $a_1 = 1, a_n = 2a_{n-1}$ $(n \geq 2)$

Set $A(x) = \sum_{n\geq 1} a_n x^n$. $2xA(x) = \sum_{n\geq 1} 2a_n x^{n+1} = \sum_{n\geq 2} 2a_{n-1}x^n = \sum_{n\geq 2} a_n x^n = A(x) - x$. We simplify to get $A(x) = x(1-2x)^{-1} = x\sum_{n\geq 0} 2^n x^n = \sum_{n\geq 0} 2^n x^{n+1} = \sum_{n\geq 1} 2^{n-1}x^n$. Hence $a_n = 2^{n-1}$.

Much as with differential equations, recurrences fall into many types, with many different strategies for solution. A *linear* recurrence relation of *order* $k$ may be written as $a_n = \star a_{n-1} + \star a_{n-2} + \cdots + \star a_{n-k} + \star$, where each $\star$ is some function of $n$. If each $\star$ is, in fact, a constant, we say that the recurrence has *constant coefficients*. In this section, we will only consider linear recurrences. Further, we will assume that all the coefficients (except possibly the final $\star$) are constants. If the final $\star$ is identically zero (i.e. $a_n = \star a_{n-1} + \star a_{n-2} + \cdots + \star a_{n-k}$) we call the relation *homogeneous*; otherwise we call it *nonhomogeneous*. In the above examples, $a_n = 2a_{n-1}$ is first-order homogeneous with constant coefficients, $b_n = b_{n-2} + 2$ is second-order nonhomogeneous with constant coefficients, $c_n = c_{n-1} + c_{n-2}$ is second-order homogeneous with constant coefficients, and $d_{n+1} = n(d_{n-1} + d_n)$ is second-order homogeneous with nonconstant coefficients.

**Homogeneous Linear Recurrence Relations with Constant Coefficients**

We consider the recurrence relation $a_n = c_{n-1}a_{n-1} + c_{n-2}a_{n-2} + \cdots + c_{n-k}a_{n-k}$. Because this is homogeneous, we may multiply a solution by any constant and it will be a solution. We may also add two solutions and get a solution. In short, the set of solutions forms a linear space. This space is of dimension $k$, because the relation is of order $k$ and requires $k$ initial conditions to fully specify the recurrence. Hence, to find the general solution, we may find $k$ linearly independent solutions, and take all their linear combinations. Caution: be sure that the $k$ specific solutions are linearly independent.

Let's guess that $a_n = x^n$ is a solution. We substitute into the recurrence to get $x^n = c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_{n-k}x^{n-k}$. Dividing by $x^{n-k}$ gives us $x^k = c_{n-1}x^{k-1} + c_{n-2}x^{k-2} + \cdots + c_{n-k}$. This is known as the *characteristic equation* of the recurrence relation. It is a polynomial of degree $k$, and therefore by the Fundamental Theorem of Algebra has $k$ complex roots, counted by multiplicity.

If the $k$ roots $r_1, r_2, \ldots, r_k$ are all distinct, then $a_n = r_1^n, a_n = r_2^n, \ldots, a_n = r_k^n$ are $k$ linearly independent solutions, and therefore span the solution space. The general solution is therefore $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$. The $k$ initial conditions allow us to determine the unknown $\alpha_1, \alpha_2, \ldots, \alpha_k$ for a particular solution.

If, on the other hand, a root is repeated (i.e. $r_1 = r_2$), then $a_n = r_1^n, a_n = r_2^n, \ldots, a_n = r_k^n$ are *NOT* $k$ linearly independent solutions. $\alpha_1 r_1^n + \alpha_2 r_2^n$ is a one-dimensional subspace, being equal to $\alpha_1 r_1^n$ alone, because $r_1 = r_2$. Fortunately, if a root is repeated, we have available to us additional solutions, that are linearly independent. If root $r_1$ has multiplicity 4, then $r_1^n, nr_1^n, n^2r_1^n, n^3r_1^n$ are four linearly independent solutions (this fact will not be proved). In this manner we again get $k$ linearly independent solutions, and therefore the general solution via linear combinations.

**Example 1c**: $a_1 = 1, a_n = 2a_{n-1}$ $(n \geq 2)$

This has characteristic equation $x = 2$; hence the general solution is $a_n = \alpha 2^n$. Substituting $n = 1$ and using the initial conditions, we have $1 = a_1 = \alpha 2^1$. We solve to find $\alpha = 1/2$; hence the specific solution is $a_n = (1/2)2^n = 2^{n-1}$.

**Example 2**: $a_1 = a_2 = 1, a_n = a_{n-1} + a_{n-2}$ $(n \geq 3)$ (Fibonacci numbers)

This has characteristic equation $x^2 = x + 1$, which has roots (using the quadratic formula) $r_1 = (1+\sqrt{5})/2$ and $r_2 = (1-\sqrt{5})/2$. Hence the general solution is $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$. We have two initial conditions: $1 = a_1 = \alpha_1 r_1 + \alpha_2 r_2, 1 = a_2 = \alpha_1 r_1^2 + \alpha_2 r_2^2$. This is a $2 \times 2$ linear system in the unknowns $\alpha_1, \alpha_2$, with solution $\alpha_1 = \frac{1}{\sqrt{5}}, \alpha_2 = \frac{-1}{\sqrt{5}}$. Hence the specific solution is $a_n = (r_1^n - r_2^n)/\sqrt{5}$.

**Example 3**: $a_0 = a_2 = 1, a_1 = 0, a_3 = 2, a_n = -a_{n-1} + 3a_{n-2} + 5a_{n-3} + 2a_{n-4}$ $(n \geq 4)$

This has characteristic equation $x^4 + x^3 - 3x^2 - 5x - 2 = 0$. We find the roots by guessing small integers (the rational root theorem helps too); if we successfully guess a root $r$, we divide by $x - r$ using long division and continue. In this manner, we find roots $-1$ (multiplicity 3), and 2. Hence, the general solution is $a_n = \alpha_1(-1)^n + \alpha_2 n(-1)^n + \alpha_3 n^2(-1)^n + \alpha_4 2^n$. We now apply our initial conditions to get:

$(n = 0) : 1 = a_0 = \alpha_1 + \alpha_4$ $\quad$ $(n = 1) : 0 = a_1 = \alpha_1(-1) + \alpha_2(-1) + \alpha_3(-1) + \alpha_4 2$

$(n = 2) : 1 = a_2 = \alpha_1 + \alpha_2 2 + \alpha_3 4 + \alpha_4 4$ $\quad$ $(n = 3) : 2 = a_3 = \alpha_1(-1) + \alpha_2(-3) + \alpha_3(-9) + \alpha_4 8$

This is a $4 \times 4$ linear system, with solution $\alpha_1 = 7/9, \alpha_2 = -3/9, \alpha_3 = 0, \alpha_4 = 2/9$. Therefore, the specific solution is $a_n = (7/9)(-1)^n - (3n/9)(-1)^n + (2/9)2^n$.

**Example 4 (Gambler's ruin)**: A gambler repeatedly plays a game against a casino, until one of them runs out of money. Each time the gambler has probability $p$ of winning \$1, and probability $q = 1 - p$ of losing \$1. The gambler starts with $n$ dollars, and the casino with $m - n$ dollars (there are $m$ total dollars to be won). What is the probability that the gambler will run out of money before the casino?

Let $a_n$ denote the desired probability, that the gambler is successful starting with $n$ dollars. For the gambler to win, either (1) gambler wins first bet, and then is successful starting with $n+1$ dollars, or (2) gambler loses first bet, and then is successful starting with $n-1$ dollars. Therefore, this sequence satisfies the recurrence relation $a_n = pa_{n+1} + qa_{n-1}(0 < n < m)$, with boundary conditions $a_0 = 1, a_m = 0$. This has characteristic equation $px^2 - x + q = 0$, with roots $r_1 = 1, r_2 = q/p$. Hence the problem breaks into two cases, depending on whether $p = q$ or not.

$(p \neq q)$: The general solution is $a_n = \alpha 1^n + \beta r_2^n = \alpha + \beta r_2^n$. We apply the boundary conditions, to get $(n = 0) : 1 = a_0 = \alpha + \beta, \quad (n = m) : 0 = a_m = \alpha + \beta r_2^m$. This has solution $\alpha = -r_2^m/(1 - r_2^m), \beta = 1/(1 - r_2^m)$. Hence, the specific solution is $(-r_2^m + r_2^n)/(1 - r_2^m) = 1 - \frac{1-r_2^n}{1-r_2^m}$.

$(p = q = 1/2)$: The general solution is $a_n = \alpha 1^n + \beta n 1^n = \alpha + \beta n$. We apply the boundary conditions, to get $(n = 0) : 1 = a_0 = \alpha, \quad (n = m) : 0 = a_m = \alpha + \beta m$. This has solution $\alpha = 1, \beta = -1/m$. Hence, the specific solution is $1 - (n/m)$.

**Nonhomogeneous Linear Recurrence Relations**

We want to solve the nonhomogeneous recurrence relation $a_n = c_{n-1}a_{n-1} + c_{n-2}a_{n-2} + \cdots + c_{n-k}a_{n-k} + b(n)$, where $b(n)$ is a function of $n$. The technique to find the general solution is in two parts. First, drop the $b(n)$ term and find the general solution to the homogeneous recurrence relation. Then, find any single solution to the nonhomogeneous recurrence (under any initial/boundary conditions). The general solution to the nonhomogeneous recurrence is the sum of these two – a $k$-dimensional term from the homogeneous part, and a single term with no constants from the nonhomogeneous part.

Finding a particular solution is, at times, an art form. The only good way to find them is to guess and check – guess a particular solution, and see if it fits the nonhomogeneous relation. If $b(n)$ is a polynomial, it's a good idea to try guessing a polynomial of the same degree; however, if the homogeneous solution has overlap with this, then increase the degree of your guess. If $b(n)$ is an exponential, it's a good idea to try a multiple of the same exponential.

**Example 5**: $a_0 = 2, a_n = 2a_{n-1} + 3^n \ (n \geq 1)$

Homogeneous version: $a_n = 2a_{n-1}$, which has characteristic equation $x = 2$ and general solution $\alpha 2^n$.
Nonhomogeneous version: Let's guess $\beta 3^n$. Plugging into the relation, we get $\beta 3^n = 2\beta 3^{n-1} + 3^n$. We divide both sides by $3^{n-1}$ to get $3\beta = 2\beta + 3$; hence $\beta = 3$. Thus $3^{n+1}$ is a specific solution to the original, nonhomogeneous, recurrence.
Putting them together, we find the general solution to the nonhomogeneous recurrence is $a_n = \alpha 2^n + 3^{n+1}$. We now consider the initial condition, $(n = 0) : 2 = a_0 = \alpha 2^0 + 3^1$. This has solution $\alpha = -1$, and so the specific solution is $a_n = 3^{n+1} - 2^n$.

**Example 6**: $a_0 = a_1 = 1, a_n = 2a_{n-1} - a_{n-2} + 5^n \ (n \geq 2)$

Homogeneous version: $a_n = 2a_{n-1} - a_{n-2}$, which has characteristic equation $x^2 - 2x + 1 = 0$. This has a double root of 1, hence has general solution $\alpha_1 1^n + \alpha_2 n 1^n = \alpha_1 + \alpha_2 n$.
Nonhomogeneous version: Let's guess $\beta 5^n$. Plugging into the relation, we get $\beta 5^n = 2\beta 5^{n-1} - \beta 5^{n-2} + 5^n$. We divide both sides by $5^{n-2}$ to get $25\beta = 10\beta - \beta + 25$. This has solution $\beta = 25/16$, so a nonhomogeneous solution is $(25/16)5^n = 5^{n+2}/16$.
Putting them together, we find the general solution to the nonhomogeneous recurrence is $a_n = \alpha_1 + \alpha_2 n + 5^{n+2}/16$. Considering the initial conditions, $(n = 0) : 1 = a_0 = \alpha_1 + 25/16, \quad (n = 1) : 1 = a_1 = \alpha_1 + \alpha_2 + 125/16$. This has solution $\alpha_1 = -9/16, \alpha_2 = -132/16$, and so the specific solution is $a_n = (-9 - 132n + 5^{n+2})/16$.

**Examle 7:** $a_0 = 2, a_n = 3a_{n-1} - 4n \ (n \geq 1)$

Homogeneous version: $a_n = 3a_{n-1}$, which has characteristic equation $x = 3$ and general solution $\alpha 3^n$.

Nonhomogeneous version: We guess a solution of $\beta_1 n + \beta_0$. Plugging into the nonhomogeneous equation, we get $(\beta_1 n + \beta_0) = 3(\beta_1(n-1) + \beta_0) - 4n$. Simplifying, we get $0 = (2\beta_1 - 4)n + (-3\beta_1 + 2\beta_0)$. If a polynomial equals zero, then each coefficient must equal zero; hence $0 = 2\beta_1 - 4$ and $0 = -3\beta_1 + 2\beta_0$. We solve this system to get $\beta_1 = 2, \beta_0 = 3$. Hence $2n + 3$ is a solution to the nonhomogeneous recurrence.
Putting them together, we find the general solution to the nonhomogeneous recurrence is $a_n = \alpha 3^n + 2n + 3$. With our initial condition, we have $(n = 0) : 2 = a_0 = \alpha 3^0 + 3$, so $\alpha = -1$. So the specific solution is $a_n = -3^n + 2n + 3$.

**Example 8 (Tower of Hanoi)**: We have three pegs and $n$ disks of different sizes. The disks all start on one peg arranged in order of size, and we must move them to another. We move one disk at a time, and may never put a larger disk onto a smaller. How many moves does it take?

Let $a_n$ represent the answer. We see that $a_1 = 1$. To move the biggest disk from peg 1 to peg 2, all the smaller disks must be in a single stack, on peg 3. Therefore, the solution must contain three steps: First, move the $n-1$ smaller disks from peg 1 to peg 3, then move the largest disk form peg 1 to peg 2, then move the $n-1$ smaller disks back onto the largest disk from peg 3 to peg 2. Hence, $a_n = a_{n-1} + 1 + a_{n-1} = 2a_{n-1} + 1$.

The homogeneous recurrence is again $a_n = 2a_{n-1}$ with general solution $\alpha 2^n$. To find a specific solution to the nonhomogeneous recurrence, consider a constant (0-th degree) polynomial in $n$, say $\beta$. Plugging into the nonhomogeneous equation, we get $\beta = 2\beta + 1$; we solve this to get $\beta = -1$. Hence the general solution to the nonhomogeneous relation is $a_n = \alpha 2^n - 1$. Our initial conditions tell us $1 = a_1 = \alpha 2^1 - 1$; hence $\alpha = 1$ and our specific solution is $a_n = 2^n - 1$.

**Example 9 (Gambler's ruin revisited):** Consider the gambler of example 4. What is the expected number of games played until either the gambler or casino is ruined?

Let $a_n$ denote the desired answer (when the gambler starts with $\$n$). If the gambler wins, then the expected number of games is one more than the expected number of games, had the gambler started with $\$(n+1)$. If the gambler loses, then the expected number of games is one more than the expected number of games, had the gambler started with $\$(n-1)$. Hence we get the relation $a_n = p(a_{n+1} + 1) + q(a_{n-1} + 1)$ $(0 < n < m)$. We have boundary conditions $0 = a_0 = a_m$, and may rewrite the relation as $pa_{n+1} = a_n - qa_{n-1} - 1$. The homogeneous recurrence has the familiar characteristic equation $px^2 - x + q = 0$; once again the problem splits into cases based on whether $q = p$.

$(p \neq q)$: The homogeneous general solution is $\alpha + \beta r_2^n$ (recall that $r_2 = q/p$). If we try to guess a 0-th degree polynomial solution to the nonhomogeneous recurrence, we will find no luck (try it and see). The reason is that all 0-th degree polynomials are already solutions of the homogeneous recurrence, and so none of them could ever solve the nonhomogeneous recurrence.

Instead let's try a first-degree polynomial $c_1n + c_0$. We plug into the nonhomogeneous equation to get $p(c_1(n+1) + c_0) = c_1n + c_0 - q(c_1(n-1) + c_0) - 1$. We collect terms to get $n(pc_1 - c_1 + qc_1) + (pc_1 + pc_0 - c_0 - qc_1 + qc_0 + 1) = 0$. The first coefficient is zero already, and the second coefficient simplifies to $(p-q)c_1 + 1 = 0$; hence $c_1 = -1/(p-q)$, and we may as well take $c_0 = 0$ although the choice is arbitrary (in fact, we could have known this since all constants are part of the homogeneous solution). Therefore, the general nonhomogeneous solution is $a_n = \alpha + \beta r_2^n - n/(p-q)$. For the particular solution, we take $0 = a_0 = \alpha + \beta, 0 = a_m = \alpha + \beta r_2^m - m/(p-q)$. This has solution $\beta = \frac{m}{(1-2p)(1-r_2^m)}, \alpha = -\beta$. We plug these into the general solution, to find $a_n = \left(n - m\frac{1-r_2^n}{1-r_2^m}\right)/(1-2p)$.

$(p = q = 1/2)$: The homogeneous general solution is $\alpha + \beta n$. We won't get very far trying low-degree polynomials, since they are all part of the homogeneous solution. So, let's try $cn^2$. We plug into the nonhomogeneous equation to get $pc(n+1)^2 = cn^2 - qc(n-1)^2 - 1$. We rewrite to get $n^2(pc - c + qc) + n(2pc - 2qc) + (pc + qc + 1) = 0$. Since $p = q = 1/2$, the first two coefficients are zero already, and the last is zero when $c = -1$. Hence the general nonhomogeneous solution is $a_n = \alpha + \beta n - n^2$. For the specific solution, we take $0 = a_0 = \alpha, 0 = a_m = \alpha + \beta m - m^2$. This has solution $\alpha = 0, \beta = m$. Hence, the specific solution is $a_n = mn - n^2 = n(m-n)$.

**Exercises**

Solve the following recurrence relations.

1. $a_0 = a_1 = 2$, $a_n = -2a_{n-1} - a_{n-2}$ $(n \geq 2)$
2. $a_0 = 0, a_1 = 1$, $a_n = 4a_{n-2}$ $(n \geq 2)$
3. $a_0 = 2, a_1 = -4, a_2 = 26, a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3}$ $(n \geq 3)$
4. $a_0 = a_1 = a_2 = 0, a_n = 9a_{n-1} - 27a_{n-2} + 27a_{n-3}$ $(n \geq 3)$
5. $a_0 = a_1 = 0, a_n = a_{n-1} + 2a_{n-2} + 3$ $(n \geq 2)$
6. $a_0 = a_1 = 0, a_n = a_{n-1} + 2a_{n-2} + n$ $(n \geq 2)$
7. $a_0 = a_1 = 0, a_n = a_{n-1} + 2a_{n-2} + e^n$ $(n \geq 2)$
8. Let $a_n$ be the number of $n$-digit nonnegative integers in which no three consecutive digits are the same. Justify that $a_{n+2} = 9a_{n+1} + 9a_n$, then find $a_n$.
9. Let $a_n$ be the number of ways to color the squares of a $1 \times n$ chessboard using the colors red, white, and blue, so that no two red squares are adjacent.
10. Let $a_n$ be the number of ways to color the squares of a $1 \times n$ chessboard using the colors red, white, and blue, so that no red square is adjacent to a white square.
11. Let $a_n$ be the number of ways to color the squares of a $1 \times n$ chessboard using the colors red, white, and blue, so that the specific sequence red-white-blue does not occur.
12. Let $a_n$ be the number of ways to climb a flight of $n$ stairs, when each of your steps may move you one, two, or three steps higher.